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Summary

This paper studies a continuous-time regulatory problem in which a firm holds persistent private information about demand and is subject to a flow limited-liability constraint. The regulator regulates prices through a dynamic mechanism that ensures truthful reporting of the evolving type. Limited liability imposes a state-dependent lower bound on the firm's instantaneous utility, inducing a reflecting boundary in continuation utility and giving rise to a tractable singular-control representation. We derive closed-form expressions for the optimal pricing rule and the associated continuation-utility function, and we characterize the optimal up-front transfer required to induce truthful revelation of the firm's initial type. The resulting contract is fully explicit and highlights how limited liability shapes information rents and regulatory distortions over time.

Keywords: Dynamic regulation, Limited liability, Adverse selection, Continuous-time contracting, Reflecting boundary, Singular control

JEL Classification: D82, D86, L51, H54, C61

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Abstract

This paper studies a continuous-time regulatory problem in which a firm holds persistent private information about demand and is subject to a flow limited-liability constraint. The regulator regulates prices through a dynamic mechanism that ensures truthful reporting of the evolving type. Limited liability imposes a state-dependent lower bound on the firm's instantaneous utility, inducing a reflecting boundary in continuation utility and giving rise to a tractable singular-control representation. We derive closed-form expressions for the optimal pricing rule and the associated continuation-utility function, and we characterize the optimal up-front transfer required to induce truthful revelation of the firm's initial type. The resulting contract is fully explicit and highlights how limited liability shapes information rents and regulatory distortions over time.

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1 Introduction

Dynamic regulation with persistent private information has long been a central topic in the theory of incentives. Much of the classical literature considers environments with static private information and emphasizes ex ante participation constraints. Starting with the dynamic analyses of Baron and Besanko (1984), Besanko (1985) and Laffont and Tirole (1988), subsequent work has shown that when private information evolves stochastically and participation must be ensured over time, the structure of optimal regulation changes in fundamental ways.

This paper studies a continuous-time regulatory relationship in which a firm holds persistent but imperfectly informative private information about demand conditions. The firm's type follows a driftless geometric Brownian motion, generating a dynamic signal that affects both current pricing incentives and the informational content relevant for future regulatory decisions. Building on the dynamic mechanism design literature (see Bergemann and Välimäki, 2019), we use continuous-time envelope methods and the notion of consistent deviations (Bergemann and Strack, 2015) to characterize incentive-compatible regulatory mechanisms in the presence of state-dependent uncertainty.

The envelope characterization and the restriction to consistent deviations follow the now-standard approach to dynamic screening in continuous-time stochastic environments. These tools allow us to represent local incentive compatibility in terms of the evolution of continuation utility. Our contribution does not lie in this local characterization per se. Rather, it lies in showing that when incentive compatibility is combined with a pointwise flow limited-liability constraint, the dynamic screening problem acquires an endogenous state constraint on continuation utility that alters the global structure of the contract.

More formally, the flow limited-liability constraint induces a reflecting boundary in continuation utility. This reflection is not imposed exogenously, but emerges from the interaction between the envelope condition and the non-negativity requirement on instantaneous utility. As a result, a standard dynamic screening problem admits a singular-control representation with endogenous reflection.

A central feature of our environment is a flow limited-liability constraint, which requires the firm's instantaneous utility to remain non-negative at all times. This constraint can be interpreted as an ex post flow participation or liquidity requirement (Ollier and Thomas, 2013; Krähmer and Strausz, 2015, 2025) and has two key implications for contract design. First, it restricts the admissible set of transfers at every instant, thereby introducing distortions relative to the unconstrained regulatory benchmark. Second, the flow limited-liability constraint imposes an endogenous lower bound on instantaneous utility. Because continuation utility aggregates these flow payoffs over time, this pointwise constraint translates into a restriction on its admissible path. Whenever the unconstrained allocation would drive flow utility below zero, the contract must adjust prices and transfers so as to keep continuation utility at the boundary implied by the flow limited-liability constraint.

In standard dynamic screening models without flow constraints, continuation utility evolves smoothly in response to shocks and incentive provision. Here, however, whenever demand realizations would imply negative flow utility, the regulator must intervene immediately through prices or transfers to prevent a violation of the flow limited-liability constraint. This intervention keeps instantaneous utility at zero and prevents further downward movement of continuation utility. The boundary therefore acts as a minimal intervention rule: the regulator adjusts instruments only when necessary to maintain feasibility, and otherwise lets the incentive scheme operate in the interior.

The interaction between flow limited liability and dynamic incentive provision is subtle. As shown by Krähmer and Strausz (2015) and Bergemann, Castro and Weintraub (2020), ex post participation constraints may collapse dynamic screening into static contracts or induce rich history dependence. Other contributions show that flow limited-liability constraints can simplify contractual forms while generating dynamic path dependence in utilities and allocations (Grillo and Ortner, 2018; Krasikov and Lamba, 2021). In our setting, limited liability does not eliminate dynamic incentives but instead reshapes the propagation of information rents over time.

In addition to dynamic reporting, the regulator faces an initial adverse-selection problem. Because future demand realizations depend on the initial state, the mechanism requires a type-contingent up-front transfer, whereby the firm makes a payment to the regulator upon contract signing, economically an entry fee, in order to induce truthful revelation. Using dynamic envelope arguments (Milgrom and Segal, 2002), we obtain an explicit representation of the firm's intertemporal utility and show that the flow limited-liability constraint increases the cost of implementing truthful revelation. Intuitively, higher initial types are less informative about future demand, leading the regulator to optimally extract more surplus through higher up-front payments.

Taken together, dynamic reporting, initial adverse selection, and the flow limited-liability constraint yield a continuous-time screening problem in which continuation utility solves a reflected boundary-value problem. While the demand process remains an uncontrolled stochastic process, the participation constraint endogenously imposes a lower boundary on utility. This structure delivers closed-form expressions for the optimal pricing rule, the continuation-utility process, and the associated up-front transfer.

The framework applies naturally to regulated utilities, public-private partnerships, and procurement environments with liquidity constraints. More broadly, it clarifies how the flow limited-liability constraint modifies the standard continuous-time envelope representation of dynamic screening by introducing endogenous state constraints on continuation utility. It also complements continuous-time models of dynamic contracting with limited liability and financial constraints (DeMarzo and Sannikov, 2006), offering insights particularly suited to settings in which regulation operates primarily through price controls.

The remainder of the paper is organized as follows. Section 2 reviews the related literature. Section 3 introduces the regulatory environment and describes the evolution of demand and the informational structure. Section 4 characterizes the optimal dynamic contract, deriving the optimal pricing rule and the associated transfer schedule under flow limited liability. Section 5 analyzes the optimal up-front transfer and the incentives for truthful revelation of the firm's initial type. Section 6 discusses robustness to alternative demand dynamics and specifications and presents an extension in which up-front transfers are infeasible. Section 7 concludes. All proofs are contained in the Appendix.

2 Related literature

This paper is related to several strands of the literature on regulation and dynamic contracting.

First, it builds on the classical theory of regulation under asymmetric information developed by Myerson (1981), Baron and Myerson (1982), Laffont and Tirole (1986, 1993) and Lewis and Sappington (1983, 1988a, 1988b) among others, in which a regulator uses nonlinear tariffs and transfers to trade off rent extraction and allocative efficiency in static environments. Our contribution departs from this framework by allowing the firm's private information to evolve stochastically over time and by imposing a flow limited-liability constraint, which fundamentally alters the structure of optimal regulation.

Second, our work is connected to the growing literature on dynamic mechanism design. Pavan, Segal and Toikka (2014) and Bergemann and Välimäki (2019) provide general characterizations of dynamically incentive-compatible mechanisms using envelope methods and recursive approaches, and Bergemann and Strack (2015) develop a continuous-time framework for dynamic revenue maximization.[1]

Whereas Bergemann and Strack (2015) allow continuation utilities to evolve freely subject only to incentive constraints, our environment introduces a pointwise flow limited-liability constraint that restricts continuation utility and gives rise to a boundary-value structure with a singular-control representation.

[1]Related contributions include Athey and Segal (2013), Battaglini (2005), and Williams (2011), who study dynamic incentive provision with persistent private information, as well as Sannikov (2008), which develops continuous-time incentive methods closely related to our flow limited-liability constraint. Explicit bridges to regulation include Arve and Martimort (2016), Arve and Zwart (2023), Di Corato et al. (2018), Buso et al. (2021, 2026).

Third, the paper contributes to the literature on dynamic contracts with limited liability and ex post participation. Ollier and Thomas (2013), Grillo and Ortner (2018), and Krasikov and Lamba (2021) study moral-hazard environments in which flow limited-liability constraints shape the intertemporal allocation of effort and rents.[2] Relatedly, Krähmer and Strausz (2015, 2025) analyze dynamic adverse-selection models in which the agent privately observes his type and the principal must satisfy either limited-liability or ex post participation constraints. They show that such constraints can fundamentally alter the nature of dynamic screening.

Our analysis differs by focusing on regulated demand and price control, and by showing that when limited liability binds pointwise in continuous time, it introduces an endogenous state constraint in continuation utility. By combining continuous-time envelope methods with a flow limited-liability constraint that operates at every instant, the paper highlights a general mechanism through which dynamic screening problems acquire reflected boundary structures and singular-control representations.

3 The model

Consider the following regulatory environment. At time $t = 0$, a public entity (henceforth, the regulator, *he*) grants a firm (*she*) the exclusive right to sell, for all $t > 0$, a non-storable good at a unit price p_t. The contractual horizon is sufficiently long to be approximated as infinite. Demand is given by:

$$Q_t = \theta_t - p_t,$$

where θ_t represents the realized demand state, privately observed by the firm, and thus constitutes its time-t type.

Production costs are given by

$$C_t = c \cdot Q_t + F,$$

where c is a publicly known constant marginal cost and $F \geq 0$ is a publicly known fixed operating cost incurred in each period.

The regulator does not observe θ_t when contracting but knows that it evolves as a driftless

[2] These papers focus on hidden-action settings rather than evolving private information, but they highlight how limited-liability constraints can generate reflecting boundaries and nonlinear continuation utilities, features that also arise in our dynamic adverse-selection framework

geometric Brownian motion starting from θ_0:[3]

$$d\theta_t = \sigma \cdot \theta_t \cdot dZ_t, \tag{1}$$

where Z_t is a standard Wiener process, and σ is the known constant instantaneous volatility. Integrating (1) yields:

$$\theta_t \equiv \phi(\theta_0, t, Z_t) = \theta_0 \cdot e^{-\frac{1}{2}\sigma^2 t + \sigma \int_0^t dZ_s} \tag{2}$$

The initial state of demand θ_0 is drawn from a continuous distribution function $G(\theta_0)$ with support $[\underline{\theta}, \overline{\theta}]$, density $g(\theta_0)$, and $g(\underline{\theta})$, $g(\overline{\theta}) > 0$. We assume that[4]

$$\frac{H(\theta_0)}{\theta_0} = \frac{1 - G(\theta_0)}{g(\theta_0) \cdot \theta_0}$$

is decreasing, with $\underline{\theta} \cdot g(\underline{\theta}) > 1$.

Throughout the paper, θ_0 denotes the firm's initial private information, while θ_t represents the realized state generated by the stochastic evolution $\theta_t \equiv \phi(\theta_0, t, Z_t)$. In this sense, private information enters only through the initial condition, while subsequent uncertainty is purely stochastic.

Equation (2) yields two important properties of demand dynamics. First, higher initial types lead to higher future demand, $\phi_\theta(\theta_0, t, Z_t) = \frac{\partial \phi(\theta_0, t, Z_t)}{\partial \theta_0} > 0$,[5] while the persistence of θ_0 diminishes over time, $\phi_{t\theta}(\theta_0, t, Z_t) = \frac{\partial^2 \phi(\theta_0, t, Z_t)}{\partial t \partial \theta_0} < 0$. Second, the relative influence of the initial type compared with contemporaneous shocks satisfies: $\frac{\phi_\theta(\theta_0, t, Z_t)}{\phi_Z(\theta_0, t, Z_t)} = \frac{1}{\sigma \cdot \theta_0}$. Thus, high initial types provide limited predictive power for future demand, and their informational relevance further decreases as volatility σ rises.

To screen types, the regulator designs an incentive scheme that induces truthful revelation. At each $t > 0$, he chooses a price p_t and may grant or collect monetary transfer T_t (with $T_t > 0$ denoting a subsidy and $T_t < 0$ a payment by the firm). Given (p_t, T_t), the firm's instantaneous utility is

$$u_t = (p_t - c) \cdot Q(p_t, \theta_t) - F + T_t \tag{3}$$

[3] The assumption of a driftless geometric Brownian motion isolates the role of uncertainty. Allowing for a nonzero drift would not qualitatively affect the analysis. Brownian-motion processes are standard in continuous-time models of dynamic contracting; see, among others, DeMarzo and Sannikov (2006), Sannikov (2008), Dosi and Moretto (2015), Di Corato et al. (2018), Bergemann and Strack (2022) and Arve and Zwart (2023).

[4] This assumption is weaker than the standard increasing hazard-rate condition; see, for instance, Guesnerie and Laffont (1984) and Jullien (2000).

[5] This derivative is known in the theory of stochastic differential equations as the stochastic flow associated with the diffusion (Kunita, 1997). It serves as the continuous-time analogue of the impulse response function introduced by Pavan, Segal and Toikka (2014) in discrete-time dynamic mechanism design.

Assuming the regulator is benevolent and utilitarian, his instantaneous utility is

$$w_t = S(p_t, \theta_t) - (c \cdot Q(p_t, \theta_t) + F) - \lambda \cdot T_t, \tag{4}$$

where $S(p_t, \theta_t) \equiv Q(p_t, \theta_t) \cdot (\theta_t - \frac{Q(p_t,\theta_t)}{2})$ is consumer surplus and $\lambda > 0$ is the marginal cost of public funds (Laffont and Tirole, 1986).

The contract includes a flow limited-liability clause requiring that the regulator choose (p_t, T_t) such that the firm's instantaneous utility (3) remains non-negative for all $t > 0$:

$$u_t \geq 0, \text{ for all } t > 0. \tag{LL}$$

The firm is therefore subject to a flow limited-liability constraint (LL constraint), which must hold pointwise in time. This constraint is imposed at every instant and represents an ex-post flow participation (or liquidity) requirement.[6]

Unlike standard ex-ante participation constraints, which restrict only expected utility at the contracting stage, the LL constraint operates at every instant and directly constrains instantaneous utility.

We assume full commitment by both parties throughout.

4 The optimal contract

Equation (2) implies two key features. First, the parties' instantaneous utilities (u_t, w_t) at any $t > 0$ are independent of past utilities. Second, these instantaneous utilities depend only on the initial type θ_0 and the current demand state θ_t. This structure allows the regulatory problem to be organized into two steps.

Step 1 For any given initial type θ_0, the regulator offers, at each time $t > 0$, a single-period contract $(p(\theta_0, \theta_t), T(\theta_0, \theta_t))$ designed to induce the firm to truthfully report the current demand state θ_t. Because of the LL constraint, this contract must ensure that the firm's instantaneous utility satisfies $u(\theta_0, \theta_t) \geq 0$, for all $t > 0$.

Step 2 Since each future realization θ_t depends on the initial state θ_0 and on the contemporaneous shock through the mapping $\theta_t = \phi(\theta_0, Z_t)$,[7] the contract must specify at $t = 0$ a type-contingent transfer $K(\theta_0)$ that induces the firm to truthfully disclose its initial type.

[6] See, for instance, Ollier and Thomas (2013), Krähmer and Strausz (2025), and Martimort et al. (2025).
[7] For notational convenience, we omit the explicit dependence of $\phi(\theta_0, t, Z_t)$ on time t.

The firm's expected intertemporal utility at time $t = 0$ is therefore

$$U(\theta_0) = E_0 \left[\int_0^{\infty} e^{-rt} \cdot u(\theta_0, \theta_t) \cdot dt \right] + K(\theta_0) \tag{5}$$

and the regulator's expected utility is

$$W(\theta_0) = E_0 \left[\int_0^{\infty} e^{-rt} \cdot w(\theta_0, \theta_t) \cdot dt \right] - \lambda \cdot K(\theta_0) \tag{6}$$

where the discount rate r is common to both parties.

4.1 Incentive-compatibility conditions

We begin by characterizing the conditions under which a direct mechanism is incentive-compatible. Our analysis relies on the notion of consistent deviations introduced by Bergemann and Strack (2015).[8] Consistent deviations restrict attention to misreports that preserve the law of motion of the reported state, allowing the regulator to evaluate truthful and deviating paths under the same probability measure.[9]

As usual, it is convenient to proceed backwards, beginning with $t > 0$. Time separability of the allocation implies that, for any given initial report, incentive compatibility at each $t > 0$ reduces to a static reporting problem in the current state.

Suppose the regulator has already elicited a truthful report of the initial type θ_0. Then, for any given θ_0, the firm's decision at time $t > 0$ concerns only the current demand state θ_t, which is privately observed and therefore constitutes the firm's type at time t. In particular, given a report $\hat{\theta}_t$, the firm's instantaneous utility is

$$u(\theta_0, \theta_t, \widehat{\theta}_t) = (p(\theta_0, \widehat{\theta}_t) - c) \cdot Q(p(\theta_0, \widehat{\theta}_t), \theta_t) - F + T(\theta_0, \widehat{\theta}_t) \tag{7}$$

where θ_t denotes the realized demand state and $\hat{\theta}_t$ the firm's report.

The following Lemma provides the conditions for truth-telling.[10]

Lemma 1 *By the Revelation Principle, necessary and sufficient conditions for incentive*

[8] A deviation is consistent if a firm of type θ_0 misreports at $t = 0$ by announcing some $\widehat{\theta}_0$ and thereafter reports a path $\theta_t \equiv \phi(\widehat{\theta}_0, Z_t)$ that evolves according to the same diffusion as the true type. Since the induced report follows the same law of motion, the deviation cannot be detected ex post, and prices and transfers are determined by the misreported initial type and its consistent evolution.

[9] While Pavan, Segal and Toikka (2014) do not use the term "consistent deviations," they rely on the same restriction on deviations, which must respect the state dynamics and the reporting strategy.

[10] The fixed operating cost F does not affect the incentive constraints below, since it enters flow utility additively and is independent of the report.

compatibility at each $t > 0$ are:

$$\frac{\partial u(\theta_0, \theta_t)}{\partial \theta_t} = p(\theta_0, \theta_t) - c, \tag{8}$$

and

$$\frac{\partial p(\theta_0, \theta_t)}{\partial \theta_t} \geq 0, \text{ for all } t > 0. \tag{9}$$

Proof. See Appendix A. ■

Once the optimal schedule $(p(\theta_0, \theta_t), T(\theta_0, \theta_t))$ is determined, we move backward to $t = 0$ to determine the optimal initial transfer $K(\theta_0)$. At this stage, the principal faces another static adverse-selection problem, this time with respect to the initial type.

Given a reported type $\widehat{\theta}_0$, the firm's intertemporal utility at $t = 0$ is:

$$U(\theta_0, \widehat{\theta}_0) = E_0 \left\{ \int_0^\infty e^{-rt} \cdot \left[\begin{array}{c} \left(p(\widehat{\theta}_0, \phi(\widehat{\theta}_0, Z_t)) - c\right) \cdot Q(p(\widehat{\theta}_0, \phi(\widehat{\theta}_0, Z_t)), \phi(\theta_0, Z_t)) + \\ -F + T(\widehat{\theta}_0, \phi(\widehat{\theta}_0, Z_t)) \end{array} \right] \cdot dt \right\} + K(\widehat{\theta}_0). \tag{10}$$

The transfer $K(\theta_0)$ must induce truthful disclosure, i.e. $\widehat{\theta}_0 = \theta_0$. The next Lemma characterizes the corresponding incentive-compatibility conditions.

Lemma 2 *By the Revelation Principle, the mechanism is incentive-compatible if and only if:*

$$\frac{dU(\theta_0)}{d\theta_0} = E_0 \left\{ \int_0^\infty e^{-rt} \cdot (p(\theta_0, \phi(\theta_0, Z_t)) - c) \cdot \frac{\partial \phi(\theta_0, Z_t)}{\partial \theta_0} \cdot dt \right\} \tag{11}$$

and

$$E_0 \left\{ \int_0^\infty e^{-rt} \cdot \frac{dp(\theta_0, \phi(\theta_0, Z_t))}{d\theta_0} \cdot \frac{\partial \phi(\theta_0, Z_t)}{\partial \theta_0} \cdot dt \right\} \geq 0 \tag{12}$$

Proof. See Appendix A. ■

Equation (11) follows from an envelope argument applied to the firm's intertemporal utility with respect to the initial type. As in Bergemann and Strack (2015), restricting attention to consistent deviations is sufficient to characterize the derivative of the value function. The right-hand side of (11) is the expected discounted value of the firm's instantaneous markup $(p(\theta_0, \theta_t) - c)$, weighted by the term, $\frac{\partial \phi(\theta_0, Z_t)}{\partial \theta_0}$, which measures the persistence over time of a marginal change in the initial state θ_0.

4.2 The optimal time-varying prices and transfers

Given a truthful report of the initial type θ_0, the regulator's problem is to determine the optimal price-transfer pair $(p(\theta_0, \theta_t), T(\theta_0, \theta_t))$, solving

$$\max_{p(\theta_0,\theta_t),T(\theta_0,\theta_t)} \int_{\underline{\theta}}^{\overline{\theta}} \left[\begin{array}{c} E_0 \left\{ \int_0^\infty e^{-rt} \cdot \left(\begin{array}{c} S(Q(p(\theta_0,\theta_t),\theta_t)) - (c \cdot Q(p(\theta_0,\theta_t),\theta_t) + F) + \\ -\lambda \cdot T(\theta_0,\theta_t) \end{array} \right) \cdot dt \right\} + \\ -\lambda \cdot K(\theta_0) \end{array} \right] \cdot g(\theta_0) \cdot d\theta_0. \tag{13}$$

This maximization is subject to:

1. the incentive-compatibility conditions (8)-(9);
2. the LL constraint
$$u(\theta_0, \theta_t) \geq 0, \text{ for all } t > 0; \tag{14}$$
3. the ex-ante participation constraint[11]
$$U(\theta_0) \geq 0. \tag{15}$$

Before solving (13), observe that the LL constraint restricts the admissible levels of the firm's instantaneous utility schedule. Integrating (8) with respect to θ_t yields

$$u(\theta_0, \theta_t) = \int_0^{\theta_t} (p(\theta_0, z) - c) \cdot dz + u(\theta_0, 0),$$

where $u(\theta_0, 0)$ collects all terms independent of θ_t. Under (8)-(9), $u(\theta_0, \theta_t)$ is convex in θ_t and therefore attains its minimum at a point $a(\theta_0)$ defined implicitly by $p(\theta_0, a(\theta_0)) = c$. The LL constraint requires that this minimum be non-negative, i.e. $u(\theta_0, a(\theta_0)) \geq 0$. In the optimal contract, transfers are chosen so that the constraint binds at the minimum, implying $u(\theta_0, a(\theta_0)) = 0$.

With these observations in hand, solving (13) yields the following proposition.

Proposition 1 *For any $t > 0$, the firm's optimal regulated price is:*

$$p^*(\theta_0, \theta_t) = \begin{cases} c + \frac{\lambda}{1+2\lambda} \cdot \left[\theta_t \cdot (1 - \frac{H(\theta_0)}{\theta_0}) - c\right], & \textit{for } \theta_t > a(\theta_0) \\ c, & \textit{for } \theta_t \leq a(\theta_0) \end{cases} \tag{16}$$

[11] Constraint (14) is an ex-post (flow) participation requirement that must hold at $t > 0$, whereas (15) is an ex-ante participation constraint that is imposed only once at $t = 0$.

where:

$$a(\theta_0) = \frac{c}{1 - \frac{H(\theta_0)}{\theta_0}} \tag{17}$$

Proof. See Appendix B. ■

Equation (16) shows that when demand is sufficiently high, i.e. $\theta_t > a(\theta_0)$, the regulator must leave information rents to induce truthful reporting. These rents, reflected in the markup over the marginal cost, increase in both θ_t and in θ_0. Hence, the monotonicity conditions (9) and (12) are satisfied, since $\frac{\partial p^*(\theta_0,\theta_t)}{\partial \theta_t} > 0$ and $\frac{\partial p^*(\theta_0,\theta_t)}{\partial \theta_0} > 0$, whenever $\theta_t > a(\theta_0)$.

Equation (16) can be rewritten as

$$\frac{p^*(\theta_0, \theta_t) - c}{p^*(\theta_0, \theta_t)} = \frac{\lambda}{1+\lambda} \cdot \left[\frac{1}{\varepsilon(p_t^*, \theta_t)} - \frac{H(\theta_0) \cdot \frac{\partial \phi(\theta_0, Z_t)}{\partial \theta_0}}{p^*(\theta_0, \theta_t)} \right] \tag{18}$$

where $\varepsilon(p_t^*, \theta_t) = \left| \frac{dQ^*(\theta_0,\theta_t)}{dp^*(\theta_0,\theta_t)} \cdot \frac{p^*(\theta_0,\theta_t)}{Q^*(\theta_0,\theta_t)} \right|$. Expression (18) defines an incentive-compatible Lerner index (or virtual markup) (see Laffont and Tirole, 1993; Aguirre and Beitia, 2004). The term involving $\frac{\partial \phi(\theta_0, Z_t)}{\partial \theta_0}$ captures the dynamic virtual valuation associated with the persistence of private information. In contrast, when demand is sufficiently low, i.e. $\theta_t \leq a(\theta_0)$, the optimal price schedule coincides with marginal cost, $p^*(\theta_0, \theta_t) = c$, and no information rents are paid. This region corresponds to states in which the dynamic virtual utility is non-positive and therefore the optimal allocation coincides with the efficient benchmark. In this region, conditions (9) and (12) hold trivially, since $\frac{\partial p^*(\theta_0,\theta_t)}{\partial \theta_t} = \frac{\partial p^*(\theta_0,\theta_t)}{\partial \theta_0} = 0$.

To characterize transfers, note from (16) that information rents accumulate according to

$$u^*(\theta_0, \theta_t) = \begin{cases} \int_{a(\theta_0)}^{\theta_t} (p^*(\theta_0, z) - c) \cdot dz, & \text{for } \theta_t > a(\theta_0), \\ 0, & \text{for } \theta_t \leq a(\theta_0). \end{cases} \tag{19}$$

Substituting (19) into the definition of flow utility (3), transfers are chosen to implement the desired utility profile, yielding the optimal transfer schedule

$$T^*(\theta_0, \theta_t) = \begin{cases} \int_{a(\theta_0)}^{\theta_t} (p^*(\theta_0, z) - c) \cdot dz - (p^*(\theta_0, \theta_t) - c) \cdot Q^*(\theta_0, \theta_t) + F, & \text{for } \theta_t > a(\theta_0), \\ F, & \text{for } \theta_t \leq a(\theta_0). \end{cases} \tag{20}$$

From (16) and (19), we obtain:

Corollary 1 *For all $\theta_t > a(\theta_0)$, the optimal periodic transfer $T^*(\theta_0, \theta_t)$ is strictly decreasing in θ_t. Moreover, its sign depends on the magnitude of the fixed operating F : for sufficiently large F, the transfer may be positive for low values of $\theta_t > a(\theta_0)$ and becomes negative only for higher demand realizations.*

Proof. See Appendix B. ■

This result reflects the fact that, in the presence of a fixed operating cost, the regulator may optimally subsidize the firm when demand is low in order to satisfy the flow participation constraint, while progressively reducing the subsidy, or imposing a tax, as demand increases.

4.3 Evaluating intertemporal utility under limited liability

The firm's expected intertemporal utility (5) can now be written as

$$U^*(\theta_0) = E_0 \left\{ \int_0^{\infty} e^{-rt} \cdot u^*(\theta_0, \theta_t) \cdot dt \right\} + K(\theta_0) \tag{21}$$

where $u^*(\theta_0, \theta_t)$ is given by (19).

The LL constraint restricts the realized flow utility to be non-negative at all times. Under the optimal schedule above, this restriction is implemented by a state-dependent lower bound that is attained at $\theta_t = a(\theta_0)$.

Equation (21) can be evaluated by representing continuation utility through a stochastic process reflected at the endogenous boundary $a(\theta_0)$.[12] Under the envelope condition, instantaneous utility evolves proportionally to the realized markup $p^*(\theta_0, \theta_t) - c$. In the interior region $\theta_t > a(\theta_0)$, the markup is strictly positive and continuation utility fluctuates smoothly with the demand state. When demand falls sufficiently so that the unconstrained markup would become negative, the regulator must adjust the price so that $p^*(\theta_0, \theta_t) = c$, implying zero marginal rents. At that point instantaneous utility stops decreasing and remains equal to zero. Further downward movement of continuation utility is therefore prevented by construction. In continuous time, this minimal intervention rule is equivalent to a reflecting boundary at the cutoff $a(\theta_0)$: continuation utility evolves freely in the interior and is adjusted only when it reaches the boundary through instantaneous control. This reflection captures the idea that the regulator intervenes only to prevent violations of the LL constraint and otherwise lets the incentive scheme operate without distortion. The boundary therefore emerges endogenously from the interaction between the envelope condition and the non-negativity constraint on flow utility.

Solving the associated boundary-value problem yields the following result.[13]

Lemma 3 *Under $r > \sigma^2$, the firm's intertemporal utility (21) admits the representation*

$$U^*(\theta_0) = m(\theta_0) + A(a(\theta_0)) \cdot \theta_0^{\beta} + K(\theta_0), \tag{22}$$

[12] See Harrison (1985, 2013) and Stokey (2009). In the stochastic control literature, *reflected* stochastic processes are often referred to as *regulated* stochastic processes, following Harrison's terminology.

[13] The restriction $r > \sigma^2$ guarantees convergence of the discounted expectation under geometric Brownian motion, ensuring that the value function remains finite.

where

$$m(\theta_0) = \frac{\lambda}{1+2\lambda} \cdot \left[\frac{1}{2} \cdot \frac{\theta_0}{r-\sigma^2} \cdot (1 - \frac{H(\theta_0)}{\theta_0}) - \frac{c}{r}\right] \cdot \theta_0, \tag{22.1}$$

$$A(a(\theta_0)) = -\frac{\lambda}{1+2\lambda} \cdot \frac{\sigma^2}{r-\sigma^2} \cdot \frac{c}{r} \cdot \frac{a(\theta_0)^{1-\beta}}{\beta} \tag{22.2}$$

and $\beta < 0$ *is the negative root of* $\frac{\sigma^2}{2} \cdot \beta \cdot (\beta - 1) - r = 0$.

Proof. See Appendix C. ■

In (22), the term $m(\theta_0)$ corresponds to the expected present value of information rents in the absence of limited liability, while the term $A(a(\theta_0)) \cdot \theta_0^{\beta}$ captures the additional utility generated by the reflecting boundary induced by limited liability. Since this term is strictly positive, the LL constraint increases the firm's lifetime utility for any given θ_0. This term is strictly positive because the reflecting boundary prevents the firm's continuation utility from drifting into negative regions in low-demand states. Limited liability provides an insurance component against adverse demand realizations: whenever demand falls sufficiently, the regulator adjusts the price to marginal cost and eliminates further rent losses. The reflection term therefore captures the value generated by this state-contingent protection.

5 The optimal up-front transfer

Since the pair $(p^*(\theta_0, \theta_t), T^*(\theta_0, \theta_t))$ induces truthful revelation of the contemporaneous type θ_t, the problem at $t = 0$ reduces to a static mechanism-design problem aimed at eliciting the initial type θ_0, subject to the incentive-compatibility conditions (11) and (12).

Integrating (11) over the interval $[\underline{\theta}, \theta_0]$ yields the envelope representation of the firm's intertemporal utility. This representation follows directly from the dynamic envelope theorem of Bergemann and Strack (2015, Theorem 2).

$$U^*(\theta_0) = \int_{\underline{\theta}}^{\theta_0} \frac{dU^*(y)}{dy} \cdot dy + U^*(\underline{\theta}) \tag{23}$$

The derivative

$$\frac{dU^*(y)}{dy} = E_0\left[\int_0^\infty e^{-rt} \cdot (p^*(y, \theta_t) - c) \cdot \frac{\theta_t}{y} \cdot dt\right] \tag{23.1}$$

represents the expected present value of the inframarginal rents accruing to a firm that reports type y at $t = 0$. Since the optimal pricing rule satisfies $p^*(\theta_0, \theta_t) \geq c$ for all θ_t, it follows that $\frac{dU^*(\theta_0)}{d\theta_0} \geq 0$. To satisfy the participation constraint at the lowest information cost, we therefore set $U^*(\underline{\theta}) = 0$.

Evaluating $dU^*(y)/dy$ requires computing the expected discounted value of the flow markup along the stochastic path of demand. Under the optimal price schedule, this expectation can be represented as the value of a stochastic process with reflection at the lower boundary $a(y) = c/(1 - \frac{H(y)}{y})$, where $H(y)$ is sufficiently smooth so that $a(y) = c/(1 - \frac{H(y)}{y})$ is differentiable and $da(y)/dy < 0$ (see Appendix C).

Solving the associated reflected stochastic-process problem yields the following decomposition.

Lemma 4 *Under $r > \sigma^2$, the envelope integral satisfies*

$$\frac{dU^*(y)}{dy} = n(y) + B(a(y)) \cdot y^{\beta}, \tag{24}$$

where

$$n(y) = \frac{\lambda}{1 + 2\lambda} \cdot \left[\frac{y}{r - \sigma^2} \cdot \left(1 - \frac{H(y)}{y}\right) - \frac{c}{r}\right], \tag{24.1}$$

$$B(a(y)) = -\frac{\lambda}{1 + 2\lambda} \cdot \left(\frac{2}{r - \sigma^2} - \frac{1}{r}\right) \cdot \frac{c}{y} \cdot \frac{a(y)^{1-\beta}}{\beta}. \tag{24.2}$$

and $\beta < 0$ is the negative root of $\frac{\sigma^2}{2} \cdot \beta \cdot (\beta - 1) - r = 0$.

Proof. See Appendix C. ■

The decomposition (24) shows that the LL constraint raises the marginal value of the initial type y by adding a strictly positive reflection term to the baseline envelope. The baseline term $n(y)$ corresponds to the contribution obtained under the interior pricing rule and the unconstrained evolution of the demand process, while the term $B(a(y)) \cdot y^{\beta}$ collects the boundary correction induced by limited liability.

We are now ready to determine the optimal up-front transfer.

Proposition 2 *The optimal up-front transfer is*

$$K^*(\theta_0) = \int_{\underline{\theta}}^{\theta_0} \left[n(y) + B(a(y)) \cdot y^{\beta}\right] \cdot dy - \left[m(\theta_0) + A(a(\theta_0)) \cdot \theta_0^{\beta}\right]. \tag{25}$$

In particular,

$$K^*(\underline{\theta}) = -\left[m(\underline{\theta}) + A(a(\underline{\theta})) \cdot \underline{\theta}^{\beta}\right] < 0, \text{ and } \frac{dK^*(\theta_0)}{d\theta_0} < 0. \tag{25.1}$$

Proof. See Appendix C. ■

This Proposition has two implications. First, the optimal up-front transfer $K^*(\theta_0)$ is strictly negative, implying that the firm makes a payment to the regulator upon signing the contract, which economically corresponds to an entry fee. Second, this fee increases with θ_0.

The intuition is straightforward. The informational value of the initial type θ_0 depends on how predictive it is of future demand realizations. When θ_0 is relatively high, it is less informative about future demand (i.e. $\frac{\phi_\theta(\theta_0,t,Z_t)}{\phi_Z(\theta_0,t,Z_t)} = \frac{1}{\sigma\cdot\theta_0}$), reducing the regulator's informational gains from screening at $t = 0$. As a result, the regulator optimally extracts more surplus up front, leading to a larger up-front transfer $K^*(\theta_0)$.

Substituting (22.1) and (24.1) into (25), the up-front transfer can be decomposed as

$$K^*(\theta_0) = K^{no-LL}(\theta_0) + \Delta(\theta_0) \tag{26}$$

where:

$$K^{no-LL}(\theta_0) = \frac{\lambda}{1+2\lambda}\cdot\left\{\begin{array}{c} \int_{\underline{\theta}}^{\theta_0}\left[\frac{y}{r-\sigma^2}\cdot\left(1-\frac{H(y)}{y}\right)-\frac{c}{r}\right]\cdot dy+ \\ -\left[\frac{1}{2}\cdot\frac{\theta_0}{r-\sigma^2}\cdot(1-\frac{H(\theta_0)}{\theta_0})-\frac{c}{r}\right]\cdot\theta_0+ \end{array}\right\} \tag{26.1}$$

coincides with the optimal up-front transfer that would arise in the absence of limited liability, as characterized in Bergemann and Strack (2015, pp. 830-831).

The term

$$\Delta(\theta_0) = -\frac{\lambda}{1+2\lambda}\cdot\left[\begin{array}{c} \int_{\underline{\theta}}^{\theta_0}\left(2\cdot\frac{1}{r-\sigma^2}-\frac{1}{r}\right)\cdot\frac{c}{y}\cdot\frac{a(y)}{\beta}\cdot\left(\frac{y}{a(y)}\right)^{\beta}\cdot dy+ \\ -\frac{\sigma^2}{r-\sigma^2}\cdot\frac{c}{r}\cdot\frac{a(\theta_0)}{\beta}\cdot\left(\frac{\theta_0}{a(\theta_0)}\right)^{\beta} \end{array}\right] \tag{26.2}$$

isolates the intertemporal effect of limited liability on initial screening. By raising continuation utility in low-demand states through reflection, limited liability increases the informational rents associated with truthful revelation at $t = 0$. In this sense, the participation constraint propagates backward to the contracting stage.

Since a higher θ_0 reduces the probability that the demand process ever reaches the boundary $a(\theta_0)$, the impact of limited liability on the up-front transfer weakens with θ_0.

Corollary 2 *An increase in θ_0 reduces the impact of the limited-liability constraint on the optimal up-front transfer:*

$$\frac{d\Delta(\theta_0)}{d\theta_0} > 0 \tag{27}$$

Proof. See Appendix C. ■

6 Remarks and Extensions

This section collects remarks and extensions that clarify the structure of the problem and demonstrate the robustness and flexibility of the framework.

6.1 Robustness to Alternative Demand Dynamics

The analysis has been conducted under the assumption that the state of demand follows a driftless geometric Brownian motion. This specification isolates the role of volatility and allows for closed-form solutions. The boundary structure implied by flow limited liability does not rely on this particular stochastic process.

What matters for the mechanism is that the state of demand evolves as a continuous stochastic process with non-degenerate local uncertainty. The envelope condition links instantaneous utility to the realized demand state, and the LL constraint requires utility to remain non-negative at every instant. Whenever the demand intercept follows a regular stochastic process with continuous paths, continuation utility evolves smoothly in the interior and is constrained to remain within an admissible region. The non-negativity constraint therefore induces a lower boundary independently of the specific law of motion adopted for demand.

More generally, the geometric Brownian motion assumption can be replaced by any regular continuous-time stochastic process for the demand intercept, allowing, for instance, for a non-zero drift or for mean reversion, without altering the qualitative structure of the optimal contract. In the absence of limited liability, the continuation-utility function would solve the interior differential equation associated with the underlying stochastic process. The LL constraint adds a boundary condition and smooth-pasting at the cutoff level, yielding a boundary-value problem.

Closed-form expressions are generally no longer available outside the geometric Brownian motion specification. However, the qualitative structure of the optimal contract, consisting of interior stochastic evolution and minimal intervention at an endogenous lower boundary, is preserved for any regular continuous-time stochastic process.

6.2 Robustness to Alternative Demand Specifications

The analysis has relied on a linear demand specification in which the state variable enters as the intercept of inverse demand. This assumption simplifies the characterization of prices and transfers and allows for explicit expressions. The boundary structure induced by flow limited liability does not depend on linearity per se.

The key requirement is that, for each realized demand state, the regulator's static problem admits a well-defined interior solution satisfying the usual monotonicity conditions. More generally, suppose inverse demand is given by a twice continuously differentiable function $D(Q, \theta_t)$, strictly decreasing in output and smoothly parameterized by the demand state θ_t. Provided that consumer surplus is concave in output and that the envelope condition remains valid, the continuation-utility function remains differentiable in the interior of the state space.

Under these regularity conditions, the LL constraint continues to impose a state-dependent lower bound on instantaneous utility. The non-negativity requirement therefore generates a cutoff below which intervention keeps utility at zero. Nonlinear demand affects the quantitative shape of the optimal pricing rule and the magnitude of information rents, but not the qualitative boundary structure.

Closed-form solutions need not be available under nonlinear specifications. The boundary-value structure of the optimal contract is preserved under standard smoothness and monotonicity conditions. In particular, the envelope condition continues to link marginal rents to the state variable, while the pointwise limited-liability constraint continues to impose an endogenous lower bound on instantaneous utility. The resulting dynamic program therefore retains the same structural form, consisting of an interior characterization supplemented by smooth pasting at an endogenous boundary, independently of the specific curvature of demand.

6.3 Regulation without up-front transfers

In much of the mechanism-design literature, the principal can commit to state-contingent transfers. In contrast, the delegation literature typically rules them out. As argued by Baron (1989), Laffont and Tirole (1993), and Armstrong and Sappington (2007), regulators often lack the authority to make or receive monetary transfers.[14]

We consider a mixed environment in which the regulator may use periodic transfers $T(\theta_0, \theta_t)$ but cannot impose up-front fees; that is, $K(\theta_0) = 0$.

Imposing $K(\theta_0) = 0$ implies, by Proposition 2, that

$$\int_{\underline{\theta}}^{\theta_0} \left[n(y) + B(a(y)) \cdot y^{\beta}\right] \cdot dy < m(\theta_0) + A(a(\theta_0)) \cdot \theta_0^{\beta},$$

so that the informational rents associated with truthful revelation of the initial type cannot be fully extracted through an initial transfer.

To restore equality, the regulator must limit the firm's informational rents by constraining the price schedule. Fix a maximum admissible utility $\bar{u} > 0$. From Lemma 1, since $\frac{\partial u(\theta_0,\theta_t)}{\partial \theta_t} = p(\theta_0, \theta_t) - c$ and $\frac{\partial p(\theta_0,\theta_t)}{\partial \theta_t} \geq 0$, there exist a unique threshold $b(\theta_0, \bar{u}) > a(\theta_0)$ such that $u(\theta_0, b) = \bar{u}$, with associated $\bar{p} = p^*(\theta_0, b) > c$. Using (19), the threshold is given by

$$b(\theta_0, \bar{u}) = a(\theta_0) + \sqrt[2]{\frac{2 \cdot (1 + 2\lambda)}{\lambda} \cdot \frac{\bar{u}}{1 - \frac{H(\theta_0)}{\theta_0}}} > a(\theta_0). \tag{28}$$

[14] See also Alonso and Matouschek (2008), Amador and Bagwell (2013, 2022), and Wang et al. (2023), among others.

This construction yields the following result.

Proposition 3 *When up-front transfers are infeasible, the optimal regulatory price policy is*

$$p^{**}(\theta_0, \theta_t) = \begin{cases} p^*(\theta_0, \theta_t), & 0 < \theta_t < b(\theta_0, \bar{u}) \\ \bar{p} & b(\theta_0, \bar{u}) \leq \theta_t \end{cases} \tag{29}$$

where the price cap is

$$\bar{p} = p^*(\theta_0, b) = c + \sqrt[2]{2 \cdot \frac{\lambda}{1 + 2\lambda} \cdot \bar{u} \cdot (1 - \frac{H(\theta_0)}{\theta_0})} \tag{29.1}$$

Proof. See Appendix D. ■

In the absence of up-front transfers, the price cap reduces the set of feasible allocations and therefore entails a loss of surplus relative to the unconstrained benchmark. The induced information rents are given by

$$u^{**}(\theta_0, \theta_t) = \begin{cases} u^*(\theta_0, \theta_t), & \text{for } 0 < \theta_t < b(\theta_0, \bar{u}), \\ \bar{u}, & \text{for } b(\theta_0, \bar{u}) \leq \theta_t, \end{cases} \tag{31}$$

so that the continuation-utility function is subject to both a lower boundary (induced by limited liability) and an upper boundary (induced by the absence of up-front transfers).

From a technical perspective, the problem therefore features a lower reflecting boundary induced by the LL constraint and an upper reflecting boundary induced by the restriction $K(\theta_0) = 0$.

The maximum utility level $\bar{u}$ is chosen so that the firm's expected intertemporal utility equals the informational rents implied by truthful revelation of the initial type:

$$E_0 \left\{ \int_0^\infty e^{-rt} \cdot u^*(\theta_0, \theta_t) \cdot dt \right\} = \int_{\underline{\theta}}^{\theta_0} \frac{dU^*(y)}{dy} \cdot dy \tag{32}$$

The left-hand side can be written as

$$E_0 \left\{ \int_0^\infty e^{-rt} \cdot u^*(\theta_0, \theta_t) \cdot dt \right\} = m(\theta_0) + A_1 \cdot \theta_0^{\beta_1} + A_2 \cdot \theta_0^{\beta_2} \tag{32.1}$$

while

$$\frac{dU^*(y)}{dy} = n(y) + B_1 \cdot y^{\beta_1} + B_2 \cdot y^{\beta_2} \tag{32.2}$$

where $\beta_1 > 1$ and $\beta_2 < 0$ are the two roots of $\frac{\sigma^2}{2} \cdot \beta \cdot (\beta - 1) - r = 0$.

The constants A_1, A_2, B_1 and B_2 are obtained by solving the associated boundary-value problems for stochastic processes with lower and upper reflecting boundaries $(a(\theta_0), b(\theta_0, \bar{u}))$ and $(a(y), b(y, \bar{u}))$, respectively.

7 Conclusions

This paper develops a continuous-time model of price regulation under dynamic adverse selection and flow limited liability. The flow limited-liability constraint endogenously transforms the dynamic screening problem into a boundary-value problem in continuation utility, giving rise to a singular-control representation that admits a closed-form solution. We derive explicit expressions for the optimal pricing rule, periodic and up-front transfers that induce truthful revelation of both the firm's current demand state and its initial type. The analysis highlights how the flow limited-liability constraint reshapes the intertemporal propagation of information rents and generates state-dependent regulatory distortions, while preserving full analytical tractability.

Although the closed-form characterization relies on specific functional forms, the qualitative insights regarding endogenous reflection and singular control extend to broader classes of continuous-time stochastic processes and demand specifications.

Beyond its formal contribution, the framework provides a transparent economic interpretation of the role of limited liability in dynamic regulation. The boundary mechanism induced by the participation constraint is reminiscent of minimum-revenue guarantees commonly embedded in public-private partnership agreements, where prices or transfers are adjusted to ensure the regulated firm's viability in low-demand states. More broadly, pointwise participation constraints in continuous time transform dynamic screening problems into boundary-value formulations.

Several extensions offer promising directions for future research. One natural avenue is to replace price regulation with quantity regulation, allowing the regulator to adjust output rather than prices in order to satisfy the limited-liability constraint. A further extension would introduce explicit fiscal restrictions on transfers, reflecting the possibility that the regulator faces binding budget constraints or political limits on the provision of subsidies. Such constraints would further restrict the feasible set of contracts and may interact non-trivially with both limited liability and the reflecting structure of continuation utilities. Additional extensions include incorporating investment or cost dynamics, so that the firm's technology or operating costs evolve jointly with demand. More broadly, the framework could be adapted to study alternative regulatory instruments or institutional constraints in environments where liquidity considerations and dynamic information play a central role.

A Appendix A

Throughout Appendix A, derivatives with respect to reported types are taken holding fixed the initial report and the realization of the stochastic process.

A.1 Lemma 1

Taking as given a truthful report of the initial type θ_0, the firm's periodic utility function is

$$u(\theta_0, \theta_t, \widehat{\theta}_t) = (p(\theta_0, \widehat{\theta}_t) - c) \cdot Q(p(\theta_0, \widehat{\theta}_t), \theta_t) - F + T(\theta_0, \widehat{\theta}_t) \tag{A.1}$$

where $\widehat{\theta}_t$ is the report by a firm-type θ_t.

The first-order condition with respect to $\widehat{\theta}_t$ is

$$\frac{\partial p(\theta_0, \widehat{\theta}_t)}{\partial \widehat{\theta}_t} \cdot \left[\theta_t - (2 \cdot p(\theta_0, \widehat{\theta}_t) - c)\right] + \frac{\partial T(\theta_0, \widehat{\theta}_t)}{\partial \widehat{\theta}_t} = 0 \tag{A.2}$$

A truthful report is optimal if (A.2) holds at $\widehat{\theta}_t = \theta_t$, i.e.

$$\frac{\partial p(\theta_0, \theta_t)}{\partial \theta_t} \cdot [\theta_t - (2 \cdot p(\theta_0, \theta_t) - c)] + \frac{\partial T(\theta_0, \theta_t)}{\partial \theta_t} = 0 \tag{A.3}$$

Further, the following local second-order condition must hold:

$$\left.\frac{\partial^2 p(\theta_0, \widehat{\theta}_t)}{\partial \widehat{\theta}_t^2} \cdot \left[\theta_t - (2 \cdot p(\theta_0, \widehat{\theta}_t) - c)\right] - 2 \cdot \left(\frac{\partial p(\theta_0, \widehat{\theta}_t)}{\partial \widehat{\theta}_t}\right)^2 + \frac{\partial^2 T(\theta_0, \theta_t)}{\partial \widehat{\theta}_t^2}\right|_{\widehat{\theta}_t = \theta_t} \leq 0 \tag{A.4}$$

Differentiating (A.3) with respect to θ_t yields

$$\frac{\partial p(\theta_0, \theta_t)}{\partial \theta_t} + \frac{\partial^2 p(\theta_0, \theta_t)}{\partial \theta_t^2} \cdot [\theta_t - (2 \cdot p(\theta_0, \theta_t) - c)] - 2 \cdot \left(\frac{\partial p(\theta_0, \theta_t)}{\partial \theta_t}\right)^2 + \frac{\partial^2 T(\theta_0, \theta_t)}{\partial \theta_t^2} = 0 \tag{A.5}$$

Substituting (A.5) into (A.4) yields

$$\frac{\partial p(\theta_0, \theta_t)}{\partial \theta_t} \geq 0. \tag{A.6}$$

Since local incentive constraints imply global incentive constraints, a truthful revelation mechanism can be characterized by the following conditions (see, e.g., Laffont and Martimort, 2002, pp. 134-136):

$$\frac{\partial u(\theta_0,\theta_t)}{\partial \theta_t} = p(\theta_0,\theta_t) - c, \tag{A.7}$$

$$\frac{\partial p(\theta_0,\theta_t)}{\partial \theta_t} \geq 0, \quad \text{for all } t > 0 \tag{A.8}$$

A.2 Lemma 2

As implied by (2), $\theta_t = \phi(\theta_0, Z_t)$. At time $t = 0$, the expected present value of the firm's intertemporal utility flow is:

$$U(\theta_0,\widehat{\theta}_0) = E_0\left\{\int_0^\infty e^{-rt}\cdot\left[\begin{array}{c}\left(p(\widehat{\theta}_0,\phi(\widehat{\theta}_0,Z_t)) - c\right)\cdot Q(p(\widehat{\theta}_0,\phi(\widehat{\theta}_0,Z_t)),\phi(\theta_0,Z_t)) - F+ \\ +T(\widehat{\theta}_0,\phi(\widehat{\theta}_0,Z_t))\end{array}\right]\cdot dt\right\} + K(\widehat{\theta}_0) \tag{A.9}$$

where $\widehat{\theta}_0$ is the report by a firm-type θ_0.

The first-order condition with respect to $\widehat{\theta}_0$ is:

$$E_0\left\{\int_0^\infty e^{-rt}\cdot\left\{\begin{array}{c}\frac{dp(\widehat{\theta}_0,\phi(\widehat{\theta}_0,Z_t))}{d\widehat{\theta}_0}\cdot\left[\phi(\theta_0,Z_t) - (2\cdot p(\widehat{\theta}_0,\phi(\widehat{\theta}_0,Z_t)) - c)\right] \\ +\frac{dT(\widehat{\theta}_0,\phi(\widehat{\theta}_0,Z_t))}{d\widehat{\theta}_0}\end{array}\right\}\cdot dt\right\} + \frac{dK(\widehat{\theta}_0)}{d\widehat{\theta}_0} = 0 \tag{A.10}$$

A truthful report is optimal if (A.10) holds at $\widehat{\theta}_0 = \theta_0$, i.e.

$$E_0\left\{\int_0^\infty e^{-rt}\left\{\begin{array}{c}\frac{dp(\theta_0,\phi(\theta_0,Z_t))}{d\theta_0}\cdot\left[\phi(\theta_0,Z_t) - (2\cdot p(\theta_0,\phi(\theta_0,Z_t)) - c)\right] \\ +\frac{dT(\theta_0,\phi(\theta_0,Z_t))}{d\theta_0}\end{array}\right\}dt\right\} + \frac{dK(\theta_0)}{d\theta_0} = 0 \tag{A.11}$$

The following local second-order condition must also hold:

$$E_0\left\{\int_0^\infty e^{-rt}\left\{\begin{array}{c}\frac{d^2p(\widehat{\theta}_0,\phi(\widehat{\theta}_0,Z_t))}{d\widehat{\theta}_0^2}\cdot\left[\phi(\theta_0,Z_t) - (2\cdot p(\widehat{\theta}_0,\phi(\widehat{\theta}_0,Z_t)) - c)\right] \\ -2\cdot\left(\frac{dp(\widehat{\theta}_0,\phi(\widehat{\theta}_0,Z_t)}{d\widehat{\theta}_0}\right)^2 + \frac{d^2T(\widehat{\theta}_0,\phi(\widehat{\theta}_0,Z_t))}{d\widehat{\theta}_0^2}\end{array}\right\}dt\right\} + \left.\frac{d^2K(\widehat{\theta}_0)}{d\widehat{\theta}_0^2}\right|_{\widehat{\theta}_0=\theta_0} \leq 0 \tag{A.12}$$

Differentiating (A.11) with respect to θ_0 yields:

$$E_0\left\{\int_0^\infty e^{-rt}\cdot\left\{\begin{array}{c}\frac{dp(\theta_0,\phi(\theta_0,Z_t))}{\partial\theta_0}\cdot\left(\frac{\partial Q(p(\theta_0,\phi(\theta_0,Z_t)),\phi(\theta_0,Z_t))}{\partial\phi(\theta_0,Z_t)}\cdot\frac{\partial\phi(\theta_0,Z_t)}{\partial\theta_0}\right)+ \\ +\frac{d^2p(\theta_0,\phi(\theta_0,Z_t))}{d\theta_0^2}\cdot\left[\phi(\theta_0,Z_t) - (2\cdot p(\theta_0,\phi(\theta_0,Z_t)) - c)\right] + \\ -2\cdot\left(\frac{dp(\theta_0,\phi(\theta_0,Z_t)}{d\theta_0}\right)^2 + \frac{d^2T(\theta_0,\phi(\theta_0,Z_t))}{d\theta_0^2}\end{array}\right\}\cdot dt\right\} + \frac{d^2K(\theta_0)}{d\theta_0^2} = 0 \tag{A.13}$$

Substituting (A.13) into (A.12) yields:

$$E_0\left\{\int_0^\infty e^{-rt}\cdot\left(\frac{dp(\theta_0,\phi(\theta_0,Z_t))}{d\theta_0}\cdot\frac{\partial\phi(\theta_0,Z_t)}{\partial\theta_0}\right)\cdot dt\right\}\geq 0 \tag{A.14}$$

A truthful revelation mechanism is therefore characterized by the following conditions:

$$\begin{aligned}\frac{dU(\theta_0)}{d\theta_0}=\frac{\partial U(\theta_0,\widehat{\theta}_0)}{\partial\theta_0}&=E_0\left\{\int_0^\infty e^{-rt}\cdot(p(\theta_0,\phi(\theta_0,Z_t))-c)\cdot\frac{\partial\phi(\theta_0,Z_t)}{\partial\theta_0}\cdot dt\right\}\\&=E_0\left\{\int_0^\infty e^{-rt}\cdot(p(\theta_0,\theta_t)-c)\cdot\frac{\theta_t}{\theta_0}\cdot dt\right\}\end{aligned} \tag{A.15}$$

$$E_0\left\{\int_0^\infty e^{-rt}\cdot\frac{dp(\theta_0,\phi(\theta_0,Z_t))}{d\theta_0}\cdot\frac{\partial\phi(\theta_0,Z_t)}{\partial\theta_0}\cdot dt\right\}=E_0\left\{\int_0^\infty e^{-rt}\cdot\frac{dp(\theta_0,\phi(\theta_0,Z_t))}{d\theta_0}\cdot\frac{\theta_t}{\theta_0}\cdot dt\right\}\geq 0 \tag{A.16}$$

B Appendix B

B.1 Derivation of the optimal price and transfer schedules

By the Envelope Theorem (see Milgrom and Segal, 2002, Theorems 1-2), integrating (A.15) with respect to the initial type θ_0 yields the firm's intertemporal utility:

$$U(\theta_0)=\int_{\underline{\theta}}^{\theta_0}E_0\left\{\int_0^\infty e^{-rt}\cdot(p(y,\theta_t)-c)\cdot\frac{\theta_t}{y}\cdot dt\right\}\cdot dy+U(\underline{\theta}) \tag{B.1}$$

Using (A.9), we obtain:

$$\begin{aligned}&\int_{\underline{\theta}}^{\overline{\theta}}\left[E_0\left\{\int_0^\infty e^{-rt}\cdot T(\theta_0,\theta_t)\cdot dt\right\}+K(\theta_0)\right]\cdot g(\theta_0)\cdot d\theta_0\\&=\int_{\underline{\theta}}^{\overline{\theta}}\left[U(\theta_0)-E_0\left\{\int_0^\infty e^{-rt}\cdot((p(\theta_0,\theta_t)-c)\cdot Q(p(\theta_0,\theta_t),\theta_t)-F)\cdot dt\right\}\right]\cdot g(\theta_0)\cdot d\theta_0\end{aligned} \tag{B.2}$$

Moreover, integrating (B.1) by parts over $[\underline{\theta},\overline{\theta}]$ gives:

$$\int_{\underline{\theta}}^{\overline{\theta}}U(\theta_0)\cdot g(\theta_0)\cdot d\theta_0=\int_{\underline{\theta}}^{\overline{\theta}}E_0\left\{\int_0^\infty e^{-rt}\cdot(p(\theta_0,\theta_t)-c)\cdot\frac{\theta_t}{\theta_0}\cdot dt\right\}\cdot\frac{1-G(\theta_0)}{g(\theta_0)}\cdot g(\theta_0)\cdot d\theta_0+U(\underline{\theta}). \tag{B.3}$$

Substituting (B.2) and (B.3) into the regulator's objective (13) and collecting terms yields the reduced-form problem:

$$\max_{p(\theta_0,\theta_t)} \int_{\underline{\theta}}^{\overline{\theta}} E_0 \left\{ \int_0^{\infty} e^{-rt} \cdot \left[\begin{array}{c} S(Q(p(\theta_0,\theta_t),\theta_t) - c \cdot Q(p(\theta_0,\theta_t),\theta_t) - (1+\lambda)\cdot F + \\ -\lambda \cdot (p(\theta_0,\theta_t) - c) \cdot \left(\frac{\theta_t}{\theta_0} \cdot \frac{1-G(\theta_0)}{g(\theta_0)} - Q(p(\theta_0,\theta_t),\theta_t) \right) \end{array} \right] \cdot dt \right\} \cdot g(\theta_0) \cdot d\theta_0 +$$
$$-\lambda \cdot U(\underline{\theta}), \tag{B.4}$$

where $S(Q(p(\theta_0, \theta_t), \theta_t)) = Q(p(\theta_0, \theta_t), \theta_t) \cdot (\theta_t - \frac{Q(p(\theta_0,\theta_t),\theta_t)}{2})$.

Abstracting from monotonicity (A.7)-(A.8), limited-liability (14) and participation (15), the first-order condition for $p(\theta_0, \theta_t)$ gives the interior candidate:

$$p^*(\theta_0, \theta_t) = c + \frac{\lambda}{1+2\lambda} \cdot \left[\theta_t \cdot (1 - \frac{H(\theta_0)}{\theta_0}) - c \right] \tag{B.5}$$

The first-order condition characterizes the interior candidate price schedule. Monotonicity and the limited-liability constraint then determine the admissible truncation at the boundary.

Integrating (A.7) gives:

$$u^*(\theta_0, \theta_t) = \int_0^{\theta_t} (p^*(\theta_0, z) - c) \cdot dz + u^*(\theta_0, 0),$$

which (under A.7-A.8) is convex in θ_t. Its minimum is attained at the point $a(\theta_0)$ satisfying

$$p(\theta_0, a(\theta_0)) = c \rightarrow a(\theta_0) = \frac{c}{1 - \frac{H(\theta_0)}{\theta_0}} \geq c. \tag{B.5.1}$$

Limited liability requires $u^*(\theta_0, a(\theta_0)) \geq 0$. In the optimum transfers are chosen so that the constraint binds: $u^*(\theta_0, a(\theta_0)) = 0$.

However, (B.5) implies $p^*(\theta_0, \theta_t) < c$ for $\theta_t < a(\theta_0)$, so that $\frac{\partial u^*(\theta_0,\theta_t)}{\partial \theta_t} = p^*(\theta_0, \theta_t) - c < 0$ on $[0, a(\theta_0))$. Given $u^*(\theta_0, a(\theta_0)) = 0$, this would generate strictly positive rents for low realizations $\theta_t < a(\theta_0)$, which is welfare worsening. Consider instead the modified schedule that sets $p^*(\theta_0, \theta_t) = c$ for $\theta_t \leq a(\theta_0)$ while keeping the interior rule for $\theta_t > a(\theta_0)$. Relative to (B.5), this modification

i) eliminates rents on $[0, a(\theta_0)]$, since $\frac{\partial u^*(\theta_0,\theta_t)}{\partial \theta_t} = 0$ there, and

ii) increases welfare by avoiding the need for costly transfers when price equals marginal cost.

Therefore, the optimal regulatory price is

$$p^*(\theta_0,\theta_t) = \begin{cases} c + \frac{\lambda}{1+2\lambda} \cdot \left[\theta_t \cdot (1 - \frac{H(\theta_0)}{\theta_0}) - c\right] & \text{for } \theta_t > a(\theta_0), \\ c, & \text{for } \theta_t \le a(\theta_0), \end{cases} \tag{B.6}$$

The associated quantity is

$$Q^*(\theta_0,\theta_t) = \begin{cases} \theta_t - p^*(\theta_0,\theta_t), & \text{for } \theta_t > a(\theta_0), \\ \theta_t - c, & \text{for } c < \theta_t \le a(\theta_0), \\ 0 & \text{for } \theta_t \le c \end{cases} \tag{B.7}$$

and for $\theta_t > a(\theta_0)$,

$$\theta_t - p^*(\theta_0,\theta_t) = (\theta_t - c) - (p^*(\theta_0,\theta_t) - c) < \theta_t - c.$$

Monotonicity conditions (A.8) and (A.16) hold; in particular,

$$\frac{\partial p^*(\theta_0,\theta_t)}{\partial \theta_t} = \begin{cases} \frac{\lambda}{1+2\lambda} \cdot (1 - \frac{H(\theta_0)}{\theta_0}) & \text{for } \theta_t > a(\theta_0), \\ 0, & \text{for } \theta_t \le a(\theta_0), \end{cases} \tag{B.6.1}$$

and

$$\frac{\partial p^*(\theta_0,\theta_t)}{\partial \theta_0} = \begin{cases} \frac{\lambda}{1+2\lambda} \cdot \theta_t \cdot (1 - \frac{H(\theta_0)}{\theta_0}) \cdot \left(\frac{1}{\theta_0} + \frac{d \ln\left(1 - \frac{H(\theta_0)}{\theta_0}\right)}{d\theta_0}\right), & \text{for } \theta_t > a(\theta_0), \\ 0, & \text{for } \theta_t \le a(\theta_0), \end{cases} \tag{B.6.2}$$

Moreover,

$$\frac{da(\theta_0)}{d\theta_0} = -a(\theta_0) \cdot \frac{d \ln\left(1 - \frac{H(\theta_0)}{\theta_0}\right)}{d\theta_0} < 0 \tag{B.6.3}$$

B.2 Transfers

Given $u^*(\theta_0, a(\theta_0)) = 0$ and the optimal price rule (B.6), the firm's flow rent is

$$u^*(\theta_0,\theta_t) = \begin{cases} \int_{a(\theta_0)}^{\theta_t} (p^*(\theta_0,z) - c) \cdot dz, & \text{for } \theta_t > a(\theta_0), \\ 0, & \text{for } \theta_t \le a(\theta_0). \end{cases} \tag{B.8}$$

Substituting into (A.1) yields the optimal periodic transfer

$$T^*(\theta_0,\theta_t) = \begin{cases} \int_{a(\theta_0)}^{\theta_t} (p^*(\theta_0,z) - c) \cdot dz - (p^*(\theta_0,\theta_t) - c) \cdot Q^*(\theta_0,\theta_t) + F, & \text{for } \theta_t > a(\theta_0), \\ F, & \text{for } \theta_t \le a(\theta_0). \end{cases} \tag{B.9}$$

Differentiating $T^*(\theta_0, \theta_t)$ for $\theta_t > a(\theta_0)$ gives:

$$\begin{aligned}\frac{\partial T^*(\theta_0, \theta_t)}{\partial \theta_t} &= \frac{\partial p^*(\theta_0, \theta_t)}{\partial \theta_t} \cdot [2 \cdot (p^*(\theta_0, \theta_t) - c) - (\theta_t - c)] \\ &= \frac{\partial p^*(\theta_0, \theta_t)}{\partial \theta_t} \cdot \left\{ \frac{2\lambda}{1+2\lambda} \cdot \left[\theta_t \cdot (1 - \frac{H(\theta_0)}{\theta_0}) - c \right] - (\theta_t - c) \right\} < 0. \end{aligned} \tag{B.9.1}$$

where the inequality follows from $\frac{\partial p^*(\theta_0, \theta_t)}{\partial \theta_t} > 0$ and $p^*(\theta_0, \theta_t) > c$ for $\theta_t > a(\theta_0)$. The monotonicity of $T^*(\theta_0, \theta_t)$ with respect to θ_t is independent of F, which affects only the level of transfers.

C Appendix C

C.1 Reflected process

To evaluate intertemporal utility under the LL constraint, it is convenient to represent the demand process as a stochastic process reflected at the endogenous lower boundary $a(\theta_0)$.

We define the reflected process ϑ_t by $\vartheta_t = L_t \cdot \theta_t$, following Harrison (1985, 2013) and Bentolila and Bertola (1990). The construction satisfies the following properties:

1. θ_t follows a driftless geometric Brownian motion, $d\theta_t = \sigma \cdot \theta_t \cdot dZ_t$, with initial condition $\theta_0 > a$.

2. L_t is a continuous, non-decreasing process with $L_0 = 1$.

3. L_t increases only when $\vartheta_t = a$, where $a > 0$ is a fixed reflecting boundary.

4. $\vartheta_t \in [a, \infty)$ for all $t \geq 0$.

Since $L_0 = 1$, the original and reflected processes share the same initial condition, $\theta_0 = \vartheta_0$. Applying Ito's Lemma to ϑ_t, we obtain:[15]

$$d\vartheta_t = dL_t \cdot \theta_t + L_t \cdot d\theta_t. \tag{C.1}$$

Substituting for $d\theta_t$ yields

$$d\vartheta_t = \sigma \cdot \vartheta_t \cdot dZ_t + \vartheta_t \cdot \frac{dL_t}{L_t}. \tag{C.2}$$

The term dL_t/L_t represents the regulating increment required to prevent ϑ_t from crossing the boundary a. In particular, $dL_t = 0$ whenever $\vartheta_t > a$, and $dL_t > 0$ only when ϑ_t reaches the boundary a.

[15] Because L_t is of finite variation, $dL_t^2 = 0$; see Harrison (2013, p. 92).

From (C.2), it follows that

$$
\begin{aligned}
d \ln \vartheta_t &= d \ln \theta_t + d \ln L_t \\
&= -\frac{\sigma^2}{2} \cdot dt + \sigma \cdot dZ_t + d \ln L_t
\end{aligned} \tag{C.3}
$$

Integrating (C.3) from 0 to t gives

$$
\ln \vartheta_t - \ln \vartheta_0 = -\frac{\sigma^2}{2} \cdot t + \sigma \cdot \int_0^t dZ_s + \ln L_t - \ln L_0
$$

Exponentiating both sides and using $L_0 = 1$, we obtain the explicit representation

$$
\vartheta_t = \vartheta_0 \cdot e^{-\frac{\sigma^2}{2} \cdot t + \sigma \cdot \int_0^t dZ_s} \cdot L_t = \phi(\theta_0, L_t, Z_t). \tag{C.4}
$$

Finally, since the regulating process L_t is memoryless in the sense of Harrison (2013, Proposition 2.3), the effect of the initial condition ϑ_0 on ϑ_t persists over time in the same way as for the unconstrained stochastic process. The reflected representation therefore preserves the propagation of initial conditions while ensuring that the process remains above the boundary.

C.2 Expected present value of flow rents

Under the optimal price schedule (B.5), define

$$
M(\theta_0) \equiv E_0 \left\{ \int_0^\infty e^{-rt} \cdot u^*(\theta_0, \vartheta_t) \cdot dt \right\} = E_0 \left\{ \int_0^\infty e^{-rt} \cdot \left[\int_{a(\theta_0)}^{\vartheta_t} (p^*(\theta_0, z) - c) \cdot dz \right] \cdot dt \right\}.
$$

Using (B.5), this can be written as

$$
M(\theta_0) = \frac{\lambda}{1+2\lambda} \cdot E_0 \left\{ \int_0^\infty e^{-rt} \cdot \left[\frac{\vartheta_t}{2} \cdot (1 - \frac{H(\theta_0)}{\theta_0}) - c \right] \cdot \ \vartheta_t \cdot dt \ \right\}. \tag{C.5}
$$

For a generic state $\vartheta_s \geq a(\theta_0)$, let $M(\vartheta_s)$ denote the continuation value starting at ϑ_s. Standard dynamic programming for reflected stochastic processes implies that $M(\vartheta_s)$ satisfies the ODE:

$$
\sigma^2 \cdot \frac{\vartheta_s^2}{2} \cdot M_{\vartheta\vartheta}(\vartheta_s) - r \cdot M(\vartheta_s) = -\frac{\lambda}{1+2\lambda} \cdot \left[\frac{\vartheta_s}{2} \cdot (1 - \frac{H(\theta_0)}{\theta_0}) - c \right] \cdot \vartheta_s, \ \vartheta_s \geq a(\theta_0) \tag{C.6}
$$

with reflecting boundary condition:

$$
M_\vartheta(a(\theta_0)) = 0. \tag{C.7}
$$

We impose the usual transversality/growth restriction that rules out the explosive solution associated with the positive root of the characteristic equation; hence only the term in $\beta < 0$ is admissible.

The general solution is:

$$M(\vartheta_s) = m(\vartheta_s) + A \cdot \vartheta_s^{\beta}, \tag{C.8}$$

where $\beta < 0$ is the negative root of $\frac{\sigma^2}{2} \cdot \beta \cdot (\beta - 1) - r = 0$, and a quadratic particular solution $m(\vartheta_s) = h_0 + h_1 \cdot \vartheta_s + h_2 \cdot \vartheta_s^2$ yields

$$h_0 = 0,\ h_1 = -\frac{\lambda}{1+2\lambda} \cdot \frac{c}{r},\ h_2 = \frac{1}{2} \cdot \frac{\lambda}{1+2\lambda} \cdot \frac{1}{r-\sigma^2} \cdot (1 - \frac{H(\theta_0)}{\theta_0}).$$

The constant A is pinned down by smooth pasting $M_{\vartheta}(a(\theta_0)) = 0$, giving

$$A = -\frac{\lambda}{1+2\lambda} \cdot \frac{\sigma^2}{r-\sigma^2} \cdot \frac{c}{r} \cdot \frac{a(\theta_0)^{1-\beta}}{\beta}.$$

Evaluating at $\vartheta_0 = \theta_0$ yields

$$M(\theta_0) = m(\theta_0) + A \cdot \theta_0^{\beta}. \tag{C.9}$$

which corresponds to (22.1)-(22.2) in the main text.

C.3 Singular-control representation

The reflected construction above admits a natural singular-control representation in the sense of Harrison (1985, 2013).

Under the optimal contract, instantaneous utility satisfies the envelope condition

$$\frac{\partial u(\theta_0, \theta_t)}{\partial \theta_t} = p(\theta_0, \theta_t) - c,$$

and is subject to the state-dependent constraint $u(\theta_0, \theta_t) \geq 0$.

Convexity of the utility profile implies that this constraint binds at a unique cutoff $a(\theta_0)$, so that the flow utility is constant below the boundary and increasing above it.

The regulating process L_t is of finite variation and increases only when the boundary is reached. The reflected stochastic process ϑ_t therefore evolves according to a stochastic differential equation with a minimal finite-variation adjustment that prevents violations of the state constraint. In this precise sense, the process has a singular-control interpretation: intervention occurs only at the boundary and only to enforce feasibility.

Importantly, the stochastic evolution of demand remains entirely exogenous. The regulator does not control the underlying process θ_t; rather, prices and transfers adjust continuation utility so as to ensure compliance with the non-negativity constraint. The reflecting structure thus emerges endogenously from the optimal enforcement of a state constraint, rather than from the solution of an explicit dynamic control problem in which a singular control is chosen as a primitive instrument.

C.4 The value of the initial report

Following the Bergemann-Strack envelope approach (Bergemann and Strack, 2015, Theorem 2), define the marginal value of the initial report

$$N(y) = \frac{dU^*(y)}{dy} = E_0 \left\{ \int_0^\infty e^{-rt} \cdot (p^*(y, \vartheta_t) - c) \cdot \frac{\vartheta_t}{y} \cdot dt \right\}, \ \vartheta_0 = y > a(y) \tag{C.10}$$

Let $N(\vartheta_s)$ denote the continuation value starting from $\vartheta_s \geq a(y)$, holding fixed the reported initial type y. Then $N(\vartheta_s)$ satisfies

$$\sigma^2 \cdot \frac{\vartheta_s^2}{2} \cdot N_{\vartheta\vartheta}(\vartheta_s) - r \cdot N(\vartheta_s) = -\left[\vartheta_s \cdot (1 - \frac{H(y)}{y}) - c\right] \cdot \frac{\vartheta_s}{y}, \ \vartheta_s \geq a(y) \tag{C.11}$$

with reflecting boundary condition

$$N_\vartheta(a(y)) = 0, \tag{C.12}$$

and the same admissibility restriction that rules out the explosive root. The solution takes the form

$$N(\vartheta_s) = n(\vartheta_s) + B \cdot \vartheta_s^\beta, \tag{C.13}$$

where a quadratic particular solution $n(\vartheta_s) = j_0 + j_1 \cdot \vartheta_s + j_2 \cdot \vartheta_s^2$ yields

$$j_0 = 0, \ j_1 = -\frac{\lambda}{1+2\lambda} \cdot \frac{c}{r} \cdot \frac{1}{y}, \ j_2 = \frac{1}{r-\sigma^2} \cdot \frac{\lambda}{1+2\lambda} \cdot \frac{1}{y} \cdot \left(1 - \frac{H(y)}{y}\right).$$

Imposing $N_\vartheta(a(y)) = 0$ pins down

$$B = -\frac{\lambda}{1+2\lambda} \cdot \frac{c}{y} \cdot (\frac{2}{r-\sigma^2} - \frac{1}{r}) \cdot \frac{a(y)^{1-\beta}}{\beta}.$$

Evaluating at $\vartheta_s = y$ gives the decomposition (24) in the main text.

C.5 The up-front transfer

From the incentive constraints at $y = 0$ and the envelope condition

$$\frac{dU^*(\theta_0)}{d\theta_0} = N(\theta_0),$$

so integrating over $[\underline{\theta}, \theta_0]$ yields

$$U^*(\theta_0) = \int_{\underline{\theta}}^{\theta_0} N(y) \cdot dy + U^*(\underline{\theta}). \tag{C.15a}$$

We set $U^*(\underline{\theta}) = 0$ (binding participation at the bottom type).

By definition of (21), total intertemporal utility equals the discounted flow rent component plus the up-front transfer:

$$U^*(\theta_0) = M(\theta_0) + K(\theta_0). \tag{C.15b}$$

Combining (C.15a)-(C.15b) gives

$$K(\theta_0) = \int_{\underline{\theta}}^{\theta_0} N(y) \cdot dy - M(\theta_0),\ \ K(\underline{\theta}) = -M(\underline{\theta}) < 0. \tag{C.15}$$

Substituting (A.3) into (A.11) yields

$$\frac{dK(\theta_0)}{d\theta_0} = -E_0 \left\{ \int_0^{\infty} e^{-rt} \cdot \left(\int_{a(\theta_0)}^{\theta_t} \frac{\partial p^*(\theta_0, z)}{\partial \theta_0} \cdot dz \right) \cdot dt \right\} < 0 \tag{C.16}$$

Using (C.9) and (C.14) in (C.15) yields the decomposition as

$$K(\theta_0) = K^{no-LL}(\theta_0) + \Delta(\theta_0) \tag{C.17}$$

where $K^{no-LL}(\theta_0) = \int_{\underline{\theta}}^{\theta_0} n(y) \cdot dy - m(\theta_0)$ is the benchmark absent limited liability, and

$$\Delta(\theta_0) = \frac{\lambda}{1+2\lambda} \cdot \left[\begin{array}{c} -\int_{\underline{\theta}}^{\theta_0} \left(2 \cdot \frac{1}{r-\sigma^2} - \frac{1}{r}\right) \cdot \frac{c}{y} \cdot \frac{a(y)}{\beta} \cdot \left(\frac{y}{a(y)}\right)^{\beta} dy + \\ + \frac{\sigma^2}{r-\sigma^2} \cdot \frac{c}{r} \cdot \frac{a(\theta_0)}{\beta} \cdot \left(\frac{\theta_0}{a(\theta_0)}\right)^{\beta} \end{array} \right], \tag{C.18}$$

collects the reflecting-boundary correction. Differentiating $\Delta(\theta_0)$ yields

$$\frac{d\Delta(\theta_0)}{d\theta_0} = \frac{\lambda}{1+2\lambda} \cdot \frac{1}{r-\sigma^2} \cdot \frac{c}{r} \cdot \frac{1}{\beta} \cdot \left\{ \begin{array}{c} [\sigma^2 \cdot (\beta - 1) - r] \cdot \frac{a(\theta_0)}{\theta_0} + \\ + \sigma^2 \cdot (1-\beta) \cdot \frac{da(\theta_0)}{d\theta_0} \end{array} \right\} \cdot \left(\frac{\theta_0}{a(\theta_0)}\right)^{\beta} > 0. \tag{C.19}$$

D Appendix D

For $\theta_t > a(\theta_0)$, from (19) and (16), the firm's flow utility is given by

$$u^*(\theta_0, \theta_t) = \int_{a(\theta_0)}^{\theta_t} (p^*(\theta_0, z) - c) \cdot dz.$$

Imposing the upper bound $u(\theta_0, b) = \bar{u}$ yields

$$\int_{a(\theta_0)}^{b} (p^*(\theta_0, z) - c) \cdot dz = \bar{u}.$$

Using the optimal pricing rule (B.5), this condition becomes

$$\frac{\lambda}{1+2\lambda} \cdot \int_{a(\theta_0)}^{b} \left[z \cdot (1 - \frac{H(\theta_0)}{\theta_0}) - c \right] \cdot dz = \bar{u},$$

which can be written as

$$\frac{\lambda}{1+2\lambda} \cdot \left[\frac{b + a(\theta_0)}{2} \cdot (1 - \frac{H(\theta_0)}{\theta_0}) - c \right] \cdot (b - a(\theta_0)) = \bar{u}.$$

Rearranging terms yields

$$(b - a(\theta_0))^2 = \frac{2 \cdot (1+2\lambda)}{\lambda} \cdot \frac{\bar{u}}{(1 - \frac{H(\theta_0)}{\theta_0})},$$

hence,

$$b(\theta_0, \bar{u}) = a(\theta_0) + \sqrt[2]{\frac{2 \cdot (1+2\lambda)}{\lambda} \cdot \frac{\bar{u}}{(1 - \frac{H(\theta_0)}{\theta_0})}} > a(\theta_0).$$

Finally, substituting $b(\theta_0, \bar{u})$ into the optimal pricing rule yields the price cap

$$\begin{aligned} \bar{p} = p^*(\theta_0, b(\theta_0, \bar{u})) &= c + \frac{\lambda}{1+2\lambda} \cdot \left[b(\theta_0, \bar{u}) \cdot (1 - \frac{H(\theta_0)}{\theta_0}) - c \right] \\ &= c + \sqrt[2]{2 \cdot \frac{\lambda}{1+2\lambda} \cdot (1 - \frac{H(\theta_0)}{\theta_0}) \cdot \bar{u}} > c. \end{aligned}$$

E Declaration of generative AI and AI-assisted technologies in the manuscript preparation process

Statement: During the preparation of this work, the authors used ChatGPT 5.2 to improve language and readability, with caution. After using this tool, the authors reviewed and edited the content as needed and took full responsibility for the content of the publication.

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