June 2024

Fondazione Eni Enrico Mattei

Working Paper

11.2024

Effort Provision and Incentivisation in Tullock Group-Contests with Many Groups: An Explicit Characterisation

Davide Bosco, Mario Gilli

Effort Provision and Incentivisation in Tullock Group-Contests with Many Groups: An Explicit Characterisation

Davide Bosco
(Department of Economics, Management and Statistics, University of Milano-Bicocca, Milano and Center for European Studies - CefES)
Mario Gilli
(Department of Economics, Management and Statistics, University of Milano-Bicocca, Milano and Center for European Studies - CefES)

Summary

We study effort provision and incentivisation in a Tullock group-contest with m ≥ 2 groups that differ in size. A novel algorithmic procedure is presented that, under a symmetry assumption, explicitly characterises the equilibrium. Endogenous, optimal incentivisation schemes are then determined. Four results ensue. First, strategic interactions endogenously come in mean-field form: individual effort provision responds to the aggregate effort and average egalitarianism across groups. Therefore, the game is aggregative. Second, individuals endlessly cycle between zero and positive aggregative. Second, individuals endlessly cycle between zero and positive effort provision at some incentivisation schemes: no pure-strategy equilibria exist in these cases. Third, group size determines whether the egalitarianism of endogenous schemes increases or decreases in the average egalitarianism across groups. Fourth, all groups provide effort at the endogenous schemes if incentivisation is properly restricted.

Keywords: Collective-action problem, Conflict, Selective incentives, Strategic complements and substitutes

JEL classification: C72, D71, D74

Corresponding Author:
Mario Gilli
Full Professor of Economics,
Department of Economics, Management and Statistics, University of Milano-Bicocca
Piazza dell'Ateneo Nuovo, 1, Milan (Italy)
e-mail: mario.gilli@unimib.it

Effort Provision and Incentivisation in Tullock Group-Contests with Many Groups: An Explicit Characterisation*

Davide Bosco[†,a,b]

[a] Department of Economics, Management and Statistics (DEMS), University of Milano-Bicocca, Milano (IT)

[b] Center for European Studies (CefES), Milano (IT)

Mario Gilli[‡,a,b]

[a] Department of Economics, Management and Statistics (DEMS), University of Milano-Bicocca, Milano (IT)

[b] Center for European Studies (CefES), Milano (IT)

Abstract

We study effort provision and incentivisation in a Tullock group-contest with $m \geq 2$ groups that differ in size. A novel algorithmic procedure is presented that, under a symmetry assumption, explicitly characterises the equilibrium. Endogenous, optimal incentivisation schemes are then determined. Four results ensue. First, strategic interactions endogenously come in mean-field form: individual effort provision responds to the aggregate effort and average egalitarianism across groups. Therefore, the game is aggregative. Second, individuals endlessly cycle between zero and positive effort provision at some incentivisation schemes: no pure-strategy equilibria exist in these cases. Third, group size determines whether the egalitarianism of endogenous schemes increases or decreases in the average egalitarianism across groups. Fourth, all groups provide effort at the endogenous schemes if incentivisation is properly restricted.

JEL classification: C72, D71, D74

Keywords: Collective-action problem, Conflict, Selective incentives, Strategic complements and substitutes

* We are very grateful to Stefano Barbieri for extremely useful comments and suggestions.

† *E-mail address:* ⟨davide.bosco@unimib.it⟩

‡ *E-mail address:* ⟨mario.gilli@unimib.it⟩

1 Introduction

Group-contest games provide a tractable yet rich analytical framework for the study of social environments in which the allocation of a surplus (or a rent) among individuals is determined by the conflict among pre-existing social groups to which individuals belong. Key to the analysis of group contests is the understanding of the deep mechanics that govern decentralised effort provision by individual group members. Because of the social nature of conflict, the probability to share into the surplus faced by any single individual crucially depends on the rent-seeking effort collectively spent by his/her group. A natural tension arises between *within-group* and *across-groups* competition. Within-group competition invites free-riding via the substitutability of individual effort across members of the same group – the larger the effort spent by other group members, the smaller the impact, *ceteris paribus*, of individual effort provision on the group probability of success, the larger the incentive to free-ride on others' effort. Across-groups competition spurs effort provision via the strategic complementarity that arises at the aggregate level – the larger the effort spent by other group members, the higher the probability of success, *ceteris paribus*, the larger the incentive to participate into collective effort provision to secure a bigger share of the prize in case of victory.

It is well-known that, when the surplus is a private good, heterogeneity in group size and incentivisation schemes play a role in shaping equilibrium outcomes, in general, and effort provision at any scale of aggregation, in particular. Less known is the exact extent of such a role: contest games with many heterogeneous groups are complicated mathematical objects, and their analysis often relies on implicit equilibrium characterisations. Aim of this paper is to characterise *explicitly* how heterogeneity affects effort provision and incentivisation in group-contest games with a large number of groups.

We study a two-stage Tullock group-contest game with $m \geq 2$ competing groups and linear costs of effort. Groups differ in size, and distribute surplus among their members in accordance with heterogeneous incentivisation schemes. In the first stage – the *incentivisation stage-game* – (the utilitarian leaders of) all groups simultaneously determine the incentivisation schemes. In the second stage – the *effort stage-game* – all members of all groups simultaneously decide the extent of their effort provision, conditioning on the group-specific schemes determined at the incentivisation stage.

At the effort stage-game we focus on within-group symmetric equilibria (*WGS equilibria*, henceforth), where effort provision is evenly distributed among the members of the same group but may well differ across different groups.[1] Albeit constrained by the WGS assumption,[2] the equilibrium structure that emerges at this stage is rich, and its analysis provides interesting, sometimes counter-intuitive insights about the role and relevance of heterogeneity.

Building on a simple algorithmic procedure, we explicitly characterise the equilibrium of the effort stage-game for any number of groups and any arbitrary profile of incentivisation schemes. We then explicitly characterise the unique subgame-perfect WGS equilibrium of the two-stage game by identifying the optimal schemes at the incentivisation stage. Our analysis highlights that, at some incentivisation schemes, no pure-strategy WGS equilibrium exists because individual group members endlessly cycle between zero and positive effort provision – they prefer to free-ride when all other members are providing effort, and to provide effort when all other members are free-riding. Existence and uniqueness of pure-strategy equilibria is re-established by restricting the space of admissible schemes so as to eliminate those that punish contributors and reward free-riders.

Two further results ensue. First, we identify an important property of WGS equilibria: at any level of aggregation, the equilibrium provision of effort can be expressed as a function of the aggregate effort and average egalitarianism across groups. In other words: we prove that, in a WGS environment, the strategic interaction among individuals and groups endogenously comes in *mean-field* form. This is an important result: while aggregative[3] and global games[4] postulate mean-field interactions *a priori* to leverage on their superior tractability,

[1]Accordingly, the assumption that utilitarian leaders and not group members themselves decide on within-group incentivisation is without any loss of generality.

[2]Cavalli et al. (2018) prove that asymmetries in the equilibrium provision of effort may arise also in symmetric contest games. However, some contributions suggest that within-group symmetry can be rationalised as either: (*i*) successful coordination in a correlated equilibrium – where effort provision is decentralised, but all the members of the same group peg their strategies to a coordination device that generates public signals; (*ii*) the result of pre-play communication, defined over a proper message space. On this, see Konrad and Kovenock (2009) and, more recently, Barbieri and Topolyan (2023).

[3]Aggregative games are games where, *"for any player, payoffs depend on her own action and an aggregate that encapsulates all interactions in the game"* (see Corchón (2021)). The idea was first introduced by Selten (1970), and has been used in industrial organisation for the study of Cournot games.

[4]Pioneered by Carlsson and Van Damme (1993) and popularised by Morris and Shin (1998), global games are "*a class of incomplete information games where small uncertainty about payoffs implies a significant failure of common knowledge*" (see Morris (2008), p. 1). The global-games approach has been successfully used to study currency attacks, bank runs and political protests – see Morris and Shin (2003) for an excellent primer on this topic.

mean-field interactions arise here as a by-product of decentralised behaviour – restricted only by within-group symmetry. Essential to this result is the explicit form of our equilibrium characterisation. Second, we show that both strategic complementarity and substitutability can arise in the equilibrium level of egalitarianism across groups when the incentivisation schemes are endogenously determined. If complementarity prevails, groups have a strong incentive to match the egalitarianism of their competitors, while they try to undercut each other when substitutability prevails – they increase egalitarianism when they expect competitors to be less egalitarian, and *vice versa*. What we find is that group size unambiguously determines which of the two configurations prevails in equilibrium: substitutability drives the egalitarianism of those groups that are small with respect to the equilibrium 'coalition' of effort providers, whereas complementarity governs the egalitarianism of the single group, if any, whose size accounts for more than a half of the size of the same coalition.

On the theoretical side, the contribution of this paper is threefold. First, it provides an equilibrium characterisation in explicit form for a linear-symmetric setup with more than two groups.[5] In so doing, it complements the analysis of Ueda (2002), that characterises analytically but *implicitly* the set of pure-strategy equilibria of a Tullock group-contest game with linear incentivisation schemes and costs of effort. Second, because of the explicit characterization it provides, this paper allows for a deeper understanding of the driving forces that govern effort provision in group-contest games, highlighting the crucial role played by across-group heterogeneity. In particular, it shows that, in a WGS environment: (*i*) complex strategic feedbacks reduce to simple, tractable mean-field interactions; (*ii*) weak oligopolisation in the sense of Ueda (2002) occurs when the incentivisation schemes are endogenously determined. Third, it extends the analysis of Ueda (2002) by allowing for *unrestricted* incentivisation schemes. Restricted schemes imply that redistribution is the only instrument available to group leaders to induce effort provision by group members – in the worst case, free-riders get nothing even upon success. Unrestricted schemes extend the strategy space of leaders by allowing for the expropriation and redistribution of members' *private* resources. To fix ideas: an unrestricted scheme may require free-riders to pay a 'tax' for their inaction, and the proceeds may be transferred, as a reward, to the active contributors of the same group.[6] Besides

[5] To the best of our knowledge, this it the first paper to succeed in this enterprise.

[6] Possibly, the extent of such within-group transfers can be made contingent to the (relative) amount of effort provided by the receiver.

being more general from a mathematical standpoint, and of practical interest for the study of optimal incentivisation within groups, unrestricted schemes bring about an unexpected theoretical result: pure-strategy equilibria do not exist at some schemes, because the incentives faced by individual group members generate cycles between zero and positive effort provision.

On the methodological side, this paper contributes in two ways. First, it proves that the convoluted tangle of strategic interactions that arises in m-group environments can be reduced to a simple mean-field (aggregative) game.[7] Second, it develops a simple but elegant resolution procedure that builds on block-recursion and takes on a handy algorithmic structure. Block recursion allows for the explicit characterisation of equilibrium strategies by solving the game 'top-down' – the aggregate effort across groups is determined first; the latter then determines group effort that, in turn, determines individual effort provision.[8] The algorithmic structure allows for the identification of some important structural properties of equilibria – number of active groups and degree of cross-sectional heterogeneity of the endogenous incentivisation schemes. In this respect, our approach complements that of Hartley (2017) – that tackles the same tractability issue, but takes another route: he shows that a group-contest game can be *represented* as a simpler contest game played by single agents (interpreted as the groups' leaders), each minimising an 'artificial cost function' representative of a group's 'typical' preferences.

This paper is organised as follows. Section 2 outlines the model for an arbitrary number of groups $m \geq 2$. Section 3 briefly outlines a two-group example to fix ideas. Section 4 characterises the WGS equilibria of the effort stage-game for any number of groups and any arbitrary profile of incentivisation schemes, and describes the resolution procedure. Section 5 identifies the endogenous schemes at the incentivisation stage-game and characterises the unique subgame-perfect equilibrium of the two-stage game. Section 6 wraps up and concludes the paper.

[7] In so doing, it potentially paves the way for future research on large group-contest games with continua of players (on the applied side), and on alternative resolution procedures (on the technical one).

[8] This block-recursive structure is a by-product of the aggregative form of the game under the WGS assumption.

2 The Model

2.1 Players, Strategies and Payoffs

Consider a group-contest with N risk-neutral, self-interested individuals partitioned into $m \geq 2$ groups. Each group is indexed by i and has $n_i \geq 2$ members, each indexed by j.[9] M indicates the group set. Groups compete in effort for the appropriation of a private good with a common value of $v > 0$ utils (the *prize*). Each member exerts an individual effort level $x_{ij} \in \mathbb{R}_+$, and the x_{ij} jointly determine the group effort $X_i \geq 0$ via the additive *impact function*

$$X_i := \sum_{j=1}^{n_i} x_{ij}\,, \qquad \text{for all groups } i = 1, 2, \ldots, m\,. \tag{1}$$

We call $\mathbf{x_i} := (x_{i1}, x_{i2}, \ldots, x_{in_i}) \in \mathbb{R}_+^{n_i}$ the profile of individual effort levels x_{ij} in group i, $\mathbf{x} := (\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_m}) \in \mathbb{R}_+^N$ the profile of *all* individual effort levels, and $\mathbf{X} := (X_1, X_2, \ldots, X_m) \in \mathbb{R}_+^m$ the corresponding profile of group effort levels X_i. Moreover, we say that a group is *active* if $X_i > 0$. The effort across groups $\mathbf{X}$ affects a single group's probability of success[10] $p_i(\mathbf{X})$ via the (Tullock) *contest success function*

$$p_i(\mathbf{X}) := \begin{cases} \dfrac{X_i}{\sum_{\ell=1}^m X_\ell} & \text{if } \displaystyle\sum_{\ell=1}^m X_\ell > 0\,, \quad (2a) \\[2ex] \dfrac{1}{m} & \text{otherwise}\,, \quad (2b) \end{cases}$$

but not the extent of the prize. Accordingly, conflict among groups is interpreted as pure rent-seeking. The prize is allocated to the j-th member of the winning group in accordance with the *incentivisation scheme*

$$q_{ij}(\mathbf{x_i}) := \begin{cases} (1 - a_i)\,\dfrac{x_{ij}}{X_i} + a_i\,\dfrac{1}{n_i} & \text{if } X_i > 0\,, \quad (3a) \\[2ex] \dfrac{1}{n_i} & \text{otherwise}\,, \quad (3b) \end{cases}$$

where x_{ij}/X_i measures the relative effort provision of agent j in group i, and a_i parameterises the level of egalitarianism within the same group. The individual

[9] So that $N = \sum_{i=1}^m n_i$ holds by definition, and the subscript ij indicates the j-th member of the i-th group.

[10] That is, the probability to win the prize.

cost of effort is

$$C(x_{ij}) := x_{ij} ,$$

for all all members $j = 1, 2, \ldots, n_i$ of all groups $i = 1, 2, \ldots, m$, so that the expected utility π_{ij} of the generic member ij can be written as

$$\pi_{ij}(x_{ij}, \mathbf{X}) = p_i(\mathbf{X}) \, [\, v \, q_{ij}(\mathbf{x}_i) \,] - x_{ij} \, . \tag{4}$$

Information about all payoff-relevant variables is complete and symmetric.

2.2 Incentivisation Schemes: A Closer Look

It is immediate to check from expression (3) that the parameter a_i summarises all salient characteristics of the incentivisation scheme deployed by group i. We say that a scheme is *restricted* if $a_i \in [0, 1]$ applies, and *unrestricted* otherwise.

Restricted schemes amount to pure within-group redistribution. In this light, a_i can be interpreted as the share of the prize to be allocated, in case of success, to all members of group i independently of the effort they provided. Accordingly, we refer to it as the *equalising part* of the scheme. Symmetrically, $1 - a_i$ is the effort-contingent share of the prize, distributed only to the members of group i that actively contributed to the group's success. Accordingly, we refer to it as the *incentivising part* of the scheme. Restricted schemes are more and more 'egalitarian' as a_i approaches one, and more and more 'meritocratic' as a_i approaches zero.

Unrestricted schemes go beyond pure redistribution by allowing for within-group cross-subsidisation financed by the expropriation of members' private resources. To spur effort provision, free-riders can be punished by levying a 'tax' on their private resources ($a_i < 0$), and contributors can be rewarded, on top of their fair share of the prize, with the corresponding 'tax revenues'. Symmetrically, to disincentivise effort provision the private resources of contributors can be taxed ($a_i > 0$) to reward free-riders. Unrestricted schemes are less and less 'meritocratic' as a_i diverges to $+\infty$, and more and more so as a_i diverges to $-\infty$. In general, the a_i component of any scheme determines how appealing free riding is to the typical member of that group – the larger a_i, the higher the level of within-group egalitarianism, the larger the incentive to free-ride.

Building on Nitzan (1991a,b) and Lee (1995), in this paper we go beyond pure redistribution[11] and focus instead on more general, unrestricted incentivisation schemes. Following Ueda (2002), we henceforth refer to the generic scheme via the summary statistic

$$\gamma_i \;:=\; 1 - a_i + \frac{a_i}{n_i}\,, \qquad \text{for all groups } i = 1, 2, \ldots, m\,, \tag{5}$$

that can be interpreted as the stand-alone incentive to provide effort in group i induced by that scheme. Accordingly, we indicate with $\boldsymbol{\gamma} := (\gamma_1, \gamma_2, \ldots, \gamma_m) \in \mathbb{R}^m$ the arbitrary profile of unrestricted schemes. Note that, since γ_i strictly decreases in a_i, a larg(er) γ_i indicates a low(er) level of egalitarianism in group i, and *vice versa*.

2.3 Sequential Structure

We consider a two-stages game with the following timing. At $t = 1$ the incentivisation stage-game is played: all group leaders simultaneously select incentivisation schemes that solve the following welfare-maximisation problem

$$\gamma_i^* \in \underset{\{\gamma_i\}}{\operatorname{argmax}} \; \sum_{j=1}^{n_i} \pi_{ij}(x_{ij}, \mathbf{X}) \;=\; p_i(\mathbf{X})\,[\,v\, q_{ij}(\mathbf{x}_{\boldsymbol{i}})\,] - x_{ij}\;,$$

and the equilibrium profile $\gamma^* \in \mathbb{R}^m$ of schemes is established. At $t = 2$ the effort stage-game is played: after observing the profile γ^*, all members j of all groups i simultaneously choose an effort level $x_{ij}^* : \mathbb{R}^m \longmapsto \mathbb{R}_+$. The prize is subsequently assigned in accordance with (2), all payoffs are realised and the game ends.

3 Effort Stage-Game: Two-Group Example

To ease the exposition and fix ideas, we discuss first a simple two-group example – the related proofs and derivations are collected into the Appendix B. In Section 4 we generalise the analysis to an arbitrary number $m \geq 2$ of groups.

Consider a group-contest game identical to that outlined in Section 2, but for the fact that $m = 2$ and $\sum_{i=1}^m X_i = X_1 + X_2$. Without loss of generality, assume further that $n_1 \geq n_2$. Since, via (2), the expected utilities $\pi_i j$ of all group members are not continuous at $\mathbf{X} = \mathbf{0}$, we must analyse separately the zero-effort candidate equilibria (where $X_1^* = X_2^* = 0$) and their positive-effort

[11] See also Baik and Lee (1997).

counterparts (where $X_i^* > 0$ for *at least* one group $i = 1, 2$). Focus first on the zero-effort equilibria: abiding by the prescribed course of action, the generic member j of group i gets $v/2n_i$; deviating to a positive but arbitrarily small effort level, the same group member gets $\lim_{x_{ij} \longrightarrow 0} v\gamma_i - x_{ij} = v\gamma_i$. Therefore, a necessary and sufficient condition for the existence of a zero-effort equilibrium is

$$\gamma_i \leq \frac{1}{2n_i}, \qquad \text{for both groups } i = 1, 2\,. \tag{6}$$

Note that $\gamma_i \leq 1/2n_i$ entails that $a_i \geq (2n_i - 1)/2(n_i - 1) > 1$. Therefore, condition (6) essentially says that a zero-effort equilibrium ensues if all groups adopt incentivisation schemes that punish contributors and reward free-riders. Now turn to the positive-effort equilibria, and arbitrarily focus on group 1: differentiating with respect to x_{1j} the expected utility (4) of its generic member j and imposing within-group symmetry, we obtain

$$X_1^* = \begin{cases} (X_1^* + X_2^*) \left[\gamma_1 - \dfrac{1}{v}\,(X_1^* + X_2^*)\right] & \text{if } X_1^* > 0 \text{ and } X_2^* > 0\,, \\ \dfrac{v}{n_1}\left(\gamma_1 - \dfrac{1}{n_1}\right) & \text{if } X_1^* > 0 \text{ and } X_2^* = 0\,, \end{cases}$$

with

$$X_1^* + X_2^* = v\left(\frac{n_1\gamma_1 + n_2\gamma_2}{N} - \frac{1}{N}\right)$$

when both groups are active, and with $x_{1j}^* = X_1^*/n_1$ via the WGS assumption. Note that, in equilibrium, individual and group effort respond solely to the aggregate effort across all groups and the average egalitarianism among the active ones. Note indeed that: (i) when both groups are active, the element $(n_1\gamma_1 + n_2\gamma_2)/N$ is their average stand-alone incentive – weighted by relative group sizes; (ii) when only group 1 is active, γ_1 is by definition the same average stand-alone incentive.[12] The non-negativity constraints

$$n_1\gamma_1 + n_1\gamma_2 \geq 1\,, \tag{7a}$$

$$\gamma_1 \geq \frac{1}{n_1} \tag{7b}$$

ensure, at the same time, that $X_1^* + X_2^* > 0$ indeed holds and that no profitable unilateral deviations from the prescribed play are available to individual group members.

Checking for the mutual consistency of constraints (6)-(7), a unique WGS

[12] A symmetric reasoning applies when group 2 is active and group 1 is inactive.

equilibrium can be characterised for every arbitrary profile $\boldsymbol{\gamma} \in \mathbb{R}^2$ of unrestricted schemes, except for two subsets of the state space of $\boldsymbol{\gamma}$ where individual group members endlessly cycle between zero and positive effort. The following proposition outlines the characterisation.

PROPOSITION 1 (Two-group example: equilibrium of the effort stage-game)**.** *Let $(\gamma_1, \gamma_2) =: \boldsymbol{\gamma} \in \mathbb{R}^2$ be a pair of arbitrary incentivisation schemes for group 1 and 2 respectively. Moreover, let $P^*(\mathbb{R}^2) := \{ \boldsymbol{G_{00}}, \boldsymbol{G_{+0}}, \boldsymbol{G_{0+}}, \boldsymbol{G_{++}}, \boldsymbol{G_{\nexists}} \}$ be a partition of the state space of $\boldsymbol{\gamma}$ defined as follows:*

$$
\begin{aligned}
\boldsymbol{G_{00}} &= \left\{ (\gamma_1, \gamma_2) \in \mathbb{R}^2 \, : \, \gamma_1 \leq \frac{1}{2n_1} \, , \gamma_2 \leq \frac{1}{2n_2} \right\}, \\
\boldsymbol{G_{+0}} &= \left\{ (\gamma_1, \gamma_2) \in \mathbb{R}^2 \, : \, \gamma_1 > \frac{1}{n_1} \, , \gamma_2 \leq \gamma_1 - \frac{1}{n_1} \right\} \\
\boldsymbol{G_{0+}} &= \left\{ (\gamma_1, \gamma_2) \in \mathbb{R}^2 \, : \, \gamma_2 > \frac{1}{n_2} \, , \gamma_1 \leq \gamma_2 - \frac{1}{n_2} \right\}, \\
\boldsymbol{G_{++}} &= \left\{ (\gamma_1, \gamma_2) \in \mathbb{R}^2 \, : \, \gamma_1 \geq \gamma_2 - \frac{1}{n_2} \, , \gamma_2 \geq \gamma_1 - \frac{1}{n_1} \, , \gamma_2 \geq \frac{1}{n_1} - \frac{n_2}{n_1}\gamma_1 \right\}, \\
\boldsymbol{G_{\nexists}} &= \mathbb{R}^2 \setminus \boldsymbol{G_{00}} \setminus \boldsymbol{G_{+0}} \setminus \boldsymbol{G_{0+}} \setminus \boldsymbol{G_{++}} \neq \emptyset \, .
\end{aligned}
$$

Then, the effort stage-game of the two-group example has:

- *a unique WGS equilibrium with $X_1^* = X_2^* = 0$ for every $\boldsymbol{\gamma} \in \boldsymbol{G_{00}}$;*
- *a unique WGS equilibrium with $X_1^* > 0$ and $X_2^* = 0$ for every $\boldsymbol{\gamma} \in \boldsymbol{G_{+0}}$;*
- *a unique WGS equilibrium with $X_1^* = 0$ and $X_2^* > 0$ for every $\boldsymbol{\gamma} \in \boldsymbol{G_{0+}}$;*
- *a unique WGS equilibrium with $X_1^* > 0$ and $X_2^* > 0$ for every $\boldsymbol{\gamma} \in \boldsymbol{G_{++}}$;*
- *no pure-strategy WGS equilibria for every $\boldsymbol{\gamma} \in \boldsymbol{G_{\nexists}}$.*

Proof. See Appendix B. □

Figure 1 provides the graphical intuition behind the results of Proposition 1. Note that, for all schemes $(\gamma_1, \gamma_2) \in \boldsymbol{G_{\nexists}}$ the individual members of one or both groups have cyclical preferences about effort provision: they prefer to exert some effort when all other members of their group are free-riding; at the same time, they prefer to free-ride when all other members of their group are exerting effort. We call such individuals *contrarians*. By symmetry, all members of a group are contrarians if at least one of them is a contrarian. Accordingly, we call contrarian, too, a group whose members are contrarians.

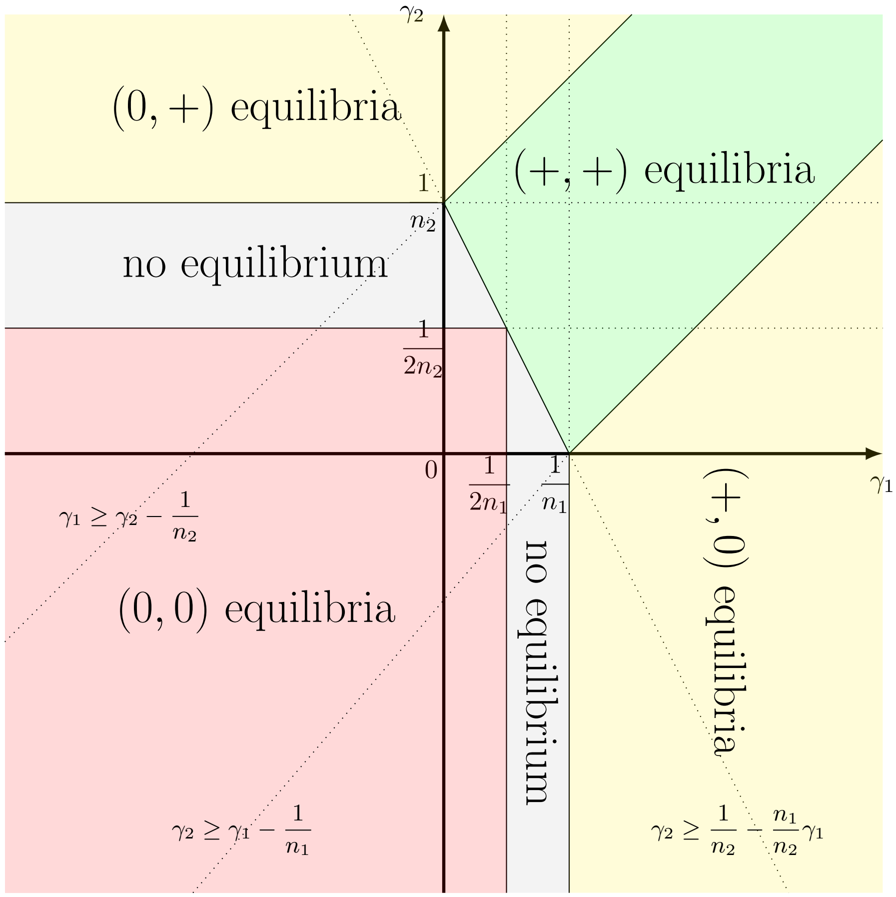

Figure 1: Partition $P^*(\mathbb{R}^2)$ of Proposition 1 on a (γ_1, γ_2) plane. The partition marks the existence regions for the pure-strategy WGS equilibria of the effort stage-game: $\boldsymbol{G_{00}}$ (in red); $\boldsymbol{G_{+0}}$ and $\boldsymbol{G_{0+}}$ (in yellow); $\boldsymbol{G_{++}}$ (in green); $\boldsymbol{G_{\nexists}}$ (in grey).

The intuition behind the presence of contrarians is the following. The condition $\gamma_i \geq 1/n_i$ ensures that it is virtually possible to sustain a positive-effort equilibrium with group i as the sole active group: if $\gamma_i < 1/n_i$, group i is too large to spur effort provision by its members in the absence of some competitive pressure from the competing group $-i$. At the same time, the condition $\gamma_i > 1/2n_i$ is sufficient for a zero-effort equilibrium not to be sustainable: the generic member of group i finds it profitable, by definition, to unilaterally deviate to positive effort provision when all other members of his/her group and all members of the competing group are not exerting any effort. Therefore, the subset $\gamma_i \in (1/2n_i, 1/n_i)$ is a grey area for the incentivisation schemes of group i: the competitive pressure within-group they brings about is sufficiently high to generate a unilateral incentive to escape from a zero-effort equilibrium, but not high enough to actually sustain effort provision if all group member were indeed to implement their desired deviations. Cycles arise as a byproduct, and no pure-strategy WGS equilibria can exist. As we will show in the next section, this result extends to the generalised m-group specification.

4 Effort Stage-Game: $m \geq 2$ Groups

Building on the intuitions provided by the two-group example, in this section we characterise the WGS equilibria of the effort stage-game for any arbitrary profile of incentivisation schemes and any arbitrary number $m \geq 2$ of groups. As in Section 3, we focus first on zero-effort equilibria, and then turn to their positive-effort counterparts.

4.1 Zero-Effort Equilibria

The necessary and sufficient condition for the existence of zero-effort equilibria is a one-to-one extension of that identified in the two-group example. The following lemma highlights the result.

LEMMA 1 (Necessary and sufficient condition for zero-effort equilibria)**.**
Let

$$\gamma_i \leq \underline{\underline{\gamma_i}} := \frac{1}{mn_i} \tag{8}$$

hold for all groups $i = 1, 2, \ldots, m$. Then, an equilibrium exists for the effort-stage-game played at $t = 2$ such that $x^*_{ij} = 0$ holds for all members of all groups, whereby $\sum_{i=1}^{m} X_i = 0$.

Proof. See Appendix A.1. □

In words: as it was the case for the two-group example, a zero-effort equilibrium ensues if all groups adopt incentive schemes that reward free-riders at the expense of contributors.[13] The underlying intuition is simple: $\underline{\underline{\gamma_i}}$ is *by definition* the critical value of the stand-alone incentive γ_i below which, in a zero-effort

[13]Since any scheme with $a_i > 1$ entails an "inverse cross-subsidisation" from contributors to free riders: in case of success, an amount $a_i - 1 > 0$ of private resources of those group member that provided effort are expropriated by the leader and distributed to those who instead opted for free riding.

environment, no individual group member has an unilateral incentive to deviate to some positive effort level, however small. If $\gamma_i > \underline{\underline{\gamma_i}}$ were to hold for one single group, a zero-effort equilibrium would not be sustainable – every member of that group would find it profitable to deviate to some positive effort level.[14]

We will prove in Lemma 7 that, for every profile $\boldsymbol{\gamma}$ of incentivisation schemes that meets condition (8), the zero-effort equilibrium of Lemma 1 is the unique WGS equilibrium of the effort stage-game.

4.2 Positive-Effort Equilibria

Even under the WGS assumption, positive-effort equilibria are intrinsically more difficult to characterise, in a m-group environment, than their zero-effort counterparts. Difficulties boil down to the self-sustaining nature of the asymmetric behaviour across groups that characterise most of such equilibria: the inaction of those groups that, in equilibrium, opt for zero effort, is sustained by the common belief that the active groups will indeed take action, and *vice versa*. This self-fulfilling element renders size and composition of the equilibrium subsets of active and inactive groups interdependent.

To overcome the interdependency problem, we adopt a three-step procedure. First, we characterise *constrained equilibria*, where the group set M is arbitrarily partitioned into two subsets: the *active coalition* $A \subseteq M$, whose members optimally and strategically choose an effort level $x_{ij} \geq 0$ in a decentralised fashion; the *inactive coalition* $\overline{A} := M \setminus A$, whose member must exert zero effort regardless of its optimality.[15] Slightly abusing language, we henceforth use the term 'coalition' to indicate a generic set of groups.[16] Second, we check under which conditions (if any) coalitions are in equilibrium – no member of any group assigned to the active coalition unilaterally prefers to switch to zero effort, and no member of any group assigned to the inactive coalition unilaterally prefers to deviate to some positive effort level. Third, we determine the equilibrium coalitions and the corresponding, unconstrained equilibrium play.[17]

[14] The argument holds true *a fortiori* if $\gamma_i \geq \underline{\underline{\gamma_i}}$ holds for more than one group.

[15] We qualify such equilibria as 'constrained' since only a subset of groups is actually free to choose its effort level optimally.

[16] In Sánchez-Pagés (2007), for instance, a coalition is a group whose size is endogenously determined by the affiliation choices of individual contestants. In our model, a coalition is a set of groups, each of pre-determined size, that autonomously adopt affine equilibrium strategies.

[17] Note that we take exogenously-given coalitions as a point of departure to characterise which incentives drive deviations, at the individual level, for any arbitrary coalitional structure.

The rationale for this procedure is the following: by abstracting from the interdependence of coalitional sizes, constrained equilibria uncover the hidden structure of positive-effort WGS equilibria, thus allowing for a clear-cut characterisation of the *ex ante* expected payoffs, hence of the unilateral incentives to deviate.

4.2.1 Constrained WGS Equilibria

Let $\mathcal{P}(M)$ be the set of all binary partitions of the group set M, with $P=\langle A, \overline{A}\,\rangle$ its generic element. Moreover, let A_k be the arbitrary coalition of $k=1,\ldots,m$ groups whose members are free to choose optimally their effort levels – i.e. A_k is the arbitrary active coalition. Differentiating the expected utility (4) of the j-th member of the i-th active group with respect to the individual effort level x_{ij}, imposing within-group symmetry via $x^c_{i1}=x^c_{i2}=\ldots=x^c_{in_i}=x^c_i$ and rearranging, we obtain the tuple of k simultaneous FOCs

$$\begin{aligned} n_1\, x^c_1 &= n_1\,\gamma_1 \sum_{\ell\in A_k} X^c_\ell - \frac{n_1}{v}\left(\sum_{\ell\in A_k} X^c_\ell\right)^2 , \\ \vdots \quad & \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ n_k\, x^c_k &= n_k\,\gamma_k \sum_{\ell\in A_k} X^c_\ell - \frac{n_k}{v}\left(\sum_{\ell\in A_k} X^c_\ell\right)^2 , \end{aligned} \tag{9}$$

where the subscripts 'c' highlight that the equilibrium play we are characterising is subject to exogenous behavioural constrains. Note that the left-hand sides of (9) coincide, via the WGS assumption, with the group effort levels $X^c_i=n_i x^c_i$. Notice further that $\sum_{\ell\in A_k} X^c_\ell=\sum_{\ell=1}^m X^c_\ell$ holds since, by assumption, $x^c_{ij}=X^c_i=0$ is mandated to all members of the $m-k$ groups assigned to the inactive coalition $\overline{A_k}$. Hence, the generic FOC in (9) becomes

$$X^c_i \;=\; n_i\,\gamma_i \sum_{\ell=1}^m X^c_\ell - \frac{n_i}{v}\left(\sum_{\ell=1}^m X^c_\ell\right)^2 . \tag{10}$$

Expression (10) entails that the constrained equilibrium, if any, has a block-recursive structure: the FOCs (9) jointly determine the aggregate effort across groups

$$\sum_{\ell=1}^m X^*_\ell = \sum_{\ell=1}^m X^*_\ell\left(\sum_{\ell\in A_k} n_\ell\gamma_\ell\right) - \frac{1}{v}\left(\sum_{\ell=1}^m X^*_\ell\right)^2\left(\sum_{\ell\in A_k} n_\ell\right),$$

that determines 'backwards' all group effort levels X^c_i of the active groups, thus determining, in turn, the individual effort levels of the members of these groups as $x_i^c = X_i^c/n_i$ via the WGS assumption. We summarise the characterisation of the generic constrained equilibrium into the following lemma.

LEMMA 2 (Constrained WGS equilibria)**.**
Let $P_k = \langle A_k, \overline{A_k} \rangle$ be an arbitrary binary partition of the group set M, with $k \in \{1, \ldots, m\}$, such that:

- *$x_{ij}^c \geq 0$ for every j-th member of every group $i \in A_k$;*
- *$x_{ij}^c = 0$ for every j-th member of every group $i \in \overline{A_k}$.*

Moreover, let $A_k^+ \subseteq A_k$ be the subset of groups $i \in A_k$ with $x_{ij}^c > 0$, with

$$\mathcal{N}[A_k] := \sum_{\ell \in A_k^+} n_\ell$$

the size of this subset and

$$\bar{\gamma}[A_k^+] \;:=\; \frac{1}{\mathcal{N}[A_k^+]} \sum_{\ell \in A_k^+} n_\ell \gamma_\ell \tag{11}$$

the average level of egalitarianism among its groups. Then, if

$$\sum_{\ell \in A_k^+} n_\ell \gamma_\ell \;>\; 1\,, \tag{12a}$$

$$\gamma_i \;\geq\; \bar{\gamma}[A_k^+] - 1/\mathcal{N}[A_k^+] \quad \textit{for all groups } i \in A_k^+\,, \tag{12b}$$

the effort-stage-game played at $t = 2$ has a constrained pure-strategy WGS equilibrium in which

$$x_{ij}^c \;=\; \begin{cases} \max\left\{\, \hat{x}_i^c\,,\, 0 \,\right\} & \textit{for all members of all groups } i \in A_k\,, \\ \quad 0 & \textit{for all members of all groups } i \in \overline{A_k}\,, \end{cases}$$

with

$$\hat{x}_i^c \;=\; \left(\sum_{\ell=1}^{m} X_\ell^c\right)\left[\gamma_i - \frac{1}{v}\left(\sum_{\ell=1}^{m} X_\ell^c\right)\right] \tag{13}$$

and $X_i^c = n_i x_i^c$ for all groups $i = 1, 2, \ldots, m$, and where the fixed-point condition $\sum_{\ell=1}^m X_\ell^c = \sum_{\ell \in A_k^+} n_\ell \hat{x}_\ell^c$ unambiguously identifies the aggregate effort across groups

$$\sum_{i=1}^{m} X_i^c \;=\; v\left(\bar{\gamma}[A_k^+] - \frac{1}{\mathcal{N}[A_k^+]}\right) \;>\; 0\,. \tag{14}$$

Proof. See Appendix A.2. □

Not to excessively weigh down the exposition, we henceforth assume that $A_k \equiv A_k^+$. Since this holds true by definition in equilibrium, the assumption is without significant loss of formal rigour.

Three observation ensue. First: when positive, the optimal effort level of the j-th member of the i-th active group is solely affected by averages and aggregates – namely, by the aggregate effort $\sum_{i=1}^m X_i^c$ and average level of egalitarianism $\bar{\gamma}[A_k]$ among the other active groups. Therefore, under the WGS assumption, strategic interactions endogenously come in a mean-field form and the game is naturally *aggregative* – note that this is the same aggregative structure highlighted by the two-group example of Section 3. Second: the aggregate effort across groups strictly increases in the average level of egalitarianism $\bar{\gamma}[A_k]$ among those groups that, in equilibrium, are indeed active. The intuition is straightforward. The larger γ_i, the less egalitarian (equiv. the more meritocratic) group i is: to get a larger share of the prize upon success, its generic j-th member must exert more effort, *ceteris paribus*, since the effort-contingent share is larger. Anticipating that competition will be harsh because the other active groups are, on average, more meritocratic, the members of each group understand that a greater endeavour is required to have a chance to succeed. Third: effort within- and across-groups are strategic complements or substitutes in the sense of Bulow et al. (1985) depending on relative the intensity of within-group competition. More precisely: individual and group effort provision increase in the aggregate effort across groups (complementarity) if within-group competition is high with respect to average of the other active groups; the reverse is true (substitutability) if within-group competition is low with respect to average of the other active groups. The following corollary highlights the result.

COROLLARY 1.
The individual effort x_{ij}^c exerted by the generic member j of an active group $i \in A_k^+$ increases in the aggregate effort level across groups $\sum_{i=1}^m X_i^c$ if $\gamma_i > 2(\bar{\gamma}[A_k^+] - 1/\mathcal{N}[A_k^+])$, and decreases otherwise.

Proof. The proof is mathematically trivial: via (13) $\partial x_{ij}^c / \partial \sum_{i=1}^m X_i^c = \gamma_i - 2/v \sum_{i=1}^m X_i^c$ holds that, via (14), is strictly positive if $\gamma_i > 2(\bar{\gamma}[A_k^+] - 1/\mathcal{N}[A_k^+])$ holds. □

4.2.2 Unilateral Deviations

Lemma 2 characterised a generic constrained equilibrium where strategies are optimally selected only by a subset of individuals – those whose groups have been assigned to the active coalition A_k. We now take a step further towards the characterisation of its unconstrained counterpart by checking under which conditions, if any, unilateral profitable deviations are available to some group member in any coalition. What we find is that the relative egalitarianism of any group with respect to the average egalitarianism of the active coalition unambiguously determines whether or not individual group members have an incentive to deviate from the prescribed (constrained play). The following lemma summarises the analysis.

LEMMA 3 (Unilateral deviations)**.**
For every arbitrary binary partition $P_k = \langle A_k, \overline{A_k} \rangle$ of the group set M:

- *the j-th member of the i-th group within the active coalition A_k deviates to zero effort if*
$$\gamma_i \leq \bar{\gamma}[A_k] - \frac{1}{\mathcal{N}[A_k]}\,, \tag{15}$$
and abides by the prescribed strategy $x^c_{ij} > 0$ otherwise;[18]

- *the j-th member of the i-th group within the inactive coalition $\overline{A_k}$ deviates to some positive effort level if*
$$\gamma_i > \bar{\gamma}[A_k] - \frac{1}{\mathcal{N}[A_k]}\,, \tag{16}$$
and abides by the prescribed strategy $x^c_{ij} = 0$ otherwise.

Proof. See Appendix A.3. □

In words: unilateral deviations are strictly ordered by the level of within-group egalitarianism γ_i: (i) profitable deviations within the active coalition A_k, if any, are available to the members of the most egalitarian groups – those with low(er) γ_i; (ii) profitable deviations within the inactive coalition $\overline{A_k}$, if any, are available to the members of the most meritocratic groups – those with high(er) γ_i. It is worth noting that unilateral incentives to deviate are once again defined in terms of averages and aggregates: conditions (15) and (16) indeed require individual members to compare the egalitarianism of the group they belong to with (i) the average egalitarianism $\bar{\gamma}[A_k]$ of the active coalition (ii) loosened by a size-related 'tolerance margin' $1/\mathcal{N}[A_k]$ – the smaller the size of the active coalition, the larger the tolerance margin, the larger the incentive for the generic member

[18]Recall that, for the sake of simplicity, we are assuming that $\mathcal{A}_k \equiv \mathcal{A}^+_k$.

of an inactive group to deviate to positive effort provision, *ceteris paribus*. Note finally that conditions (15) and (12b) are symmetric: the absence of profitable unilateral deviations from within the arbitrary active coalition, therefore, is sufficient to guarantee that individual and group effort be strictly positive in a constrained equilibrium.

Before proceeding, it is important to highlight that, via condition (15), $\gamma_i > \bar{\gamma}[A_k] - 1/\mathcal{N}[A_k]$ guarantees at the same time that the generic member of an active group i has no incentive to deviate to zero effort *and* that the aggregate effort of group i be strictly positive. The following Corollary formally states the result.

COROLLARY 2 (Interpretation of negative equilibrium effort)**.**
The condition

$$\gamma_i \ > \ \bar{\gamma}[A_k] - \frac{1}{\mathcal{N}[A_k]}$$

is sufficient to guarantee that $X_i^c = n_i x_i^c > 0$*, with* x_i^c *the individual constrained-equilibrium effort defined by (13).*

Proof. The proof is mathematically trivial. Via (13) it is immediate to check that, if $\sum_{\ell=1}^m X_\ell^c > 0$, then $X_i^c = n_i^c > 0$ if $\gamma_i > \sum_{\ell=1}^m X_\ell^c / v = \bar{\gamma}[A_k] - 1/\mathcal{N}[A_k]$. □

Corollary 2 is important because it stresses that the non-negativity constraints have a logical content that is consistent with that of the constrained-equilibrium characterisation: individual and aggregate effort levels are positive, in equilibrium, when it is indeed optimal for group members to provide positive effort; *vice versa*, negative effort provision unambiguously signals that it is suboptimal for group members to provide effort.

4.2.3 Cyclical Behaviour

All members of all groups are identical *a priori*. Therefore, if one of them has a unilateral incentive to deviate from a prescribed play, this must be the case, too, for all other members of the group he/she belongs to. Accordingly, any WGS equilibrium must entail that: (i) all members of that group be able to implement their desired deviations; (ii) coalitions be stable after deviations are collectively implemented. This *per se* innocuous observation can be very problematic for the characterisation of WGS equilibria:

i) the two-group example of Section 3 already highlighted that, symmetry notwithstanding, cycles arise at some profiles of incentivisation schemes $\boldsymbol{\gamma}$

nearby those that sustain zero-effort equilibria – in these cases, one or both groups are contrarians: their generic member prefers to switch unilaterally to positive effort provision at a zero-effort profile, but, at the same time, he/she prefers to deviate back to zero effort if all other members switch to positive effort;

ii) by the same argument, conditions (15) and (16), too, may generate cycles. Take the perspective of the generic member of an inactive group, and assume he/she finds it optimal to switch from zero to positive effort – by symmetry, all other members of the same group must be willing to do the same. Via (16) we know that, for this to be the case, the egalitarianism of the inactive group under consideration must be sufficiently close to the average egalitarianism of the active coalition. But if all members of the inactive group actually implement their desired deviations, the average egalitarianism of the active coalition changes, for the active coalition itself changes: the formerly inactive group turns active and, in so doing, it is *de facto* included into a new, enlarged active coalition. Via (15) we also know that the generic member of the formerly inactive group does not find it optimal to switch back to zero effort if the egalitarianism of his/her group is sufficiently close to the average egalitarianism of the *new* active coalition. There is no guarantee *a priori* that the two conditions be mutually consistent and, absent such consistency, the inactive group under consideration is contrarian – its members endlessly cycle between zero and positive effort.

The existence of contrarians is relevant to the analysis in two respects. First, it highlights that, even under a strong symmetry assumption, a contest played by groups significantly differs from a contest played by individuals. Our groups are compact cohorts of individuals whose decentralised decision-making is somehow coordinated by within-group symmetry; yet, they cannot be interpreted as single 'representative' contestants, since unilateral deviations that are profitable at the individual level need not be so at the group level. Second, the presence of contrarians entails that pure-strategy WGS equilibria do not exist at some profiles of incentivisation schemes.

In the remainder of this subsection we prove that (*i*) conditions (15) and (16) do *not* generate cycles, but (*ii*) cyclical regions in the space of incentivisation schemes as those highlighted by the two-group example indeed exist in the generalised m-group setup. Therefore, to proceed with the characterisation, we finally provide a sufficient condition for the absence of cycles.

Absence of Cycles Induced by Conditions (15) and (16)

Conditions (15) and (16) may indeed be problematic, since the critical value of γ_i above which the generic member of an inactive group finds it optimal to deviate to positive effort is strictly smaller than the critical value of γ_i below which the generic member of an active group finds it optimal to deviate to zero effort. However, it is possible to prove that the presence of a unilateral incentive to deviate from zero to positive effort is *sufficient*, in the first place, to guarantee the absence of cycles – any member that is willing to deviate to positive effort when his/her group is inactive never finds it optimal to switch back to zero effort once his/her group turns active.

Let $A_k^{\not\ni i}$ be the arbitrary active coalition of k groups that does not include group i, and $A_{k+1}^{\ni i} := A_k^{\not\ni i} \cup \{i\}$ be the same coalition after the inclusion of group i. The following lemma formalises the results qualitatively discussed above.

LEMMA 4 (Conditions (15) and (16) do not induce cycles)**.**
Let all behavioural constraints of Lemma 2 hold. Moreover, let condition (16) be met for some inactive groups $i \in \overline{A_k}$ – whereby their generic member has a unilateral incentive to deviate to positive effort. Then, there exists a non empty, well-defined set of values

$$\gamma_{out \to in}(A_k^{\not\ni i}) \;=\; \bar{\gamma}[A_k^{\not\ni i}] - \frac{1}{\mathcal{N}[A_k^{\not\ni i}]}\,, \tag{17a}$$

$$\gamma_{in \to out}(A_{k+1}^{\ni i}) \;=\; \bar{\gamma}[A_{k+1}^{\ni i}] - \frac{1}{\mathcal{N}[A_k^{\not\ni i}] + n_i}\,, \tag{17b}$$

such that all members of such inactive groups: (i) *strictly prefer to remain inactive if $\gamma_i < \gamma_{out \to in}(A_k^{\not\ni i})$;* (ii) *strictly prefer to turn active if $\gamma_i > \gamma_{in \to out}(A_{k+1}^{\ni i})$;* (iii) *are contrarians if $\gamma_i \in [\gamma_{out \to in}(A_k^{\not\ni i}), \gamma_{in \to out}(A_{k+1}^{\ni i})]$.*

However, condition (16) is sufficient to guarantee that $\gamma_i > \gamma_{in \to out}(A_{k+1}^{\ni i})$ and, accordingly, that all inactive groups (if any) whose members prefer to turn active can never be contrarian groups.

Proof. See Appendix A.4. □

In words: every inactive group whose generic member has a unilateral incentive to switch to positive effort can be *in principle* a contrarian group; however, the very presence of unilateral incentive to deviate rules out the possibility of being

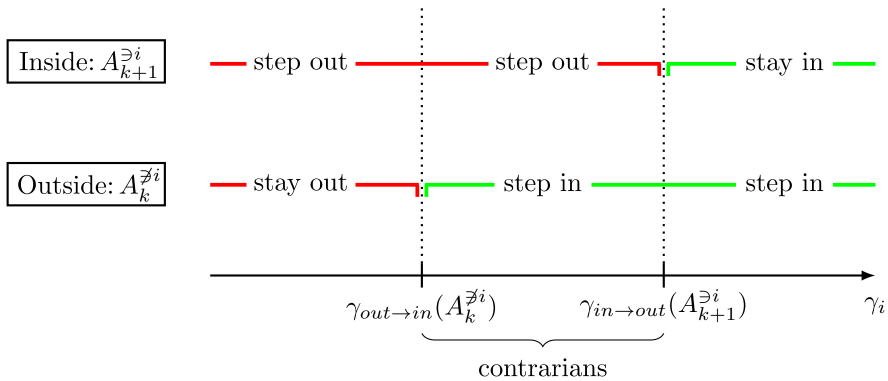

Figure 2: Range of values of γ_i such that, given the arbitrary binary partition $P_k = \langle A_k, \overline{A_k} \rangle$ of the group set, group i is contrarian.

a contrarian group. As a consequence, no cycles in effort provision are actually generated by conditions (15) and (16).

As we briefly discussed above, from a mathematical standpoint the possibility to have contrarian groups boils down to the fact that the inclusion of a $k+1$-th group into any active coalition A_k changes in opposite directions the conditions – (15) and (16) – that govern unilateral deviations. Even in an extremely orderly situation where the k active groups are those with the highest γ_i, the inclusion of a $k+1$-th group (i) unambiguously increases the average egalitarianism coalition-wide – $\bar{\gamma}[A_{k+1}^{\ni i}] < \bar{\gamma}[A_k^{\not\ni i}]$ –, thus relaxing condition (16) for outsiders, but (ii) unambiguously decreases the size-related margin – $1/\mathcal{N}[A_k^{\not\ni i}] > 1/\mathcal{N}[A_{k+1}^{\ni i}]$ –, thus rendering condition (15) more stringent. The net effect is *a priori* ambiguous. Lemma 4 essentially proves that the unilateral incentive to switch from zero to positive effort provision induced by condition entails $\gamma_{in\to out}(A_{k+1}^{\ni i}) > \gamma_{out\to in}(A_k^{\not\ni i})$ (possibility of cycles) but also $\gamma_i > \gamma_{in\to out}(A_{k+1}^{\ni i})$ (impossibility of cycles).

Presence of Cycles Nearby Zero-Effort Equilibria

The two-group example of Section 3 showed that cycles in effort provision arise when the within-group egalitarianism of one or both groups is sufficiently low to guarantee a zero-effort equilibrium cannot be sustained, but not low enough to sustain positive-effort equilibria where one or both groups are active. We now prove that this remains true in a generalised m-group environment. To fix ideas, the following three-group example outlines the argument.

EXAMPLE 1.
In a three-group environment ($m=3$) *consider an arbitrary profile of incentivisation schemes* $\boldsymbol{\gamma} := (\gamma_1, \gamma_2, \gamma_3)$ *such that* $\gamma_1 \in (1/3n_1, 1/2n_1)$, $\gamma_2 \in (1/3n_2, 1/2n_2)$ *and* $\gamma_3 < 0 < 1/3n_3$. *Then:*

- *a zero-effort equilibrium cannot be sustained, since all members of groups* 1 *and* 2 *have a unilateral incentive to deviate to positive effort provision via Lemma 1;*

- *a positive-effort equilibrium with one, two or three active groups cannot be sustained, since the aggregate effort would be negative via Lemma 2.*

Accordingly, all members of groups 1 *and* 2 *endlessly cycle between zero and positive effort provision.*

In general, for every arbitrary $k = 1, 2, dots, m$ the condition $\gamma_i \geq 1/kn_i$ for all groups $i = 1, 2, \ldots, m$ is sufficient for the stability of an active coalition of $k, k+1, \ldots, m$ groups since, via Lemma 2, it ensures that the aggregate effort at the constrained equilibrium be positive – via Corollary 2, it also ensures that no individual member of any active coalition $A_k, A_{k+1}, \ldots, A_m$ finds it profitable to deviate to zero effort. Therefore, any profile of incentivisation schemes such that $\gamma_i < 1/n_i$ for all groups $i = 1, 2, \ldots, m$ potentially generates cyclical behaviour. The following lemma proves the claim for a specific configuration of the profile of schemes.

LEMMA 5 (Cycles near zero-effort equilibria)**.**
Let all behavioural constraints of Lemma 2 hold. Moreover, let $\gamma_i < 0 < \underline{\underline{\gamma_i}}$ *hold for the first* $k \geq 1$ *groups and* $\gamma_i \in (\underline{\underline{\gamma_i}}, 1/kn_i)$ *hold for the remaining* $m-k$ *groups. Then, the effort stage-game has no pure-strategy WGS equilibrium, since all members of the first* k *groups endlessly cycle between zero and positive effort.*

Proof. See Appendix A.5. □

By identifying the problem, Lemma 5 also suggests a solution. Cycles arise when no group is meritocratic enough to sustain a 'monopolised' positive-effort equilibrium with a single active group. The presence of a single group with $\gamma_i \geq 1/n_i$, therefore, is sufficient *in principle* to rule out cyclical behaviour – the single group with $\gamma_i > 1/n_i$ is active for sure in equilibrium, and all other $-i$ groups are active or not depending on how close their egalitarianism is to $1/n_i$. *A fortiori*, the no-cycle result extends to a more stringent condition whereby

$\gamma_i \geq 1/n_i$ must hold for all groups $i = 1, 2, \ldots, m$. Besides ruling out cycles, such condition has a sensible economic interpretation: it amounts to impose the inadmissibility of those incentivisation schemes that tax the private resources of effort providers to reward free-riders.[19] The following lemma formally states the condition.

LEMMA 6 (Sufficient condition for the absence of cycles)**.**
Let

$$\boldsymbol{G^*} := \left\{ \boldsymbol{\gamma} : \gamma_i \geq \frac{1}{n_i} \text{ for all } i \in M \right\} \tag{18}$$

be the subset of unrestricted incentivisation schemes that do not reward free-riders at the expense of effort providers. Then, $\boldsymbol{\gamma} \in \boldsymbol{G^*}$ *is a sufficient condition for the absence of cycles.*

Proof. See Appendix A.6. □

Note that, by ruling out cycles, the condition outlined by Lemma 6 is sufficient, too, for the existence of pure-strategy WGS equilibria at every admissible profile of schemes $\boldsymbol{\gamma} \in \boldsymbol{G^*}$.[20]

4.2.4 Existence and Uniqueness of Equilibrium Coalitions

Lemma 3 proved that, among the inactive groups, the least egalitarian ones (those with larger γ_i) are necessarily willing to step into the active coalition, and that all other groups are too egalitarian to find it profitable to actively participate into the contest because of their inability to effectively spur effort provision by their group members – too much egalitarianism invites free-riding. Lemma 4 proved that, once in, such groups never want to step out. Lemma 6 provided a sufficient condition for the absence of cycles. Building on these observations, it is finally possible the *unconstrained* WGS equilibrium by identifying the endogenous coalition A^* and $\overline{A^*} = M \setminus A^*$, hence their cardinalities, k^* and $m - k^*$, and sizes $\mathcal{N}[A^*]$ and $\mathcal{N}[\overline{A^*}]$. The following simple algorithmic procedure serves the purpose.

ALGORITHM 1 (Equilibrium active coalitions)**.**
Let $\boldsymbol{\gamma}$ *be an arbitrary profile of incentivisation schemes, ordered so that* γ_1 *is the*

[19] Recall via definition (5) that $\gamma_i = 1/n_i$ entails $a_i = 1$ – upon victory, the prize is evenly distributed among all group members, regardless of effort provision. Since γ_i strictly decreases in a_i, $\gamma_i \geq 1/n_i$ entails $a_i \geq 1$.

[20] By the formal argument discussed above, pure-strategy WGS equilibria exist for sure also at some profiles $\boldsymbol{\gamma} \notin \boldsymbol{G^*}$.

largest and γ_T *the smallest – with* $t = 1, \ldots, T \leq m$ *an index variable defined accordingly. Moreover, let the notation*

$$A(\gamma) = \{i \in M : \gamma_i \geq \gamma, \ i \in M\}$$

indicate the active coalition of all groups with $\gamma_i \geq \gamma$*, with* $k(\gamma) = \#[A(\gamma)]$*. Then, for any arbitrary admissible scheme* $\boldsymbol{\gamma} \in \boldsymbol{G^*}$*, the following algorithm can be used to determine the equilibrium coalition* A^* *and its size* k^**:*

Step 1: start with the active coalition $A(\gamma_1)$ *and compute its size* $\mathcal{N}[A(\gamma_1)]$*;*[21]

Step 2: test all groups with $\gamma_i = \gamma_2$ *– those with the second largest stand-alone incentive – for condition (16):*

- *if the condition is not met, stop: the equilibrium coalition* A^* *includes only the groups with* $\gamma_i = \gamma_1$*;*
- *if the condition is met, continue.*

Step 3 consider the enlarged active coalition $A(\gamma_2)$*, compute its size* $\mathcal{N}[A(\gamma_2)]$ *and its average egalitarianism* $\bar{\gamma}[A(\gamma_2)]$*;*

Step 4 test all groups with $\gamma_i = \gamma_3$ *– those with the third largest stand-alone incentive – for condition (16):*

- *if the condition is not met, stop: the equilibrium coalition* A^* *includes only the groups with* $\gamma_i \geq \gamma_2$*;*
- *if the condition is met, continue.*

Iterate the procedure until either: (i) *it stops at some intermediate* $t^* = t_{stop} < T$*;* (ii) *it reaches the terminal condition* $t^* = T$*.*

The stopping point t^* of Algorithm 1 unambiguously identifies the perimeter of the equilibrium active coalition A^*: if the terminal condition $t^* = T$ is reached, the coalition A^* includes all m groups; if stopping occurs at some intermediate value $t_{stop} \in \{1, \ldots, T-1\}$, the coalition A^* only includes the groups with that match or exceed the meritocracy level of the group with the t^*-th largest stand-alone incentive.

It is immediate to check that Algorithm 1 works smoothly if and only if contrarian groups are *not* encountered in the descent from the largest to the

[21]Note that the smallest coalition $A(\gamma_1)$ is necessarily part of an equilibrium, since $\gamma_i = \bar{\gamma}[A(\gamma_1)] = \gamma_1$ holds by construction for all the groups it includes.

smallest γ_i. Otherwise, there is no guarantee *a priori* that the inactive groups included at each iteration do not deviate back to inactivity once included into the active coalition. The absence of contrarian groups, hence of cycles, guaranteed by the sufficient condition $\boldsymbol{\gamma} \in \boldsymbol{G^*}$ is therefore sufficient to guarantee, too, that a well-defined set of intermediate stopping points $t_{stop} \in \{1, \dots T-1\}$ exists whenever the Algorithm 1 does not reach its terminal condition $t^* = T$. Each of such stopping points, when they exist, is unambiguously identified by the condition

$$\gamma_{t_{stop}+1} \; < \; \bar{\gamma}[A(\gamma_{t_{stop}})] - \frac{1}{\mathcal{N}[A(\gamma_{t_{stop}})]} \; . \tag{19}$$

Since Algorithm 1 works 'downwards' from the largest to the smallest γ_t, it necessarily stops at the largest of the intermediate stopping points t_{stop} whenever it does not reach the terminal condition $t^* = T$. The following corollary formally states the result.

COROLLARY 3 (Stopping of algorithm)**.**
Let the arbitrary profile of admissible schemes $\boldsymbol{\gamma} \in \boldsymbol{G^*}$ be ordered as required by Algorithm 1. Moreover, let $\mathcal{T}_{stop}. = \{\, t$: condition (19) holds$\}$ be the (possibly empty) set of all intermediate stopping point t_{stop}. Then,

$$t^* \neq T \quad \Longrightarrow \quad \mathcal{T}_{stop} \neq \emptyset \;\; \textit{and} \;\; t^* \, = \, \sup \, \mathcal{T}_{stop} \, .$$

Proof. The result stems from Lemmas 3 to (6). □

4.3 Wrap-Up

The equilibrium structure identified by Lemmas 2 to 6 and Corollaries 1 to 3 for the generic (constrained) positive-effort WGS equilibrium implies that the necessary and sufficient condition for the existence of zero-effort equilibria identified by Lemma 1 also guarantees the uniqueness of such equilibria. The following lemma highlights the result.

LEMMA 7 (Uniqueness of zero-effort equilibria)**.**
For every profile $\boldsymbol{\gamma}$ of incentivisation schemes such that $\gamma_i < \underline{\underline{\gamma_i}}$ holds for all groups $i = 1, 2, \dots, m$, the effort stage-game played at $t = 2$ has a unique pure-strategy WGS equilibrium with $x^*_{ij} = X^*_i = \sum_{\ell=1}^{m} X^*_\ell = 0$.

Proof. See Appendix A.7. □

Direct consequence is that $\boldsymbol{\gamma} \in \boldsymbol{G^*}$ implies that the aggregate equilibrium effort is strictly positive. Moreover, Lemmas 3 to 6 and Corollary 2 guarantee that: (i) the unilateral incentives to deviate, if any, faced by the individual members of the i-th group are strictly ordered by the level of egalitarianism γ_i of that group; (ii) no cycles arise in the presence of such incentives. Therefore, at every admissible profile of schemes $\boldsymbol{\gamma} \in \boldsymbol{G^*}$ a unique equilibrium exists with positive effort – the monotonicity of unilateral incentives in within-group egalitarianism implies that, for any fixed γ_i, no group can be active at one equilibrium and inactive at another equilibrium. We are finally able to wrap-up the characterisation of the unconstrained WGS equilibria in the following proposition.

PROPOSITION 2 (Unconstrained equilibria of effort-stage-game)**.**
For every arbitrary profile $\gamma \in \boldsymbol{G^*}$ of admissible incentivisation schemes, the effort-stage-game played at $t = 2$ has a unique pure-strategy WGS equilibrium in which:

- *the active coalition is $A^* = A(\gamma^*)$ encompasses all the $k^* \geq 1$ most meritocratic groups with $\gamma_i \geq \gamma^*$;*
- *the meritocracy threshold γ^*, the number of active groups $k^* = \#[A^*]$, and the size of the active coalition $N^* = \mathcal{N}[A^*] = \sum_{\ell \in A^*} n_\ell$ are unambiguously identified by Algorithm 1;*
- *all members j of all k^* active groups exert a strictly positive effort level*

$$x^*_{ij} \;=\; v\left(\bar{\gamma}^* - \frac{1}{N^*}\right)\left[\gamma_i - \left(\bar{\gamma}^* - \frac{1}{N^*}\right)\right] \,,$$

  with $\bar{\gamma}^* = \bar{\gamma}[A^*]$, $X^*_i = n_i x^*_{ij}$ and $\sum_{i=1}^m X^*_i = v(\bar{\gamma}^* - 1/N^*)$ the aggregate effort across groups;
- *all members j of all other $m - k^*$ inactive groups exert zero effort;*
- *the probability of winning of the i-th active group is*

$$p^*_i \;=\; n_i\left[\gamma_i - \left(\bar{\gamma}^* - \frac{1}{N^*}\right)\right] \in (0,1)\,. \tag{20}$$

Proof. The results stem from Lemmas 1 to 7 and Corollaries 1 to 2. □

Note that, as it is the case for the equilibrium level of individual/group effort, the equilibrium probability of winning defined by (20) exhibits an aggregative form – since it can be rewritten as $p_i^* = n_i(\gamma_i - \sum_\ell^m X_\ell^*/v)$, i.e. as a function of the aggregate effort across groups.

5 Incentivisation Stage-Game and Subgame-Perfect Equilibrium

The characterisation outlined in Section 4 highlighted that, if incentivisation is restricted so as to forbid all schemes that tax effort providers and reward contributors, the effort stage-game has a unique pure-strategy WGS equilibrium at every admissible profile of schemes $\boldsymbol{\gamma} \in \boldsymbol{G^*}$. In this section we complete the characterisation of the two-stage game by identifying the profile of optimal (equilibrium) incentivisation schemes $\boldsymbol{\gamma}^* \in \boldsymbol{G^*}$ simultaneously selected at $t = 1$ by the utilitarian leaders of all groups.[22] Moreover, we prove that the two-stage game has a unique subgame-perfect WGS equilibrium in pure strategies.

The logical structure of the formal arguments is identical to that of Section 4: first, we characterise constrained equilibria with exogenous, pre-determined active (A_k) and inactive ($\overline{A_k}$) coalitions. Since the sufficient condition identified by Lemma 6 guarantees that the equilibrium play at the effort stage-game is uniquely pinned down by Proposition 2 at every profile $\boldsymbol{\gamma} \in \boldsymbol{G^*}$, all group leaders anticipate that their deliberations about within-group incentivisation jointly determine the equilibrium path by selecting *de facto* a future equilibrium play. In this light, we then analyse under which conditions (if any) the pre-assigned coalitions are in equilibrium – by checking for the presence of profitable unilateral deviations by single group leaders. Finally, we determine the unconstrained equilibrium play of the incentivisation stage-game and prove uniqueness.

The mutually-consistent and mutually-sustaining profiles of optimal incentivisation schemes and effort-provision choices, respectively, constitute the unique pure-strategy WGS-SPE of the two-stage game.

5.1 Constrained Equilibrium

Recall from Section 4 that, at the effort stage-game, Algorithm 1 defines the equilibrium number of active groups k^* only implicitly. Therefore, to identify an uncontrained equilibrium play at the incentivisation stage-game we

[22]Leaders maximise the aggregate utility of the groups they oversee. Since all group members are identical by assumption and, in equilibrium, they exert the same effort level via the WGS assumption, the results would not change if group-specific incentive schemes were determined by some aggregator of consensus among group members – e.g. by voting.

must again proceed via guess-and-solve. Consider again an arbitrary partition $\langle A_k, \overline{A_k}\rangle := P_k \in \mathcal{P}(M)$, that assigns to every group $i \in M$ an active or inactive status. The partition is announced to all group leaders at the beginning of date $t = 1$: all group leaders assigned to the inactive coalition $\overline{A_k}$ are forced to implement an incentivisation scheme consistent with inaction at the effort stage-game, whereas all group leaders assigned to the active coalition A_k anticipate that, at the effort stage-game, all active group members will condition their effort-provision choices to the entire profile $\boldsymbol{\gamma}$ of incentivisation schemes.

For every admissible profile $\boldsymbol{\gamma} \in \boldsymbol{G^*}$, the continuation payoff of the generic active group leader is unique and, for every $k \geq 1$, it can be written as

$$\Pi_i^* \;=\; \sum_{j=1}^{n_i} \pi_{ij}^* \;=\; n_i v \left[\gamma_i - \left(\bar{\gamma}[A_k] - \frac{1}{\mathcal{N}[A_k]}\right)\right] \left[1 - \left(\bar{\gamma}[A_k] - \frac{1}{\mathcal{N}[A_k]}\right)\right] \quad (21)$$

via Proposition 2. Note that, since $i \in A_k$ holds by assumption, the average egalitarianism coalition-wide, $\bar{\gamma}[A_k]$, contains γ_i. Consider first the case $k = 1$. Then, expression (21) reduces to[23]

$$\Pi_i^* \;=\; v\left(1 - \gamma_i + \frac{1}{n_i}\right),$$

that strictly decreases in γ_i. In this case, $\gamma_i^* = 1/n_i$ holds – in words: the leader of the unique active group selects the most egalitarian admissible scheme. Consider now the less trivial case $k > 1$. Differentiating the k expected utilities (21) with respect to γ_i, imposing the FOCs and rearranging, we obtain

$$\begin{aligned}
\gamma_1^c \;&=\; \left[\frac{\mathcal{N}[A_k]\Big(\mathcal{N}[A_k] - n_1\Big) + \mathcal{N}[A_k] - 2n_1}{n_1\mathcal{N}[A_k]}\right] - \left(\frac{\mathcal{N}[A_k] - 2n_1}{n_1}\right)\bar{\gamma}[A_k]\,, \\
&\;\;\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (22) \\
\gamma_k^c \;&=\; \left[\frac{\mathcal{N}[A_k]\Big(\mathcal{N}[A_k] - n_k\Big) + \mathcal{N}[A_k] - 2n_k}{n_k\mathcal{N}[A_k]}\right] - \left(\frac{\mathcal{N}[A_k] - 2n_k}{n_k}\right)\bar{\gamma}[A_k]\,,
\end{aligned}$$

where the subscripts 'c' again highlight that we are considering a constrained equilibrium. Expression (22) displays the same block-recursive structure highlighted in Section 4: the optimal average scheme coalition-wide, $\bar{\gamma}^*[A_k]$, can be identified first, since all optimal group-specific incentives γ_i^c solely respond to it. Once characterised, $\bar{\gamma}^c[A_k]$ determines 'backwards' all the optimal schemes $\boldsymbol{\gamma}^c$.

[23]Since, for $k = 1$, it holds that $N_k = n_i$ and $\bar{\gamma}[A_k] = \gamma_i$.

Once again, the convoluted tangle of strategic feedbacks collapses into a simple mean-field term, thus rendering the incentivisation stage-game aggregative. The following lemma summarises the result.

LEMMA 8 (Aggregate laws of optimal incentivisation)**.**
Let $P_k = \langle A_k, \overline{A_k} \rangle$ *be an arbitrary binary partition of the group set* M*. Moreover, let the behavioural constraints of Lemma 2 hold. Then, under the assumption of within-group symmetry (WGS), the equations*

$$\gamma_i^c(\bar{\gamma}[A_k]) \;=\; \begin{cases} 1/n_i & \text{if } k = 1\,, \\ \left[\dfrac{\mathcal{N}[A_k]\Big(\mathcal{N}[A_k] - n_i\Big) + \mathcal{N}[A_k] - 2n_i}{n_i \mathcal{N}[A_k]}\right] + \\ \qquad - \left(\dfrac{\mathcal{N}[A_k] - 2n_i}{n_i}\right) \bar{\gamma}^c[A_k] & \text{if } k > 1\,, \end{cases} \tag{23}$$

link the group-specific optimal schemes γ_i^* *to the average (optimal) incentive across groups* $\bar{\gamma}^c[A_k]$ *consistent with equilibrium, that comes in the form*

$$\bar{\gamma}^c[A_k] \;=\; 1 + \frac{k-2}{\mathcal{N}[A_k](k-1)} \;\geq\; 1 \tag{24}$$

for every $k > 1$*.*

Proof. See Appendix A.8. □

Note that, as Lemma 8 stresses, the second expression in equation (23) cannot be interpreted as a best-response function *tout court*, since the average incentive $\bar{\gamma}^c[A_k]$ includes the group-specific incentive γ_i. Rather, it can be seen as a micro-macro law that characterises how, in equilibrium, the average incentive across groups (the statistic $\bar{\gamma}^c[A_k]$) relates to its underlying constituent parts (the group-specific incentives γ_i).

Three observations ensue. First, the driving forces that shape optimal incentive design significantly different under 'monopolisation' and 'oligopolisation' as intended by Ueda (2002).[24] If a single group is expected to be active ($k = 1$), it is optimal for its leader to implement the most egalitarian scheme available. If multiple groups are expected to be active ($k > 1$), less extreme incentive schemes are preferred. Second, if multiple groups are active, strategic complementarity or substitutability governs the equilibrium level of egalitarianism

[24]In the parlance of Ueda (2002), monopolisation occurs when effort is exerted by one single group. Conversely, oligopolisation entails effort provision by multiple groups.

across groups depending on the composition of the active coalition in term of group sizes n_i. Explicitly solving in γ_i the aggregate law in (23) we obtain the explicit best-response function

$$\begin{aligned}\gamma_i^c(\bar{\gamma}^{\not\ni i}[A_{k-1}]) \;=\; & \left[\frac{\mathcal{N}[A_{k-1}]\Big(\mathcal{N}[A_{k-1}]-n_i\Big)+\mathcal{N}[A_{k-1}]-2n_i}{2\left(\mathcal{N}[A_{k-1}]-n_i\right)}\right]+ \\ & \qquad\qquad - \left(\frac{\mathcal{N}[A_{k-1}]-2n_i}{2n_i}\right)\bar{\gamma}^{\not\ni i}[A_{k-1}]\end{aligned} \tag{25}$$

that identifies the optimal reaction of the leader of the i-th active group to the average incentivisation $\bar{\gamma}^{\not\ni i}[A_k]$ enforced by all other $-i$ active groups. Direct inspection of (25) immediately reveals that the best-response $\bar{\gamma}_i^c(\bar{\gamma}^{\not\ni i}[A_{k-1}])$ *increases* in the average egalitarianism coalition-wide $\bar{\gamma}^{\not\ni i}[A_{k-1}]$ if $n_i > \mathcal{N}[A_{k-1}]/2$, and *decreases* otherwise. In the former case, there is strategic complementarity, at the optimum, in the within-group level of egalitarianism: it is in the best interest of the group to match the incentivisation implemented by the competitors – be egalitarian when competitors are egalitarian, be meritocratic when they are meritocratic. In the latter case, there is strategic substitutability: it is in the best interest of the group to undercut the incentivisation implemented by competitors – be meritocratic when they are egalitarian, be egalitarian when they are meritocratic. Third, the sheer number of active groups k affects *per se* the average level of egalitarianism within the active coalition. Hence, the internal composition of A_k matters, but does not shape the equilibrium play exclusively.

Building on Lemma 8, we can now characterise the constrained equilibria of the incentivisation stage-game.

LEMMA 9 (Constrained equilibria of the incentivisation stage-game)**.**
For every arbitrary binary partition $P_k = \langle A_k, \overline{A_k}\rangle$ of the group set M, the constrained equilibrium profile $\boldsymbol{\gamma}^{\mathbf{c}}$ of incentivisation schemes entails

$$\gamma_i^c \;=\; \gamma_i^c(\bar{\gamma}^c[A_k]) \;=\; \begin{cases} 1/n_i & \text{if } k=1\,, \\ 1+\dfrac{\mathcal{N}[A_k]-2n_i}{n_i\mathcal{N}[A_k](k-1)} > \dfrac{1}{n_i} & \text{if } k>1\,, \end{cases} \tag{26}$$

for all active groups $i \in A_k$.

Proof. See Appendix A.9. □

Note that, for every $k > 1$, the optimal incentivisation scheme (26) induces a group-specific probability of winning

$$p_i^c = \frac{\mathcal{N}[A_k] - n_i}{\mathcal{N}[A_k](k-1)}, \tag{27}$$

that strictly decreases in group size n_i – relatively small(er) groups enjoy higher probabilities of winning in equilibrium. The result is consistent with both the extant literature on group-contest games and the well-known Olsonian group-size paradox – see e.g. Olson (1965) and Pecorino (2023).

5.2 Unconstrained Equilibrium

We now analyse the stability of coalitions, by identifying under which conditions (if any) unilateral profitable deviations are available to single group leaders. The following lemma summarises the analysis.

LEMMA 10 (Unilateral deviations)**.**
For every arbitrary binary partition $P_k = \langle A_k, \overline{A_k} \rangle$ of the group set M, the expected utility of the generic active group $i \in A_k$ is

$$\Pi_i^c = \frac{v\, p_i^c}{\mathcal{N}[A_k](k-1)} > 0 \tag{28}$$

for every $k > 1$.

Proof. See Appendix A.10. □

Lemma 10 implies that deviating to an incentivisation scheme that triggers positive effort provision at the effort stage-game is a (strictly) dominant strategy for every leader of every inactive group. Note indeed that the aggregate expected utility (28) of the generic active group is strictly positive for every $k \geq 1$, while the aggregate expected utility of every inactive group is zero by definition. As a consequence, the incentivisation stage-game has a unique *unconstrained* equilibrium in which all group leaders select (optimal) incentivisation schemes that induce positive effort-provision at the subsequent effort stage-game. Formally, this amounts to saying that $k^* = m$ must necessarily hold that, via Lemma 9, unambiguously determines a unique equilibrium profile of (optimal) admissible incentivisation schemes $\boldsymbol{\gamma^*} \in \boldsymbol{G^*}$. The following proposition formally outlines the characterisation.

PROPOSITION 3 (Unconstrained equilibrium incentivisation)**.**
Let $\boldsymbol{\gamma} \in \boldsymbol{G^*}$ *hold, with* $\boldsymbol{G^*}$ *the set of the admissible incentivisation schemes defined by (18). Then, the incentivisation stage-game has a unique pure-strategy equilibrium in which the incentivisation schemes*

$$\gamma_i^* \;=\; 1 + \frac{N - 2n_i}{n_i N(m-1)}\,, \qquad \textit{for every group } 1, 2, \ldots, m\,,$$

optimally selected by the group leaders, trigger effort provision by all groups at the subsequent effort stage-game, and entail the average egalitarianism across groups

$$\bar{\gamma}^* \;=\; 1 + \frac{m-2}{N(m-1)}\,. \tag{29}$$

Proof. The results stem directly from Lemmas 8 to 10. □

We can finally wrap-up the results Propositions 2 and 3 to characterise the unique subgame-perfect pure-strategy equilibrium of the two-stage game under analysis. The proposition that follows summarises the characterisation.

PROPOSITION 4 (Subgame-perfect WGS equilibrium)**.**
Let $\boldsymbol{\gamma} \in \boldsymbol{G^*}$ *hold. Then, the two-stage group-contest game under analysis has a unique subgame-perfect WGS equilibrium in pure strategies, in which:*

- *at the effort stage-game played at* $t = 2$*, all members of all groups exert effort in accordance with Proposition 2 at every arbitrary profile* $\boldsymbol{\gamma}$ *of admissible incentivisation schemes;*

- *at the incentivisation stage-game played at* $t = 1$*, all leaders optimally select incentivisation schemes for the groups they oversee in accordance with Proposition 3.*

The equilibrium outcome entails that:

- *all groups are active, whereby* $k^* = m$ *and* $N^* = N$*;*

- *the probability of winning faced by the generic member of the group* i *is*

$$p_i^* \;=\; \frac{N - n_i}{N(m-1)} \in (0, 1)\,; \tag{30}$$

- *the aggregate effort across groups is*

$$\sum_{i=1}^{m} X_i^* \;=\; v\left(1 - \frac{1}{N(m-1)}\right) \;>\; 0\,; \tag{31}$$

– *the effort individually provided by the generic group member ij is*

$$x_{ij}^* = p_i^* \sum_{i=1}^{m} X_i^* \,. \tag{32}$$

Proof. The result stems directly from Propositions 2 and 3. □

In equilibrium, all groups that account for less than a half of the entire population N adopt highly meritocratic *unrestricted* incentive schemes with $\gamma_i^* > 1$ – in words: they tax free-riders and reward contributors with the proceeds. The opposite holds for the largest group, if any, that, in equilibrium, adopts a purely redistributive scheme with $\gamma_i^* \in (1/n_i, 1)$. The leader of the largest group, if any, leverages on group size and opts for a moderate incentivisation. Conversely, the leaders of all other groups try to compensate their smaller group sizes by adopting highly meritocratic schemes. In any case, the average level egalitarianism across groups is low in equilibrium, since $\bar{\gamma}^* > 1$ holds for sure via expression (29).

6 Conclusions

We studied effort provision and incentivisation in a group-contest game where an arbitrarily large number $m \geq 2$ of groups, heterogeneous in size, compete for the appropriation of a common-value prize. The cost of effort is linear and information is complete, and effort provision is perfectly coordinated in every group via a within-group symmetry (WGS) assumption. Utilitarian group leaders strategically select *unrestricted* group-specific incentivisation schemes that, in case of victory, may simply redistribute the prize to group members or tax the private resources of free-riders to provide additional rewards to contributors.

Key contribution of the paper is to provide a novel algorithmic procedure that *explicitly* characterises the equilibrium play.

Four additional results ensue. First, we prove that, in a WGS environment, strategic interactions endogenously come in mean-field form: effort provision within groups responds solely to the aggregate effort and average egalitarianism across groups. WGS group-contest games, therefore, are naturally aggregative. Second, we prove that pure-strategy WGS equilibria do not exist at some profiles of incentivisation schemes that tax contributors and reward free-riders In these cases, the members of some groups endlessly cycle between zero and positive effort provision. Third, we highlight that heterogeneity in group sizes is crucial to understand whether complementarity or substitutability arises in the equilibrium level of egalitarianism within groups: very large groups respond to a foreseen increase in the average egalitarianism of all other groups by increasing their egalitarianism (strategic complementarity), while smaller groups do the opposite (strategic substitutability). Fourth, we show that, when group-specific incentive schemes are optimally and strategically designed, all groups exert positive effort in equilibrium.

Our analysis can be extended in several directions. First, heterogeneous marginal costs across groups can be included into the analysis, to allow for an additional layer of heterogeneity across groups. Our resolution procedure readily extends, under the WGS assumption, to any group-contest where marginal effort costs are constant but group-specific. Second, unilateral deviations by single individual members can be studied under generalised cost structures. In this respect, our analysis with simple, linear costs highlights an interesting issue: the individual incentives that govern unilateral deviations from within the equilibrium coalition of active groups differ systematically from those that govern deviations from outside. In particular: for some configurations of coalitions and incentives, subsets of values for the stand-alone incentives *may* exist for some groups such that their individual members exhibit cyclical incentives about effort provision – they prefer not to exert any effort when their group is in the active coalition, and prefer to provide some effort when their group is inactive. The possibility of cycles in effort provision highlights that the incentives that govern deviations at the individual level significantly differ from those that operate at the group level – even in a WGS environment where effort provision within groups is perfectly coordinated by assumption.

References

Baik, K. H. and Lee, S. (1997). Collective rent seeking with endogenous group sizes. *European Journal of Political Economy*, 13(1):121–130.

Barbieri, S. and Topolyan, I. (2023). Diamond in the rough: A tractable team all-pay auction vis-a-vis group-conscious selection. *Mimeo*.

Bulow, J. I., Geanakoplos, J. D., and Klemperer, P. D. (1985). Multimarket oligopoly: Strategic substitutes and complements. *Journal of Political economy*, 93(3): pp. 488–511.

Carlsson, H. and Van Damme, E. (1993). Global games and equilibrium selection. *Econometrica: Journal of the Econometric Society*, pages 989–1018.

Cavalli, F., Gilli, M., and Naimzada, A. K. (2018). A new approach to contest models. *DEMS working paper no. 364*. Available at: `https://papers.ssrn.com/sol3/papers.cfm?abstract_id=2926936`.

Corchón, L. C. (2021). Aggregative games. *SERIEs*, 12(1):49–71.

Hartley, R. (2017). *Efficiency in contests between groups. In (ed. by) Buchholz, W. and Rübbelke, D., The theory of externalities and public goods. Springer.* Springer.

Konrad, K. A. and Kovenock, D. (2009). The alliance formation puzzle and capacity constraints. *Economics Letters*, 103(2):84–86.

Lee, S. (1995). Endogenous sharing rules in collective-group rent-seeking. *Public Choice*, 85(1):31–44.

Morris, S. (2008). Global games. In (ed. by) Durlauf, S. and Blume, L. E., (2020), *The new Palgrave dictionary of economics*. springer.

Morris, S. and Shin, H. S. (1998). Unique equilibrium in a model of self-fulfilling currency attacks. *American Economic Review*, 88(3):587–597.

Morris, S. and Shin, H. S. (2003). Global games: Theory and applications. In (ed. by) Dewatripont, M., Hansen, L. P. and Turnovsky, S. J., *Advances in economics and econometrics: Theory and applications*. Eighth world congress, (vol. I). Cambridge, UK: Cambridge University Press.

Nitzan, S. (1991a). Collective rent dissipation. *The Economic Journal*, 101(409):1522–1534.

Nitzan, S. (1991b). Rent-seeking with non-identical sharing rules. *Public Choice*, 71(1):43–50.

Olson, M. (1965). *The logic of collective action.* Cambride, MA: Harvard University Press.

Pecorino, P. (2023). Group size paradox. Mimeo. Available at: `https://papers.ssrn.com/sol3/papers.cfm?abstract_id=4318739`.

Sánchez-Pagés, S. (2007). Endogenous coalition formation in contests. *Review of Economic Design*, 11:139–163.

Selten, R. (1970). *Preispolitik der Mehrproduktenunternehmung in der statischen Theorie*, volume 16. Springer-Verlag.

Ueda, K. (2002). Oligopolization in collective rent-seeking. *Social Choice and Welfare*, 19:613–626.

A PROOFS

A.1 PROOF OF LEMMA 1

Let $\mathbf{x_i} = \mathbf{0} \in \mathbb{R}^{n_i}$ hold for all groups $i = 1, 2, \ldots, m$, so that $\mathbf{x} = \mathbf{0} \in \mathbb{R}^N$ and $\mathbf{X} = \mathbf{0} \in \mathbb{R}^m$ hold, too. Moreover, let $\mathbf{x_{-ij}}$ indicate the generic profile of individual effort levels excluding that of ij. Then

$$\pi_{ij}(x_{ij} = 0, \mathbf{x_{-ij}} = \mathbf{0}) = v/mn_i$$

holds via (2b). If the individual ij unilaterally deviates to some $x'_{ij} > 0$, then $X'_i = x'_{ij} > 0$ and $p_i = 1$ must hold via (1) and (2a) respectively, whereby

$$\pi_{ij}(x'_{ij} > 0, \mathbf{x_{-ij}} = \mathbf{0}) \;=\; v\left[(1 - a_i) + \frac{a_i}{n_i}\right] - x_{ij} \;\equiv\; v\gamma_i - x_{ij}\,.$$

The deviation is not profitable if $\pi_{ij}(x'_{ij} > 0, \mathbf{x_{-ij}} = \mathbf{0}) < \pi_{ij}(x_{ij} = 0, \mathbf{x_{-ij}} = \mathbf{0})$, i.e. if $v\gamma_i < v/mn_i + x_{ij}$. Taking the limit as $x_{ij} \longrightarrow 0$ we obtain the lower bound

$$\gamma_i \;\leq\; \frac{1}{mn_i} \;:=\; \underline{\underline{\gamma_i}}\,,$$

that is the critical value of condition (8) in the main text. The condition $\gamma_i \leq \underline{\underline{\gamma_i}}$ for all groups $i = 1, 2, \ldots, m$ is necessary and sufficient for the existence of a zero-effort equilibrium: if $\gamma_i > \underline{\underline{\gamma_i}}$ for some groups, all members of those groups have by definition a unilateral incentive to deviate to some positive effort level and, symmetrically, that incentive is absent if $\gamma_i \leq \underline{\underline{\gamma_i}}$ holds for all groups. *QED*

A.2 PROOF OF LEMMA 2

We proceed via guess-and-solve. Suppose that $X_i > 0$ holds for the generic active group $i \in A_k$, so that $\sum_{\ell=1}^m X_\ell > 0$ must hold, too. Then, via expressions (2a) and (3a) in the main text, the expected payoff of the j-th member of group i is

$$\pi_{ij}(x_{ij}, \mathbf{x_{-ij}}) \;=\; v\left[\left(\frac{1 - a_i}{\sum_{\ell=1}^m X_\ell}\right)x_{ij} + \left(\frac{a_i X_i}{n_i \sum_{\ell=1}^m X_\ell}\right)\right] - x_{ij}\,. \tag{A.1}$$

It is immediate to check that (B.3) has a global maximum in x_{ij}, unambiguously identified by the FOC

$$\frac{\partial}{\partial x_{ij}}\pi_{ij}(x_{ij}, \mathbf{x_{-ij}}) = v\left[(1 - a_i)\left(\frac{\sum_{\ell=1}^m X_\ell - x_{ij}}{(\sum_{\ell=1}^m X_\ell)^2}\right) + \frac{a_i}{n_i}\left(\frac{\sum_{\ell=1}^m X_\ell - X_i}{(\sum_{\ell=1}^m X_\ell)^2}\right)\right] = 1. \tag{A.2}$$

Suppose further that (B.4) identifies a positive effort level. Then, straightforward algebraic manipulations yield

$$v\left[(1 - a_i)\left(\sum_{\ell=1}^m X_\ell^c - x_{ij}^c\right) + \frac{a_i}{n_i}\left(\sum_{\ell=1}^m X_\ell^c - X_i^c\right)\right] \;=\; \left(\sum_{\ell=1}^m X_\ell^c\right)^2, \tag{A.3}$$

where the superscripts 'c' stress that we are considering a constrained equilibrium where some groups do not act optimally. Now impose within-group symmetry via $x_{ij}^c = x_i^c$ and $X_i^c = n_i x_i^c$ for all groups $i = 1, 2, \ldots, m$. Substituting for $x_i^c = X_i^c / n_i$ and solving in x_{ij}^c the generic FOC (A.3) yields

$$x_i^c \;=\; \left(\sum_{\ell=1}^{m} X_\ell^c \right) \left[\gamma_i - \frac{1}{v} \left(\sum_{\ell=1}^{m} X_\ell^c \right) \right], \tag{A.4}$$

that is expression (13) in the main text. Multiplying both sides of (A.4) by n_i we can rewrite

$$X_i^c \;=\; n_i \gamma_i \left(\sum_{\ell=1}^{m} X_\ell^c \right) - \frac{n_i}{v} \left(\sum_{\ell=1}^{m} X_\ell^c \right)^2, \tag{A.5}$$

whereby

$$\sum_{\ell=1}^{m} X_\ell^c \;=\; \sum_{\ell=1}^{m} X_\ell^c \left(\sum_{\ell=1}^{m} n_\ell \gamma_\ell \right) - \left(\sum_{\ell=1}^{m} X_\ell^c \right)^2 \left(\sum_{\ell=1}^{m} \frac{n_\ell}{v} \right). \tag{A.6}$$

Call $A_k^+ \subseteq A_k$ the set of all active groups $i \in A_k$ such that, given (A.6), expression (A.4) indeed identifies a positive effort level. Then, $X_i^c > 0$ must hold for every $i \in A_k^+$. Since $\sum_{\ell=1}^{m} X_\ell^c = \sum_{\ell \in A_k} X_\ell^c = \sum_{\ell \in A_k^+} X_\ell^c$ by definition, we can rewrite expression (A.6) as

$$\sum_{\ell=1}^{m} X_\ell^c \;=\; \left(\sum_{\ell=1}^{m} X_\ell^c \right) \left(\sum_{\ell \in A_k^+} n_\ell \gamma_\ell \right) - \frac{1}{v} \left(\sum_{\ell=1}^{m} X_\ell^c \right)^2 \left(\sum_{\ell \in A_k^+} n_\ell \right).$$

Indicating with

$$\mathcal{N}[A_k^+] \;:=\; \sum_{\ell \in A_k^+} n_\ell \,,$$

$$\bar{\gamma}[A_k^+] \;:=\; \frac{1}{\mathcal{N}[A_k^+]} \sum_{\ell \in A_k^+} n_\ell \gamma_\ell \,,$$

the number of individuals in the subset A_k^+ and the average stand-alone incentive of the groups in A_k^+, respectively, we finally obtain

$$\sum_{\ell=1}^{m} X_\ell^c \;=\; v \left(\bar{\gamma}[A_k^+] - \frac{1}{\mathcal{N}[A_k^+]} \right), \tag{A.7}$$

that is expression (14) in the main text. The aggregate effort across groups (A.7) identifies the group-specific effort levels X_i^c via (A.5), and the X_i^c in turn identify the individual effort levels via (A.4) – or, equivalently, via the fact that $x_i^c = X_i^c / n_i$ holds for all groups $i = 1, 2, \ldots, m$ via the WGS assumption. Note finally that: (i) the condition

$$\sum_{\ell \in A_k^+} n_\ell \gamma_\ell \;>\; 1 \,,$$

that is condition (12a) in the main text, is sufficient for the aggregate effort (A.7) to indeed be positive; (ii) the condition

$$\gamma_i \ \geq \ \bar{\gamma}[A_k^+] - 1/\mathcal{N}[A_k^+] \quad \text{for all groups } i \in A_k^+ \,,$$

that is condition (12b) in the main text, is sufficient for the individual and group effort (A.4) and (A.5) to indeed be positive. If $X_i^c > 0$ for all groups $i \in A_k^+$ and $\sum_{\ell=1}^m X_\ell > 0$, the initial guess and the results we derive from it are mutually consistent. Therefore, conditions (12a)-(12b) are indeed sufficient for expressions (A.4), (A.5) and (A.7) to identify a constrained WGS equilibrium with positive effort. *QED*

A.3 PROOF OF LEMMA 3

We begin with unilateral deviations from within the arbitrary active coalition A_k. Indicate with $\mathbf{x}_{-ij}^c$ the profile of all individual effort levels at the constrained equilibrium play of Lemma 2 except that of the individual ij. If that individual belongs to an active group $i \in A_k$, abiding by the prescribed equilibrium play he/she gets

$$\pi_{ij}(x_{ij}^c > 0, \mathbf{x}_{-ij}^c) \ = \ \frac{v}{n_i} \left(\frac{X_i^c}{\sum_{\ell=1}^m X_\ell^c} \right) - x_i^c \,. \tag{A.8}$$

Substituting for x_i^c and $\sum_{\ell=1}^m X_\ell^c$ as defined by (13) and (14) respectively, the expected utility (A.8) becomes

$$\pi_{ij}(x_{ij}^c > 0, \mathbf{x}_{-ij}^c) \ = \ v \left[\gamma_i - \left(\bar{\gamma}[A_k] - \frac{1}{\mathcal{N}[A_k]} \right) \right] \left[1 - \left(\bar{\gamma}[A_k] - \frac{1}{\mathcal{N}[A_k]} \right) \right] \,. \tag{A.9}$$

Unilaterally deviating to $x'_{ij} = 0$, the same individual gets

$$\pi_{ij}(x'_{ij} = 0, \mathbf{x}_{-ij}^c) \ = \ v \left(\frac{a_i}{n_i} \right) \left(\frac{X_i^c - x_i^c}{(\sum_{\ell=1}^m X_\ell^c) - x_i^c} \right)$$

that, substituting again for x_i^c and $\sum_{\ell=1}^m X_\ell^c$ and recalling that $X_i^c = n_i x_i^c$ for all groups $i = 1, 2, \ldots, m$ via the WGS assumption, reduces to

$$\pi_{ij}(x'_{ij} = 0, \mathbf{x}_{-ij}^c) \ = \ v\, a_i \left(\frac{n_i - 1}{n_i} \right) \left\{ \frac{\gamma_i - \left(\bar{\gamma}[A_k] - \frac{1}{\mathcal{N}[A_k]} \right)}{1 - \left[\gamma_i - \left(\bar{\gamma}[A_k] - \frac{1}{\mathcal{N}[A_k]} \right) \right]} \right\} \,. \tag{A.10}$$

Comparing (A.9) with (A.10) and recalling that $\gamma_i = 1 - a_i + a_i/n_i$, it is easy to check that the unilateral deviation to $x'_{ij} = 0$ is (strictly) profitable if

$$\left(\bar{\gamma}[A_k] - \frac{1}{\mathcal{N}[A_k]} \right) \left[\gamma_i - \left(\bar{\gamma}[A_k] - \frac{1}{\mathcal{N}[A_k]} \right) \right] \ < \ 0 \,.$$

Note that $\bar{\gamma}[A_k] - 1/\mathcal{N}[A_k] = \sum_{\ell=1}^m X_\ell^c / v > 0$ by definition, therefore profitable deviations exists in the active group $i \in A_k^+$ if

$$\gamma_i \ < \ \bar{\gamma}[A_k^+] - \frac{1}{\mathcal{N}[A_k^+]} \, ,$$

that is condition (15) in the main text. Now turn to unilateral profitable deviations from within the inactive coalition $\overline{A_k}$. Abiding by the prescribed course of action $x_{ij}^c = 0$, the j-th member of generic inactive group $i \in \overline{A_k}$ gets zero expected utility. Unilaterally deviating to some $x'_{ij} > 0$ he/she gets

$$\pi_{ij}(x'_{ij} > 0, \mathbf{x}^c_{-ij}) \ = \ v \left(\frac{x'_{ij}}{\left(\sum_{\ell \in A_k} X^c_\ell\right) + x'_{ij}} \right) \left[(1-a_i) \left(\frac{x'_{ij}}{x'_{ij}} \right) + \frac{a_i}{n_i} \right] - x'_{ij} \, ,$$

that, simplifying-off and rearranging, yields

$$\pi_{ij}(x'_{ij} > 0, \mathbf{x}^c_{-ij}) \ = \ v \, \gamma_i \left(\frac{x'_{ij}}{x'_{ij} + \sum_{\ell \in A_k} X^c_\ell} \right) . \tag{A.11}$$

The deviation is (strictly) profitable if (A.11) is (strictly) positive. Substituting in (A.11) for the aggregate effort $\sum_{\ell=1}^m X^c_\ell = \sum_{\ell \in A_k} X^c_\ell$ it is immediate to check that $\pi_{ij}(x'_{ij} > 0, \mathbf{x}^c_{-ij}) > 0$ if

$$x'_{ij} \left\{ v \left[\gamma_i - \left(\bar{\gamma}[A_k] - \frac{1}{\mathcal{N}[A_k]} \right) \right] - x'_{ij} \right\} \ > \ 0 \, .$$

The corresponding equation has two real roots: $x_I = 0$ and $x_{II} = v(\gamma_i - \bar{\gamma}[A_k] + 1/\mathcal{N}[A_k])$. Then: ($i$) if $x_{II} > 0$, every deviation $x'_{ij} \in (0, x_{II})$ is strictly profitable; (ii) if $x_{II} < 0$, no profitable deviations exist. Therefore, if

$$x_{II} \ = \ \gamma_i - \bar{\gamma}[A_k] + \frac{1}{\mathcal{N}[A_k]} \ > \ 0 \tag{A.12}$$

holds for some groups $i \in \overline{A_k}$, the inactive coalition $\overline{A_k}$ is *not in equilibrium*. Rearranging (A.12) we finally obtain that if

$$\gamma_i \ > \ \bar{\gamma}[A_k] - \frac{1}{\mathcal{N}[A_k]} \, ,$$

that is condition (16) in the main text, holds for the generic inactive group $i \in \overline{A_k}$, then its generic member j prefers to deviate, *ceteris paribus*, to some $x'_{ij} \in (0, x_{II})$. *QED*

A.4 PROOF OF LEMMA 4

The proof is divided in two parts: in the first part, we prove that condition (16) in the main text implies $\gamma_{in \to out}(A_{k+1}^{\ni i}) > \gamma_{out \to in}(A_k^{\not\ni i})$, hence the possibility of cycles; in the second part, we prove that condition (16) also implies that $\gamma_i > \gamma_{in \to out}(A_{k+1}^{\ni i})$, hence the absence of cycles.

Part 1: $\gamma_{in \to out}(A_{k+1}^{\ni i}) > \gamma_{out \to in}(A_k^{\not\ni i})$

Consider the generic inactive group $i \in \overline{A_k}$ whose generic member has a unilateral incentive to deviate to positive effort provision. Substituting for the definitions (17a) and (17b) and expanding, $\gamma_{in\to out}(A_{k+1}^{\ni i}) > \gamma_{out\to in}(A_k^{\not\ni i})$ entails

$$\frac{n_1}{\mathcal{N}[A_k^{\not\ni i}]+n_i}\gamma_1 + \cdots + \frac{n_k}{\mathcal{N}[A_k^{\not\ni i}]+n_i}\gamma_k + \frac{n_i}{\mathcal{N}[A_k^{\not\ni i}]+n_i}\gamma_i - \frac{1}{\mathcal{N}[A_k^{\not\ni i}]+n_i} >$$
$$> \gamma_1\frac{n_1}{\mathcal{N}[A_k^{\not\ni i}]} + \cdots + \frac{n_k}{\mathcal{N}[A_k^{\not\ni i}]}\gamma_k - \frac{1}{\mathcal{N}[A_k^{\not\ni i}]}$$

that, factorising properly and rearranging, further reduces to

$$\frac{n_i}{\mathcal{N}[A_k^{\not\ni i}]+n_i}\gamma_i > \frac{n_i}{\mathcal{N}[A_K^{\not\ni i}]\left(\mathcal{N}[A_k^{\not\ni i}]+n_i\right)}\Big[n_1\gamma_1+\cdots+n_k\gamma_k\Big] - \frac{n_i}{\mathcal{N}[A_K^{\not\ni i}]\left(\mathcal{N}[A_k^{\not\ni i}]+n_i\right)}$$

that, simplifying-off, finally yields

$$\gamma_i > \frac{1}{\mathcal{N}[A_K^{\not\ni i}]}\sum_{\ell\in A_k} n_\ell\gamma_\ell - \frac{1}{\mathcal{N}[A_k^{\not\ni i}]} = \bar{\gamma}[A_k^{\not\ni i}] - \frac{1}{\mathcal{N}[A_k^{\not\ni i}]}\,,$$

that is condition (16) in the main text, met by definition if the generic member of the inactive group i has a unilateral incentive to deviate to positive effort.

Part 2: condition (16) implies $\gamma_i > \gamma_{in\to out}(A_{k+1}^{\ni i})$

Via the definition (17b) of the critical value $\gamma_{in\to out}(A_{k+1}^{\ni i})$ we can write the condition $\gamma_i > \gamma_{in\to out}(A_{k+1}^{\ni i})$ as

$$\gamma_i > \frac{1}{\mathcal{N}[A_k^{\not\ni i}]+n_i}\Big[\gamma_1 n_i + \cdots + \gamma_i n_i + \cdots + \gamma_{k+1}n_{k+1} - 1\Big]$$

that, rearranging and simplifying off, becomes

$$\gamma_i\mathcal{N}[A_k^{\not\ni i}] > \sum_{\ell\in A_k^{\not\ni i}}\gamma_\ell n_\ell - 1\,. \tag{A.13}$$

Dividing both sides of (A.13) by $\mathcal{N}[A_k^{\not\ni i}]$ we finally obtain

$$\gamma_i > \bar{\gamma}[A_k^{\not\ni i}] - \frac{1}{\mathcal{N}[A_k^{\not\ni i}]} = \gamma_{out\to in}(A_k^{\not\ni i})\,,$$

that is condition (16) in the main text, met by assumption for the inactive group we are considering. *QED*

A.5 PROOF OF LEMMA 5

Via Lemma 2 the assumption $\gamma_i < 1/kn_i$ for $k \geq 1$ groups entails that, if all such groups were to be active and behave in accordance with (13) and all other $k - m$ groups where to exert zero effort, the aggregate effort $\sum_{i=1}^m X_i^c$ would be negative.

Note indeed that $\gamma_i = 1/kn_i$ for all k active groups implies

$$\sum_{i=1}^{m} X_i^c \;=\; \frac{v}{n_1 + n_2 + \cdots + n_k}\left(\frac{n_1}{kn_1} + \frac{n_2}{kn_2} + \cdots + \frac{n_k}{kn_k} - 1\right) \;=\; 0\,, \qquad \text{(A.14)}$$

whereby $\sum_{\ell=1}^{m} X_\ell^c < 0$ must hold if $\gamma_i < 1/kn_i$ for all k active groups. Via (A.14) the aggregate effort would be negative, *a fortiori*, if either (i) only a subset of $h < k$ groups where to behave in accordance with (13), and/or (ii) all $m - k$ groups with $\gamma_i < \underline{\underline{\gamma_i}} = 1/mn_i$ were to turn active and behave in accordance with (13). Moreover, via Corollary 2 we know that the negativity of aggregate effort can be unambiguously traced-back to the suboptimality of positive effort provision – every group member would prefer to exert zero effort instead of behaving in accordance with (13). This implies that no positive-effort can exist in such an environment. However, via Lemma 1 we know, too, that a zero-effort equilibrium cannot exist either, since $\gamma_i > \underline{\underline{\gamma_i}} = 1/mn_i$ by assumption for $k \geq 1$ groups. Therefore, all members of such groups: (i) strictly prefer to switch to positive effort provision when the prescribed play entails zero effort overall; (ii) strictly prefer to switch to zero effort when the prescribed play assigns positive effort to the group they belong to. Therefore, no pure-strategy WGS equilibrium exists. *QED*

A.6 PROOF OF LEMMA 6

Let $\gamma_i > 1/n_i$ hold for one single group. Two scenarios ensue. First, no member of any group $-i$ has a unilateral incentive to deviate to positive effort provision. In this case, the effort stage-game has a unique pure-strategy WGS equilibrium with $X_i^* > 0$ and $X_{-i}^* = 0$ for all other $m-1$ groups. Second, the members of $h = 1, 2, \ldots, m-1$ of the $-i$ groups are willing to deviate to positive effort provision. Then: (i) Lemma 4 ensures that no individual member of any of these groups switched back to zero effort once all members the group he/she belongs to collectively deviate to positive effort; (ii) Corollary 2 ensures that the aggregate equilibrium effort provided by these groups is indeed positive. A positive-effort pure-strategy WGS equilibrium exists. In both cases, no cycles arise, which proves the result. *QED*

A.7 PROOF OF LEMMA 7

Via (13) the aggregate constrained-equilibrium effort strictly increases in every γ_i. Note that

$$\sum_{i=1}^{m} X_i^c \;=\; \frac{v}{\mathcal{N}[A_k]}\left(\frac{k}{m} - 1\right) \;\leq\; 0$$

holds if $\gamma_i = \underline{\underline{\gamma_i}}$ for all k active groups. *A fortiori*, the same holds if $\gamma_i < \underline{\underline{\gamma_i}}$ for some subset of $h \leq k$ active groups. Therefore, no positive-effort WGS equilibrium can exists if $\gamma_i < \underline{\underline{\gamma_i}}$ holds for all groups $i = 1, 2, \ldots, m$. *QED*

A.8 PROOF OF LEMMA 8

Differentiating the expected utility (21) with respect to γ_i we obtain

$$\begin{aligned}\frac{\partial}{\partial \gamma_i} \Pi_i^* \;=\; n_i v \Bigg\{ \left(1 - \frac{\partial}{\partial \gamma_i}\, \bar{\gamma}[A_k]\right) \left[1 - \left(\bar{\gamma}[A_k] - \frac{1}{\mathcal{N}[A_k]}\right)\right] + \\ - \frac{\partial}{\partial \gamma_i}\, \bar{\gamma}[A_k] \left[\gamma_i - \left(\bar{\gamma}[A_k] - \frac{1}{\mathcal{N}[A_k]}\right)\right] \Bigg\}.\end{aligned} \tag{A.15}$$

Recall that $\partial\bar{\gamma}[A_k]/\partial\gamma_i = n_i/\mathcal{N}[A_k]$ via the definition (11) of $\bar{\gamma}[A_k]$. For $k = 1$, $\mathcal{N}[A_k] = n_i$ and $\bar{\gamma}[A_k] = \gamma_i$ hold, and expression (A.15) reduces to $\partial\Pi_i^*/\partial\gamma_i = -1/\mathcal{N}[A_k]$, that is strictly negative for every $\mathcal{N}[A_k] > 0$. Therefore, $\gamma_i^* = 1/n_i$ must hold. For $k > 1$, expression (A.15) can be arranged as

$$\frac{\partial}{\partial \gamma_i} \Pi_i^* \;=\; \gamma_i + \left(\frac{\mathcal{N}[A_k] - 2n_i}{n_i}\right) \bar{\gamma}[A_k] - \left[\frac{\mathcal{N}[A_k]\Big(\mathcal{N}[A_k] - n_i\Big) + \mathcal{N}[A_k] - 2n_i}{n_i \mathcal{N}[A_k]}\right],$$

that it is strictly monotone in γ_i. The corresponding (implicit) FOC yields

$$\gamma_i^c(\bar{\gamma}[A_k]) = \left[\frac{\mathcal{N}[A_k]\Big(\mathcal{N}[A_k] - n_i\Big) + \mathcal{N}[A_k] - 2n_i}{n_i \mathcal{N}[A_k]}\right] - \left(\frac{\mathcal{N}[A_k] - 2n_i}{n_i}\right) \bar{\gamma}[A_k], \tag{A.16}$$

where γ_i is both at the LHS and RHS. Since $\partial\bar{\gamma}[A_k]/\partial\gamma_i = n_i/N_k$ is positive for $\gamma_i < \gamma_i^c(\bar{\gamma}[A_k])$ and negative otherwise, $\gamma_i^c(\bar{\gamma}[A_k])$ is a global maximum of (21). Expression (A.16) indeed identifies the optimal γ_i only if it is admissible, i.e. if $\gamma_i^c(\bar{\gamma}[A_k]) > 1/n_i$. Note that $\gamma_i \geq 1/n_i$ for all $i = 1, 2, \ldots, m$ implies that $\bar{\gamma}[A_k] \geq 1/\mathcal{N}[A_k]$. Assuming that both admissibility constraints indeed holds, and solving for the equilibrium candidate $\bar{\gamma}^c[A_k]$, we obtain

$$\bar{\gamma}^c[A_k] \;=\; 1 + \frac{k-2}{\mathcal{N}[A_k](k-1)}, \tag{A.17}$$

that is $\geq 1 > 1/\mathcal{N}[A_k]$ for every $k > 1$, hence admissible, and coincides with expression (24) in the main text. Moreover, substituting for $\bar{\gamma}^c[A_k]$ as defined by (A.17) into expression (A.16 it is immediate to check that $\gamma_i^c(\bar{\gamma}^c[A_k]) \geq 1/n_i$ if

$$n_i \;\geq\; \frac{\mathcal{N}[A_k](k-2)}{\mathcal{N}[A_k](k-1) - 2},$$

that is always true since the RHS is < 1 for every $k > 1$ and $n_i \geq 2$ holds by assumption for every group $i = 1, 2, \ldots, m$. Therefore, expression (A.16) indeed identifies, at every $k \geq 1$, how the leader of the i-th active group optimally reacts to any average egalitarianism $\bar{\gamma}[A_k]$, which completes the proof. *QED*

A.9 PROOF OF LEMMA 9

For $k = 1$ it holds that $\gamma_i^c = \bar{\gamma}^c[A_k] = 1/n_i$. For $k > 1$, via the best-response (23) we have

$$\gamma_i^c \;=\; \gamma_i^c(\bar{\gamma}^c[A_k]) \;=\; \left[\frac{\mathcal{N}[A_k]\Big(\mathcal{N}[A_k]-n_i\Big)+\mathcal{N}[A_k]-2n_i}{n_i\mathcal{N}[A_k]}\right]-\left(\frac{\mathcal{N}[A_k]-2n_i}{n_i}\right)\bar{\gamma}^*[A_k]\,. \tag{A.18}$$

Substituting in (A.18) for $\bar{\gamma}^c[A_k]$ as defined by (24), and solving in γ_i^c, we obtain

$$\gamma_i^c \;=\; \frac{\mathcal{N}[A_k]\big[n_i(k-1)+1\big]-2n_i}{n_i\mathcal{N}[A_k](k-1)}$$

that, rearranged, yields

$$\gamma_i^c \;=\; 1+\frac{\mathcal{N}[A_k]-2n_i}{n_i\mathcal{N}[A_k](k-1)}\,,$$

that is $> 1 > 1/n_i$ for every $k \geq 1$, hence admissible, since we know that $\gamma_i^c(\bar{\gamma}[A_k]) > 1/n_i$ for every $\bar{\gamma}[A_k]$ via the proof of Lemma 8. *QED*

A.10 PROOF OF LEMMA 10

Consider a scenario with $k > 1$. The aggregate expected utility *at the optimum* of the generic active group $i \in A_k$ is

$$\Pi_i^c \;=\; n_i v\left[\gamma_i^c-\left(\bar{\gamma}^c[A_k]-\frac{1}{\mathcal{N}[A_k]}\right)\right]\left[1-\left(\bar{\gamma}^c[A_k]-\frac{1}{\mathcal{N}[A_k]}\right)\right]\,. \tag{A.19}$$

Substituting in (A.19) for $\bar{\gamma}^c[A_k]$ and γ_i^c as defined by (24) and (26) respectively, we obtain

$$\Pi_i^c \;=\; v\left[\frac{\mathcal{N}[A_k]-n_i}{\mathcal{N}[A_k]^2(k-1)^2}\right]\,, \tag{A.20}$$

that is strictly positive for every $k > 1$. Factorising using the definition (27) of the probability of winning at the optimum p_i^c, we finally obtain expression (28) in the main text. *QED*

B TWO-GROUP EXAMPLE: PROOFS

This appendix collects all proofs and derivations of the two-group example presented in Section 3. All formal arguments are special cases (for $m = 2$) of those presented in Appendix A, and the characterisation unfolds as in the main text: first, we characterise *constrained equilibria* where the members of some groups are not free to choose optimally their courses of action; second, we check for individual unilateral deviations and identify under which conditions (if any) the constrained equilibria are stable; third, we characterise the *unconstrained equilibria* as the set of all stable constrained equilibria.

Four 'types' of equilibrium can possibly arise in a two-group environment:

a) $(0, 0)$ equilibria: symmetric *zero-effort* equilibria in both groups are inactive – i.e. no member of any groups exerts any effort, so that the prize is randomly allocated;

b) $(+, +)$ equilibria: symmetric equilibria in which both groups are active – i.e. both groups exert strictly positive effort;

c) $(+, 0)$ and $(0, +)$ equilibria: asymmetric equilibria in which one group is active and the other is not.

We proceed with the characterisation type-by-type.

$(0, 0)$ Equilibrium

The lemma that follows identifies a necessary condition for the existence of a zero-effort equilibrium.

LEMMA B.1.
There exists an equilibrium for the effort-stage-game played at $t = 2$ such that $x^*_{ij} = 0$ for all members $j = 1, 2, \ldots, n_i$ of both groups $i = 1, 2$ if and only if

$$\gamma_i < \underline{\underline{\gamma_i}} := \frac{1}{2n_i}.$$

Proof.
Only if. Let $\mathbf{x_i} = \mathbf{0} \in \mathbb{R}^{n_i}$ hold for both groups $i = 1, 2$, so that $\mathbf{x} = \mathbf{0} \in \mathbb{R}^N$ and $\mathbf{X} = \mathbf{0} \in \mathbb{R}^2$. Then

$$\pi_{ij}(0, \mathbf{0}) = v/2n_i$$

holds via (2b). If the j-th member of the i-th groups unilaterally deviates to $x'_{ij} > 0$, then

$$\pi_{ij}(x'_{ij}, \mathbf{X}') = v\gamma_i - x'_{ij}.$$

The deviation is *not* profitable if and only if

$$\pi_{ij}(x'_{ij}, \mathbf{X}') \leq \pi_{ij}(x_{ij}, \mathbf{X}) = \pi_{ij}(0, \mathbf{0}) \Leftrightarrow v\gamma_i - x'_{ij} \leq \frac{v}{2n_i}$$

that for infinitesimal deviations means

$$\gamma_i \leq \frac{1}{2n_i} \Leftrightarrow \gamma_i \leq \underline{\underline{\gamma_i}} := \frac{1}{2n_i} < 1. \tag{B.1}$$

Solving (B.1) in a_i we obtain

$$a_i > \underline{\underline{a_i}} := \frac{2n_i - 1}{2(n_i - 1)}.$$

The two equivalent conditions are thus necessary for the existence of a zero-effort equilibrium: if they are not met for one group, every member of that group has *by definition* a unilateral incentive to deviate to some positive effort level. Accordingly, no $(0, 0)$ equilibrium can ever exist if $\gamma_i > \underline{\underline{\gamma_i}}$ holds for at least one group.

If. Suppose

$$\gamma_i \leq \frac{1}{2n_i},$$

then ij payoff is

$$\pi_{ij}(x_{ij}, \mathbf{X}) = \begin{cases} v\frac{n_i}{(n_i-1)}\left[\left(\gamma_i - \frac{1}{n_i}\right)\frac{x_{ij}}{X_i} - (\gamma_i - 1)\frac{1}{n_i}\right] - x_{ij} & X_i > 0, X_{-i} = 0 \\ 0 & X_i = 0, X_{-i} > 0 \\ v\frac{X_i}{X_i + X_{-i}}\frac{n_i}{(n_i-1)}\left[\left(\gamma_i - \frac{1}{n_i}\right)\frac{x_{ij}}{X_i} - (\gamma_i - 1)\frac{1}{n_i}\right] - x_{ij} & X_i > 0, X_{-i} > 0 \\ \frac{1}{2}v\frac{1}{n_i} & X_i = X_{-i} = 0. \end{cases}$$

Note that

$$\frac{\partial}{\partial x_{ij}}\left[\left(\gamma_i - \frac{1}{n_i}\right)\frac{x_{ij}}{X_i} - (\gamma_i - 1)\frac{1}{n_i} - x_{ij}\right] = \left(\gamma_i - \frac{1}{n_i}\right)\frac{X_i - x_{ij}}{X_i^2} - 1 \leq 0 \Leftrightarrow$$

$$\Leftrightarrow \gamma_i - \frac{1}{n_i} \leq \frac{X_i^2}{X_i - x_{ij}} \Leftrightarrow \gamma_i \leq \frac{1}{n_i} + \frac{X_i^2}{X_i - x_{ij}}$$

which is always satisfied when $\gamma_i \leq \frac{1}{2n_i}$. Hence, if

$$\gamma_i \leq \frac{1}{2n_i},$$

$$\pi_{ij}(x_{ij}, \mathbf{X} | \gamma_i \leq \frac{1}{2n_i})$$

is maximized for $x_{ij} = 0$. *QED*

Now, we consider $\gamma_i \geq \underline{\underline{\gamma_i}}$ for at least a group $i = 1, 2$, and we look for a characterization of positive-effort equilibria $(+, +)$, $(+, 0)$ and $(0, +)$ in this region.

LEMMA B.2.

There exists a WGS equilibrium for the effort-stage-game played at $t = 2$ such that for *any* $i = 1, 2$

$$X_i^* = n_i v \left(\frac{n_1\gamma_1 + n_2\gamma_2}{N} - \frac{1}{N} \right) \left[\gamma_i - \left(\frac{n_1\gamma_1 + n_2\gamma_2}{N} - \frac{1}{N} \right) \right] > 0 \tag{B.2}$$

if and only if

$$\begin{cases} n_1\gamma_1 + n_2\gamma_2 \geq 1 \\ \gamma_1 \geq \gamma_2 - \frac{1}{n_2} \\ \gamma_2 \geq \gamma_1 - \frac{1}{n_1}. \end{cases}$$

Proof. Suppose that $X_i^* = n_i x_{ij}^* > 0$ holds for both groups $i = 1, 2$, so that $X_i^* + X_{-i}^* > 0$ must hold, too. Then, the expected payoff of the generic j-th group member is

$$\pi_{ij}(x_{ij}, \mathbf{X}) = v \frac{n_i}{(n_i - 1)} \left[\left(\gamma_i - \frac{1}{n_i} \right) \left(\frac{x_{ij}}{X_i + X_{-i}} \right) - (\gamma_i - 1) \frac{1}{n_i} \left(\frac{X_i}{X_i + X_{-i}} \right) \right] - x_{ij}^* \, . \tag{B.3}$$

Then, it easy to check that (B.3) has a global maximum in x_{ij}, identified by the FOC

$$\frac{\partial \pi_{ij}(x_{ij}, \mathbf{X})}{\partial x_{ij}} = v \frac{n_i}{(n_i - 1)} \left[\left(\gamma_i - \frac{1}{n_i} \right) \left(\frac{X_i + X_{-i} - x_{ij}}{(X_i + X_{-i})^2} \right) - (\gamma_i - 1) \frac{1}{n_i} \left(\frac{X_i + X_{-i} - X_i}{(X_i + X_{-i})^2} \right) \right] - 1 \geq 0 \Leftrightarrow$$

$$\Leftrightarrow \left(\gamma_i - \frac{1}{n_i} \right) (X_i - x_{ij}) + \gamma_i \left(1 - \frac{1}{n_i} \right) X_{-i} \geq \frac{n_i - 1}{v n_i} (X_i + X_{-i})^2 \Leftrightarrow$$

using the assumption of WGS equilibria

$$\Leftrightarrow \left(\gamma_i - \frac{1}{n_i} \right) (n_i - 1) \, x_{ij} + \gamma_i \left(\frac{n_i - 1}{n_i} \right) X_{-i} \geq \frac{n_i - 1}{v n_i} (X_i + X_{-i})^2 \Leftrightarrow$$

$$\Leftrightarrow \left[\left(\gamma_i - \frac{1}{n_i} \right) n_i x_{ij} + \gamma_i X_{-i} \right] \geq \frac{1}{v} (X_i + X_{-i})^2 \Leftrightarrow - \left(\gamma_i - \frac{1}{n_i} \right) X_i \leq \gamma_i X_{-i} - \frac{1}{v} (X_i + X_{-i})^2 \Leftrightarrow \tag{B.4}$$

$$\Leftrightarrow X_i^* = n_i \left\{ (X_i^* + X_{-i}^*) \left[\gamma_i - \frac{1}{v} (X_i^* + X_{-i}^*) \right] \right\}. \tag{B.5}$$

Similarly

$$X_{-i}^* \; = \; n_{-i} \left\{ (X_i^* + X_{-i}^*) \left[\gamma_{-i} - \frac{1}{v} (X_i^* + X_{-i}^*) \right] \right\}.$$

Therefore,

$$X_i^* + X_{-i}^* \; = \; (X_i^* + X_{-i}^*) \, (n_i\gamma_i + n_{-i}\gamma_{-i}) - \frac{N}{v} (X_i^* + X_{-i}^*)^2 \, .$$

Since $X_i^* + X_{-i}^* > 0$, then, solved in $X_i^* + X_{-i}^*$, it yields

$$X_i^* + X_{-i}^* \; = \; v \left(\frac{n_i\gamma_i + n_{-i}\gamma_{-i}}{N} - \frac{1}{N} \right) \, .$$

Then

$$X_i^* + X_{-i}^* > 0 \Leftrightarrow n_i\gamma_i + n_{-i}\gamma_{-i} > 1. \tag{B.6}$$

Substituting back into the group effort (B.5) we finally obtain

$$X_i^* \;=\; n_i v \left(\frac{n_i\gamma_i + n_{-i}\gamma_{-i}}{N} - \frac{1}{N} \right) \left[\gamma_i - \left(\frac{n_i\gamma_i + n_{-i}\gamma_{-i}}{N} - \frac{1}{N} \right) \right] , \tag{B.7}$$

that is expression (B.2). Moreover, if (B.6) holds, group i effort (B.7) is strictly positive if

$$\begin{aligned} \gamma_i > \left(\frac{n_i\gamma_i + n_{-i}\gamma_{-i}}{N} - \frac{1}{N} \right) &\Leftrightarrow n_i\gamma_i + n_{-i}\gamma_i > n_i\gamma_i + n_{-i}\gamma_{-i} - 1 \Leftrightarrow \\ &\Leftrightarrow \gamma_i > \gamma_{-i} - \frac{1}{n_{-i}} . \end{aligned} \tag{B.8}$$

To complete the characterisation, we check for profitable unilateral deviations by individual group members. Since both groups are exerting positive effort and, via the WGS assumption, this holds true also for all group members, the only possible deviation is to $x'_{ij} = 0$ – in this case, the FOC (B.4) no longer holds. Abiding by the prescribed strategy $x^*_{ij} > 0$, the generic member j gets

$$\pi^*_{ij} \;=\; \pi(x^*_{ij}, \mathbf{X}^*) \;=\; v \left[\gamma_i - \left(\frac{n_i\gamma_i + n_{-i}\gamma_{-i}}{N} - \frac{1}{N} \right) \right] \left[1 - \left(\frac{n_i\gamma_i + n_{-i}\gamma_{-i}}{N} - \frac{1}{N} \right) \right] . \tag{B.9}$$

Upon deviating to x'_{ij} the generic group member j gets

$$\pi'_{ij} \;=\; \pi(0, \mathbf{X}') \;=\; v(\gamma_i - 1) \left[\frac{\gamma_i - \left(\frac{n_i\gamma_i + n_{-i}\gamma_{-i}}{N} - \frac{1}{N} \right)}{1 - \gamma_i + \left(\frac{n_i\gamma_i + n_{-i}\gamma_{-i}}{N} - \frac{1}{N} \right)} \right] . \tag{B.10}$$

Comparing (B.9) and (B.10) it is immediate to check that the deviation is profitable if

$$\left(\frac{n_i\gamma_i + n_{-i}\gamma_{-i}}{N} - \frac{1}{N} \right) \left[\gamma_i - \left(\frac{n_i\gamma_i + n_{-i}\gamma_{-i}}{N} - \frac{1}{N} \right) \right] \;<\; 0 . \tag{B.11}$$

Note that, via (B.7), the LHS of (B.11) is x^*_{ij}/v. Therefore, condition (B.11) says that the consistency condition $x^*_{ij} > 0$ is sufficient to guarantee the absence of profitable unilateral deviations. Then, $X_i^* + X_{-i}^* > 0$ requires the inequality to hold for both groups $i = 1, 2$. *QED*

$(+, \mathbf{0})$ and $(\mathbf{0}, +)$ Equilibria

Without loss of generality, suppose that

$$\begin{aligned} X_i^* \;&> 0 , \\ X_{-i}^* \;&= \; 0 , \end{aligned}$$

so that both $(+,0)$ and $(0,+)$ can be simultaneously characterised. The following lemma summarises the characterisation.

LEMMA B.3.
There exists a WGS equilibrium for the effort-stage-game played at $t=2$ such that for $i=1,2$

$$X_i^* = v\left(\gamma_i - \frac{1}{n_i}\right) > 0 \quad \textit{and} \quad X_{-i}^* = 0$$

if and only if

$$\begin{cases} \gamma_i \geq \dfrac{1}{n_i} \\ \gamma_{-i} \leq \gamma_i - \frac{1}{n_i}. \end{cases}$$

Proof. Because of lemma B.1, $X_i^* + X_{-i}^* > 0$ must hold, so that

$$\pi_{ij}(x_{ij}, \mathbf{X}) = v\frac{n_i}{(n_i-1)}\left[\left(\gamma_i - \frac{1}{n_i}\right)\frac{x_{ij}}{X_i} - (\gamma_i - 1)\frac{1}{n_i}\right] - x_{ij}$$

so that

$$\frac{\partial \pi_{ij}(x_{ij}, \mathbf{X})}{\partial x_{ij}} = v\frac{n_i}{(n_i-1)}\left(\gamma_i - \frac{1}{n_i}\right)\left(\frac{X_i - x_{ij}}{(X_i)^2}\right) - 1 \geq 0 \Leftrightarrow$$

using the assumption of WGS equilibria, it follows that

$$\Leftrightarrow v\frac{n_i}{(n_i-1)}\left(\gamma_i - \frac{1}{n_i}\right)\left(\frac{(n_i-1)\,x_{ij}}{n_i^2 x_{ij}^2}\right) \geq 1 \Leftrightarrow v\left(\gamma_i - \frac{1}{n_i}\right)\left(\frac{1}{n_i x_{ij}}\right) \geq 1 \Leftrightarrow X_i^* = v\left(\gamma_i - \frac{1}{n_i}\right)$$

so that

$$X_i^* = v\left(\gamma_i - \frac{1}{n_i}\right) \geq 0 \Leftrightarrow \gamma_i \geq \frac{1}{n_i}.$$

Moreover

$$\pi_{-ij}(x_{-ij}^*, \mathbf{X}^*) = 0$$

but a deviation to $x'_{-ij} > 0$, implies

$$\pi_{-ij}(x'_{-ij}, \mathbf{X}^*) = v\frac{x'_{-ij}}{X_i + x'_{-ij}}\frac{n_{-i}}{(n_{-i}-1)}\left[\left(\gamma_{-i} - \frac{1}{n_{-i}}\right) - (\gamma_{-i} - 1)\frac{1}{n_{-i}}\right] - x'_{-ij} =$$

$$= v\frac{x'_{-ij}}{X_i + x'_{-ij}}\frac{n_{-i}}{(n_{-i}-1)}\gamma_{-i}\frac{n_{-i}-1}{n_{-i}} - x'_{-ij} = v\gamma_{-i}\frac{x'_{-ij}}{v\left(\gamma_i - \frac{1}{n_i}\right) + x'_{-ij}} - x'_{-ij}.$$

Then

$$\pi_{-ij}(x'_{-ij}, \mathbf{X}^*) \leq \pi_{-ij}(x^*_{-ij}, \mathbf{X}^*) = 0 \Leftrightarrow v\gamma_{-i}\frac{x'_{-ij}}{v\left(\gamma_i - \frac{1}{n_i}\right) + x'_{-ij}} - x'_{-ij} \leq 0 \Leftrightarrow$$

$$\Leftrightarrow \frac{\gamma_{-i}}{\left(\gamma_i - \frac{1}{n_i}\right) + x'_{-ij}} \leq 1 \Leftrightarrow \gamma_{-i} \leq \gamma_i - \frac{1}{n_i}$$

for a small enough deviation $x'_{-ij} > 0$. Note that the individual expected utility at the optimum is

$$\pi^*_{ij} \;=\; \pi(x^*_{ij}, \mathbf{X}^*) \;=\; \frac{v}{n_i}\left[1 - \left(\gamma_i - \frac{1}{n_i}\right)\right] .$$

Deviating to $x'_{ij} = 0$, the i-th member of the active group i obtains

$$\pi' \;=\; \pi_{ij}(x'_{ij}, \mathbf{X}') \;=\; v\left(\frac{1-\gamma_i}{n_i - 1}\right) ,$$

and the deviation is *not* profitable if $\gamma_i > 1/n_i$. Therefore, condition $\gamma_i > 1/n_i$ simultaneously guarantees that (i) individual and group effort are indeed non-negative (consistency), and that (ii) no individual member of the active group i has a unilateral incentive to deviate to zero effort (incentive compatibility). *QED*

The following result follows immediately from the previous lemmas.

PROPOSITION B.1. *There exists no WGS equilibrium if and only if*

$$\begin{cases} n_1\gamma_1 + n_2\gamma_2 < 1 \\ \gamma_1 \in \left(\frac{1}{2n_1}, \frac{1}{n_1}\right) \\ \gamma_2 \in \left(\frac{1}{2n_2}, \frac{1}{n_2}\right) . \end{cases}$$

The intuition is the following. If $\gamma_i < 1/n_i$, the generic j-th member of the active group i finds it optimal to deviate to zero effort. Since all group member are identical, all group member face the same incentive to deviate. Via the WGS assumption, the only candidate equilibrium profile must therefore entail $x^*_{ij} = X^*_i = 0$. But since $X^*_{-i} = 0$ holds by assumption, a $(0, 0)$ environment ensues if all members of group i collectively implement their profitable deviations. However, $\gamma_i > \underline{\underline{\gamma_i}} = 1/2n_i$ entails that, in a $(0, 0)$ environment, the generic member j of group i has, by definition, a unilateral incentive to deviate to some positive effort level. Therefore, $(0, 0)$ cannot be an equilibrium either.

We proceed with unilateral deviations in the inactive group $-i$. If all members of group $-i$ abide by the prescribed strategies, they all get an *ex ante* expected utility $\pi^*_{-ij} = \pi(0, \mathbf{X}^*) = 0$. By deviating to positive effort level $x'_{-ij} > 0$, the generic member j of the inactive group $-i$ gets

$$\pi'_{-ij} \;=\; \pi(x'_{-ij}, \mathbf{X}') \;=\; x_{-ij}\left[v\left(\frac{\gamma_i}{X^*_i + x'_{-ij}}\right) - 1\right]$$

whose (global) maximum in $\mathbb{R}_+$ is unambiguously identified by the FOC

$$-\left(x^*_{-ij}\right)^2 - x^*_{-ij}(2X^*_i) - \left[X^*_i\left(X^*_i - v\gamma_{-i}\right)\right] ,$$

whose smaller root is always negative, and whose larger root is

$$x^*_{-ij} \; = \; \sqrt{\gamma_{-i}\,(vX^*_i)} - X^*_i\,. \tag{B.12}$$

If (B.12) is non-positive, then no profitable deviation to some positive effort level exists for the generic member of the inactive group $-i$. Note that (B.12) is non-positive if

$$\gamma_{-i} \; \leq \; \frac{1}{v} X^*_i\,. \tag{B.13}$$

Substituting in (B.13) for X^*_i we finally obtain that no profitable deviation exists if

$$\gamma_{-i} \; \leq \; \gamma_i - \frac{1}{n_i}\,.$$

Wrap-Up

To wrap-up the equilibrium characterisation, we state the conditions identified by lemmas B.1 to B.3, while Figure 1 in the main text provides a graphical representation in $\mathbb{R}^2$. In particular note that there are no overlapping between the existence regions identified by the lemmas. Therefore, in every existence region, equilibria are unique – there is only one 'type' of equilibrium in each region. This confirms that all existence conditions are *necessary and sufficient* for both existence and uniqueness of WGS equilibria.

PROPOSITION B.2.
Let $(\gamma_1, \gamma_2) = \boldsymbol{\gamma} \in \mathbb{R}^2$ *be a pair of generic stand-alone incentives for groups* 1 *and* 2, *respectively. Moreover, let define the following partition of* $\mathbb{R}^2$*:*

$$\mathbf{G}_{00} := \left\{ (\gamma_1, \gamma_2) \in \mathbb{R}^2 : \gamma_1 \leq \frac{1}{2n_1}, \gamma_2 \leq \frac{1}{2n_2} \right\}$$

$$\mathbf{G}_{+0} := \left\{ (\gamma_1, \gamma_2) \in \mathbb{R}^2 : \gamma_1 \geq \frac{1}{n_1}, \gamma_2 \leq \gamma_1 - \frac{1}{n_1} \right\}$$

$$\mathbf{G}_{0+} := \left\{ (\gamma_1, \gamma_2) \in \mathbb{R}^2 : \gamma_1 \leq \gamma_2 - \frac{1}{n_2}, \gamma_2 \geq \frac{1}{n_2}, \right\}$$

$$\mathbf{G}_{++} := \left\{ (\gamma_1, \gamma_2) \in \mathbb{R}^2 : \gamma_1 \geq \gamma_2 - \frac{1}{n_2}, \gamma_2 \geq \gamma_1 - \frac{1}{n_1}, \gamma_2 \geq -\frac{n_1}{n_2}\gamma_1 + \frac{1}{n_2} \right\}$$

$$\mathbf{G}_{\nexists} := \mathbb{R}^2 \setminus \mathbf{G}_{00} \setminus \mathbf{G}_{+0} \setminus \mathbf{G}_{0+} \setminus \mathbf{G}_{++}.$$

Then, the effort-stage-game played by groups $i = 1, 2$, *at* $t = 2$ *has:*

a) a unique WGS equilibrium with $X^*_1 = X^*_2 = 0$ *for every* $\boldsymbol{\gamma} \in \boldsymbol{G}_{00}$*;*

b) a unique WGS equilibrium with $X^*_1 > 0$ *and* $X^*_2 = 0$ *for every* $\boldsymbol{\gamma} \in \boldsymbol{G}_{+0}$*;*

c) a unique WGS equilibrium with $X^*_1 = 0$ *and* $X^*_2 > 0$ *for every* $\boldsymbol{\gamma} \in \boldsymbol{G}_{0+}$*;*

d) a unique WGS equilibrium with $X_1^* > 0$ *and* $X_2^* > 0$ *for every* $\boldsymbol{\gamma} \in \boldsymbol{G}_{++}$*;*

e) no pure-strategy WGS equilibrium for every $\boldsymbol{\gamma} \in \boldsymbol{G}_{\nexists}$*.*

Proof. The proof follows from lemmas B.1 to B.3 and proposition 1. *QED*

FONDAZIONE ENI ENRICO MATTEI WORKING PAPER SERIES

Our Working Papers are available on the Internet at the following address:
https://www.feem.it/pubblicazioni/feem-working-papers/

"NOTE DI LAVORO" PUBLISHED IN 2024

1. A. Sileo, M. Bonacina, The automotive industry: when regulated supply fails to meet demand. The Case of Italy
2. A. Bastianin, E. Mirto, Y. Qin, L. Rossini, What drives the European carbon market? Macroeconomic factors and forecasts
3. M. Rizzati, E. Ciola, E. Turco, D. Bazzana, S. Vergalli, Beyond Green Preferences: Alternative Pathways to Net-Zero Emissions in the MATRIX model
4. L. Di Corato, M. Moretto, Supply contracting under dynamic asymmetric cost information
5. C. Drago, L. Errichiello, Remote work admist the Covid-19 outbreak: Insights from an Ensemble Community-Based Keyword Network Analysis
6. F. Cappelli, Unequal contributions to CO2 emissions along the income distribution within and between countries
7. I. Bos, G. Maccarrone, M. A. Marini, Anti-Consumerism: Stick or Carrot?
8. M. Gilli, A. Sorrentino, The Set of Equilibria in Max-Min Two Groups Contests with Binary Actions and a Private Good Prize
9. E. Bachiocchi, A. Bastianin, G. Moramarco, Macroeconomic Spillovers of Weather Shocks across U.S. States
10. T. Schmitz, I. Colantone, G. Ottaviano, Regional and Aggregate Economic Consequences of Environmental Policy

FONDAZIONE ENI ENRICO MATTEI

Fondazione Eni Enrico Mattei
Corso Magenta 63, Milano – Italia

Tel. +39 02 403 36934

E-mail: letter@feem.it
www.feem.it

FONDAZIONE ENI ENRICO MATTEI