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Summary

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Keywords: Network Formation; Nash Network; Two-way Flow Network; Agent Heterogeneity; Efficient Network.

JEL Classification: C72, D85

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Abstract

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1 Introduction

In a social network, information transmission tends to be imperfect. In the literature of strategic network formation, a rigorously studied form of imperfect information transmission is the so-called information decay. It captures the idea that the worth of information decays as it traverses through each link [1]. Naturally, if agent *i* communicates with agent *j* directly, the worth of information that *i* acquired from *j* is expected to be higher than that of the situation in which *i* acquires information of *j* indirectly from another person *k*. Thus, *i* may have an incentive to bear the cost of establishing a link with *j* as such a direct communication results in less information decay. Such an incentive, though, diminishes if the rate at which information decay via each link is sufficiently small. This assumption of 'small decay' is rigorously studied in the context of two-way flow model of network formation with nonrival information by De Jaegher and Kamphorst (2015), which is a model of network formation proposed in the seminal paper of Bala and Goyal (2000a). Their major novel findings, which allow them to finely characterize the equilibrium networks, are: (i) best-informed agents, defined as agents who received more information than others, are attractive as link receivers and (ii) these best-informed agents are located 'in the middle' of other agents.

In this note, I complement these novel findings by showing that best-informed agents, in addition to being attractive as link receivers from a strategic perspective of self-interest agents, are also agents that allow information to flow *most efficiently* within a group of agents in the network. Specifically, I show that a best-informed agent within the set of minimally connected agents *M*, once chosen by another agent *i* as a link receiver, allow this set of agents *M* to maximize the quantity of information received from *i*. Put differently, a surprising similarity that Nash networks and efficent networks have in common is that every link receiver is an efficient information transmitter. This leads to an important insight: *a strategic decision of an agent to maximize his own payoff by sending a link to an agent can lead to an outcome that is socially desirable from the (collective) point of view of the group of agents to which the link receiver belongs*. This insight is established as Proposition 1 in this note. See Figure 1 for an intuitive, informal example of this insight [2].

It is the belief of this author that this aforementioned insight, albeit posited in its utmost simplicity, offers a perspective that substantially complements yet differs from the literature since "a central theme in the literature of network formation is the conflict between the set of stable networks and the set of efficient networks"

[1]This could stem from, for instance, imperfect quality of communication devices used or simply misunderstandings caused by a lack of concentration of agents when they communicate. See the second paragraph in De Jaegher and Kamphorst (2015) for more examples

[2]This Figure is inspired by Figure 1 in De Jaegher and Kamphorst (2015)

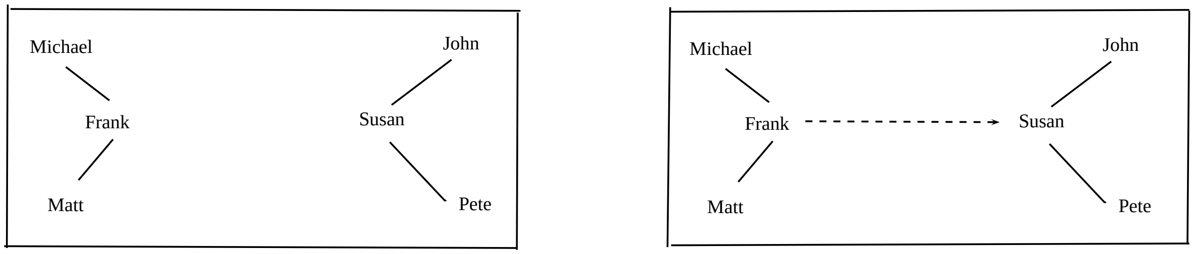


Figure 1: On the left, two groups of agents are disconnected. If the decay is small, then within the group of Susan, Pete and John it can be shown that Susan, who is in the middle, possesses more informational quantity than Pete and John do (see Lemma 1 in De Jaegher and Kamphorst (2015)). Consequently, on one hand, if Frank wants to acquire information with the group of Susan, John and Pete, then for the benefit of his own interest Frank will choose Susan as his partner, as shown on the right. On the other hand, Proposition 1 in this note further shows that, surprisingly, Susan is also an agent that efficiently transmits information of the other group (Frank, Michael and Matt) to her own group (herself, John and Pete).

(Unlu (2018)) [3] [4]. Specifically, within the context of the aforementioned two-way flow model of network formation with nonrival information Proposition 1 and 2 in De Jaegher and Kamphorst (2015) shows that the set of Nash networks can be large and does contain networks that have long diameters, albeit remaing short compared to the population size, while Proposition 5.5 in Bala and Goyal (2000a) shows that a star - a network with diameter of only 2 - is a unique minimal efficient network. Thus, these results in the exising literature give an impression that the set of stable networks and the set efficient networks do not coincide, and hence all Nash networks are not efficient except the star architecture that can be both efficient and Nash. The above insight from this note, however, show that every Nash network is partially efficient. Note that I use the word partially here because this finding shows that a best-informed agent within the set of agents M, once chosen by another agent i as a link receiver, allows this set of agents M, rather than *all agents* in the network, to maximize the quantity of information that it receives from i.

In addition, I also extend these results to a more general case of player cost het-

[3]For more elaboration on this matter, Unlu (2018) elaborates on this conflict found in Jackson and Wolinsky (1996), Bala and Goyal (2000a) and Bala and Goyal (2000b).

[4] Indeed, such a tension has also been mentioned by Breitmoser and Vorjohann (2013). In particular, Breitmoser and Vorjohann (2013) remark that stars or complete networks are efficient across various models of network formation, including two-way flow model with bilateral link consent by Jackson and Wolinsky (1996), two-way flow model with cut-off decay (Hojman and Szeidl (2008)), model with far-sighted players (Dutta et al. (2005)), model with endogenous link strength (Bloch and Dutta (2009)) and model with transfer payments between players (Bloch and Jackson (2007)). Breitmoser and Vorjohann (2013) complement the literature by showing that substantially different architectures of networks - redundant, incomplete and circular networks - are efficient if noisy communication is assumed. See the first two paragraphs in Breitmoser and Vorjohann (2013) for a comprehensive literature review.

erogeneity and a more general cost function yet no decay studied by Unlu (2018) [5]. My rationales for this extension are as follows. The case of player heterogeneity implies that the aforementioned similarity between Nash network and efficient network continues to hold, since the variation in link formation cost does not depend on the identity of the link receiver. Yet unlike the case of agent homogeneity it is no longer the case that every link sender is an efficient transmitter. Indeed, this allows us to explore the following intricate tradeoff. On one hand, by shortening the network diameter we could minimize the information decay and hence increase total quantity of information. On the other hand, lengthening the network diameter could also be a way to lower the total link formation cost [6]. How long, then, would diameters of efficient networks be? In Proposition 1, I show that (i) the diameters of efficient networks are at most only 4; (ii) if the diameter is 4, then it is a generalized interlinked center-sponsored star, with an agent that is a largest sponsor being the center of the network that bridges other centers together.

Lastly, I remark that this note also makes a technical contribution to the literature since it substantially refines the results of Unlu (2018), which assumes the same form of agent heterogeneity yet without decay. Specifically Unlu (2018) finds that the set of efficient networks is substantially large and can contain a maximal diameter network (a line), which in turn leads to the necessity to impose additional assumptions in order to restrict the set of efficent networks to be smaller [7]. These assumptions are, in the opinion of this author, strong and rather nonintuitive. On the contrary, this note achieves a fine-detail characterization of efficient networks under similar assumptions by simply introducing the assumption of small decay without any further restriction.

This note proceeds as follows. In Section 2, I introduce the model and the payoffs. In particular, Section 2.1 defines two important concepts - efficient link receiver/transmitter and best-informed agent. The latter is borrowed from De Jaegher and Kamphorst (2015). In Section 3, Proposition 1 relates these two concepts by showing that the identity of efficient link receiver coincides with the identity of best-informed agent. Subsequently, I use this proposition to establish Remark 3 which concludes that every Nash network is partially efficient in the sense that every link receiver is efficient. Remark 3 also compares differences and similarities between Nash networks and efficient networks. I then extend this result and fully characterize the set of efficient networks in the case of player heterogeneity in link formation cost in Proposition 2. My last section concludes with remarks on further potential studies.

[5] The definition of player heterogeneity (in link formation cost) is that link formation cost varies solely according to the identities of link senders, which was first defined by Galeotti et al. (2006). This form of heterogeneity is extensively studied in the literature including Goeree et al. (2009), Galeotti (2006) and Unlu (2018)

[6] I remark that to my knowledge this note is the first work in the literature that explores such a tradeoff. Hence, this is another contribution of this note to the literature.

[7] See Subsection 3.1.4 and Condition 1 in Unlu (2018)

2 The Model

This note follows primarily the notations of De Jaegher and Kamphorst (2015), since it is the paper that this note seeks to complement. Additionally the notations related to player heterogeneity follows those of Unlu (2018) for the same reason.

Link establishment and individual's strategy: Let $N = \{1, ..., n\}$ be the set of all agents. An agent $i \in N$ can form a link with another agent j without j's consent. ij denotes such a link. The set of all possible links that i forms is $L_i = \{ij : j \in N \backslash \{i\}\}$. The set of all possible links is $L \equiv \cup_{i \in N} L_i$. Naturally, $g_i \subset L_i$ is a strategy of i and $g = \cup_{i \in N} g_i$ is a strategy profile. A strategy space G, which is the set of all possible g, is $G \equiv 2^L$. Pictorially, a strategy profile g is also a network, where an arrow from agent i to j indicates that $ij \in g_i$.

Information flow. Information flow is two-way in the sense that it flows between two agents regardless of who sponsors the link, hence the term 'two-way flow model'. Accordinglys we introduce the following notations. Let $\bar{ij} \in g$ represents that either $ij \in g$ or $ji \in g$ and call $\bar{g} = \{\bar{ij} \in g\}_{i \neq j}$ the structure of information flow of g. Information can also flow via a path, which is a series of links. Specifically, a path between i and j, denoted by $P_{ij}(g)$, is a sequence of agents $\left\{\overline{i_0 i_1}, ..., \overline{i_{k-1} i_k}\right\} \subseteq g$ such that $i_0 = i, i_k = j$. If there is a path between i and j, we say that i and j are connected. A shortest path between i and j is, of course, the path(s) between i and j with the least amount of links. The distance between i and j, denoted by $d_{ij}(g)$ is defined as the amount of links of the shortest path(s). If $j = i$ then we assume, following the literature, that the distance between i and himself is 0. If j and i are not connected, then we set $d_{ij}(g) = \infty$. If two networks g^1 and g^2 are such that $\bar{g}^1$ can be transformed into $\bar{g}^2$ (and vice versa) by permuting the identities of agents, we say that g^1 and g^2 share the same structure of information flow.

Cost heterogeneity. Let c_{ij} denote the link formation cost that i bears to form a link with j. Let $\mathcal{C} = \{c_{ij}\}_{ij \in N \times N, i \neq j}$ be the cost structure. If $c_{ij} = c$ for every $i, j \in N$, then $\mathcal{C}$ is said to satisfy cost homogeneity. Similar to if $c_{ij} = c_i$ for every $i \neq j$ then $\mathcal{C}$ is said to satisfy player cost heterogeneity [8].

Information quantity Let $\sigma \in [0, 1]$ denote the decay factor, which means information decay by the proportion of $1 - \sigma$ per each link that it traverses. For example, if the value of information that an agent j possesses is 1 and the distance between i and j is k then the information that i receives from j is σ^k. Naturally, if $\sigma = 1$ then we say that there is no (information) decay. If $\sigma < 1$ then there is (information) decay.

[8] The definition of player cost heterogeneity here follows Galeotti et al. (2006).

Small decay assumption Suppose information of j flows to i via a multi-link path, then i can improve the information flow by establishing a link that results in a shorter path. Such an incentive arises if the improvement in terms of information flow exceeds the increasing link establishment cost. However, if the decay factor σ is close to 1 then the improvement in terms of information flow becomes marginal and, consequently, such an incentive to establish a link diminishes. By the same analogy, from an efficiency perspective the benefits to all agents in the network relative to the cost of establishing an extra path also diminishes if the decay is sufficiently small. As a result, there is at most only one path between any pair of agents. This small decay assumption is assumed in De Jaegher and Kamphorst (2015) and will be assumed throughout this paper [9] .

Network-related notations A subnetwork of g is a network g' such that $g' \subset g$. A network is said to be connected if there is a path between every pair of agents in the network. g', a subnetwork of g, is said to be a component of g if g' is a maximal connected subnetwork of g. A network is empty if no agent forms a link. An agent who has no link is called a singleton [10]. A non-empty component of a network or a network is minimal if there is at most one path between any pair of agents in the network. An agent i is called a link sender (receiver) if there is a link $xy \in g$ such that $x = i$ ($y = i$). A link ij is said to point towards another agent i' if j is contained in a path between i and i'. A link ij is said to point away from i' if j is not contained in a path between i and i'.

Network architectures We introduce some network architectures as follows [11] A minimally connected network is a *star* if there is an agent i such that $\bar{ij} \in g$ for every $j \neq i$ but $jk \notin g$ for every $j, k \neq i$. Such an agent i is called a central player. A *center-sponsored star* is defined likewise except that ij replaces $\bar{ij}$ in the aforementioned definition. A network such that each minimally connected group of agents is a star and a central player i of group l forms a link with the central player j of group l', where $l' \neq l$, is called an *interlinked star* network. Morever, if each star is center-sponsored then the network is said to be an interlinked center-sponsored star. An agent is a *bridge* agent if he has a link with the central agents of at least two stars. A *generalized interlinked star* network is an interlinked star network where there is a unique bridge agent between the center of each star [12].

Lastly, I introduce some notations concerning information flow. Let g be a minimally connected network. Due to the fact that there is only one path between every pair of agents in g, a removal of the link $\bar{ij} \in g$ further splits g into two discon-

[9] More rigorously, see Lemma 1 in the Appendix of this paper.

[10] The definitions of singleton and empty component follow Bala and Goyal (2000a).

[11] Two networks are said to share the same architecture if one network can be obtained from the other by permuting the strategies of agents.

[12] These definitions are borrowed from Unlu (2018).

nected subnetworks - one containing i and the other one containing j. Let $D^i_{i\bar{j}}(g)$ and $D^j_{i\bar{j}}(g)$ denote these two subnetworks respectively. Furthermore, let $N^i_{ij}(g)$ and $N^j_{ij}(g)$ be the sets of agents in these two networks respectively.

We now use these notations to introduce how we can modify a network as follows. $g - ij$ is defined as $g - ij = g \backslash \{ij\}$. That is, $g - ij$ is modified from g by simply removing the link $ij \in g$. Similarly, $g + ij = g \cup \{ij\}$ is the network g modified by adding the link ij. Of course, $g - ij + kl = (g \backslash \{ij\}) \cup \{kl\}$ is the network g modifed by removing the link ij and adding the link kl. Next, consider two disconnected networks g' and g'' and assume that agents i and j are in g' and g'' respectively, then we define $g' \oplus_{ij} g''$ as the network that results from joining the two networks g' and g'' through the addition of the link ij. That is, $g' \oplus_{ij} g'' = g' \cup g'' \cup \{ij\}$. Note that if $ij \in g$ then $D^i_{ij}(g) \oplus_{ij} D^j_{ij}(g) = g$. Lastly, we also use these notations to define specific types of agents. For any $ij \in g$ where g is minimally connected, if $N^j_{ij}(g) = \{j\}$, we say that j is a terminal agent and the link ij is a terminal link.

Quantity of information . Let V be the value of information that each agent possesses. Let $I_{ij}(g) = \sigma^{d_{ij}(g)} V$. That is, I_{ij} is the quantity of information that i receives from j. Then we can define the total information that i receives from every agent in the network as $I_i(g) = \sum_{j \in N} I_{ij}(g) = \sum_{j \in N} \sigma^{d_{ij}(g)} V$. Observe that $I_{ii}(g) = V$, which means that i also benefits from his own information.

Cost Function and the payoffs Let c_{ij} be the cost that i pays to access j. Let $C : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ be a strictly increasing function. Let $N^S_i(g) \equiv \{j \in N : ij \in g\}$ denote the set of all agents with whom i establishes a link. The total link formation of an agent is defined as $C\left(\sum_{j \in N^S_i} c_{ij}\right)$. Thus, the general form of payoff is:

$$U_i(g) = I_i(g) - C\left(\sum_{j \in N^S_i} c_{ij}\right) \tag{1}$$

In most of the literature that concerns with efficiency [13], $C(\cdot)$ is assumed to be linear, agent homogeneity in link formation cost is also assumed and value of information is set such that $V = 1$. This leads to the following payoff:

$$\begin{aligned} U_i(g) &= I_i(g) - n^S_i c \\ &= \sum_{j \in N} \sigma^{d_{ij}(g)} - n^S_i c \end{aligned} \tag{2}$$

[13]Except Unlu (2018).

where $n_i^S = |N_i^S|$. In the main analysis section, our Proposition 2 aims to refine the results of Unlu (2018), which assumes player heterogeneity and a general cost function. Hence, the payoff for this proposition is:

$$U_i(g) = I_i(g) - C\left(n_i^S c\right) \tag{3}$$

Note that, unlike Unlu (2018), we do not impose any further restriction - strict concavity or convexity - on $C()$.

Nash networks Consider a network g^* such that a strategy of i is $g_i^* \subset g^*$. Let $g_{-i}^* = g^* \backslash g_i^*$ so that $g^* = g_i^* \sqcup g_{-i}^*$. g_i^* is said to be a best response of i if $U_i(g^*) \geq U_i\left(g_i \cup g_{-i}^*\right)$ for every g_i which is a strategy of i. g^* is said to be a Nash network if every agent chooses his best response.

Efficiency of a network Let $W(g) = \sum_{i-1}^{n} U_i(g)$. A network g *dominates* another network g' if $W(g) \geq W(g')$. A network g is efficient if it dominates every other network. Consider the payoff as in Eq. 3, we can express $W(g)$ as $W(g) = \sum_{i-1}^{n} I_i(g) - \sum_{i=1}^{n} C\left(\sum_{j \in N_i^S(g)} c_{ij}\right)$. We denote the first term on the right as $\bar{I}(g) = \sum_{i=1}^{n} I_i(g)$ and call it total informational quantity of the network g. Similarly, We denote the second term on the right as $\bar{C}(g) = \sum_{i=1}^{n} C\left(\sum_{j \in N_i^S(g)} c_{ij}\right)$ and call it total cost of the network g.

Next, consider two networks g and g' with the same set of agents N, we say that g and g' are *ls*-equivalent if for every $i \in N$ $n_i^S(g) = n_i^S(g')$. For example, if we modify a network g into $g - ij + ik$ where $ij \in g$ and $k \in N_{ij}^j(g)$, then clearly g and $g - ij + ik$ are *ls*-equivalent since the only difference between these two networks is that i replaces the link ij with ik. Of course, if g and g' are *ls*-equivalent then $\bar{C}(g) = \bar{C}(g')$. A network g' is said to be an improved network of g if g' and g are *ls*-equivalent but $\bar{I}(g') > \bar{I}(g)$. This leads us to the following remark:

Remark 1. *Assuming the payoff as in Eq. 3, player cost heterogeneity and small decay, g' dominates g if g' is an improved network of g.*

This remark will play an important role in the proof of Proposition 2, which is the characterization of an efficient network under the assumption of player heterogeneity and small decay.

2.1 Efficient Link Receiver and Best Informed Agent: Definitions

Our first step is to define the concept of efficient link receiver, which will be used in Proposition 1 to show that a best response of any agent is to have an efficient link receiver as his partner. We begin by introducing the following notations. In a

(a) $g = g - i_0 j_0 + i_0 j_0$ **(b)** $g - i_0 j_0 + i_0 j_1$

Figure 2: Example 1

minimally connected network g, we define $\bar{I}_{i\to j} = \sum_{l\in N_{ij}^{i}(g)} \sum_{k\in N_{ij}^{j}(g)} I_{kl}$. That is, $\bar{I}_{i\to j}$ is the total informational quantity that the set of minimally connected agents $N_{ij}^{j}(g)$ received from the set of minimally connected agents $N_{ij}^{i}(g)$ via the link $ij \in g$.

Definition 1 (Efficient transmitter)**.** *In a network g, consider a link $x\bar{y} \in g$. j' is superior to j'' as a transmitter with respect to the link $x\bar{y}$ if: (i) $j', j'' \in N_{x\bar{y}}^{y}(g)$ and (ii)*

$$\bar{I}_{x\to j'}\left(g - x\bar{y} + x\bar{j}'\right) \geq \bar{I}_{x\to j''}\left(g - x\bar{y} + x\bar{j}''\right)$$

Moreover, j' is said to be an efficient transmitter with respect to the link $x\bar{y}$ if j' is superior to every agent in $N_{x\bar{y}}^{y}(g)$ as a transmitter.

Definition 2 (Efficient link receiver)**.** *In a network g, a link receiver j is said to be an efficient link receiver if j is an efficient transmitter with respect to every link $xy \in g$ such that $y = j$.*

Definition 3 (Efficient link sender)**.** *In a network g, a link sender i is said to be an efficient link sender if i is an efficient transmitter with respect to every link $xy \in g$ such that $x = i$.*

That is, consider an agent i that is not connected to a minimally connected subset of agents M. An efficient transmitter j is an agent in the set M that maximizes the total informational quanity that all agents in M receives from i if there happens to be a link between i and j himself. An efficient link receiver (sender) is defined likewise, except that j becomes a link receiver (sender) from(to) i. See Example 1 for an illustration.

Example 1. *Let us assume agent homogeneity and the payoff as in Eq. 2. Consider the network g in Figure 2. Observe that $N_{i_0 j_0}^{j_0} = \{j_0, j_1, j_2\}$. We have $\bar{I}_{i_0\to j_0}(g - i_0 j_0 + i_0 j_0) = \bar{I}_{i_0\to j_0}(g) = \sum_{l\in N_{i_0 j_0}^{i_0}} I_{j_0 l}(g) + \sum_{l\in N_{i_0 j_0}^{i_0}} I_{j_1 l}(g) + \sum_{l\in N_{i_0 j_0}^{i_0}} I_{j_2 l}(g) = \sigma(1+2\sigma) + \sigma^2(1+2\sigma) + \sigma^2(1+2\sigma) = (1+2\sigma)(\sigma+2\sigma)$, whereas $\bar{I}_{i_0\to j_1}(g - i_0 j_0 + i_0 j_1) = \bar{I}_{i_0\to j_2}(g - i_0 j_0 + i_0 j_2) = (1+2\sigma)\left(\sigma+\sigma^2+\sigma^3\right)$. Hence, $\bar{I}_{i_0\to j_0}(g - i_0 j_0 + i_0 j_0) > \bar{I}_{i_0\to j_1}(g - i_0 j_0 + i_0 j_1)$. This example shows that j_0 is an efficient link receiver in g.*

Due to these definitions we are able to establish the following remark.

Remark 2. *Under the assumption of agent homogeneity, small decay and payoff as in Eq. 2, the following can be said about an efficient network*

1. *An efficient network is such that every link receiver and every link sender is an efficient transmitter. That is, every agent in the network is an efficient transmitter.*

2. *Since a star is a unique efficient network within the class of nonempty minimal network (See Proposition 5.5 in Bala and Goyal (2000a)), a star is a unique network such that every link sender and every link receiver is efficient.*

Definition 4 (Best informed agent)**.** *Let $M \subset N$ be a minimally connected subset of agents and $i, j \in M$. i is better informed than j in the set M if*

$$\sum_{k \in M} I_{ik}(g) \geq \sum_{k \in M} I_{jk}(g)$$

If i is better informed than every other agent in the set M, then i is said to be a best-informed agent in the set M. Alternatively, if $M = N^{x}_{x\bar{y}}(g)$ for a link $x\bar{y} \in g$, we then say that i is better informed than j with respect to the link $x\bar{y}$ and i is best-informed with respect to the link $x\bar{y}$.

That is, within the set of minimally connected agents M an agent is a best-informed agent if he receives more information (from other agents in the set M) than every other agent in the set M does [14].

Observe the following differences between that the definitions of best-informed agent and efficient transmitter. The definition of best-informed agent revolves around the informational quantity *received by each individual*, while the definition of efficient transmitter revolves around the total informational quantity *received by a group of agents*. Another major difference is that best-informed agent is the concept that concerns information that is exchanged *within* a group of agent, while efficient transmitter is a concept that concerns information that one group of agents receive from another group. Despite these differences, it turns out that, surprisingly, the identity of an efficient transmitter and best-informed agent is identical. This is proven in the next subsection.

3 Main Analysis: Proposition 1 and 2

I now relate the concept of efficient transmitter and the concept of best informed agent by establishing a surprising result: the identity of best informed agent and the identity of efficient transmitter are identical.

[14]See Example 1 in De Jaegher and Kamphorst (2015)

Proposition 1. *Consider a link $x\bar{y} \in g$, j' is superior to j'' as a transmitter if and only if j' is better informed than j''. Consequently, (i) j' is an efficient transmitter if and only if j' is best informed with respect to the link $x\bar{y}$ and (ii) in a Nash network every link receiver is an efficient link receiver.*

Proof. By Definition 1, we simply need to show that $\bar{I}_{x\to j'}\left(g - x\bar{y} + x\bar{j}'\right) \geq \bar{I}_{x\to j''}\left(g - x\bar{y} + x\bar{j}''\right)$. To do so let $K = I_x\left(D^x_{x\bar{y}}(g)\right)$. Hence, in $g - x\bar{y} + x\bar{j}'$ agent j' receives informational quanity of σK from the group of agents in $D^x_{x\bar{y}}(g)$. Since j' is the agent that transmits information of the group $N^x_{x\bar{y}}(g)$ to the group $N^y_{x\bar{y}}(g)$, an agent $k \in N^y_{x\bar{y}}(g)$, $k \neq j'$, receives $\sigma\sigma^{d_{j'k}(g)}K$ from the group of agents $N^x_{x\bar{y}}(g)$. Hence, we conclude that $\bar{I}_{x\to j'}\left(g - x\bar{y} + x\bar{j}'\right) = \sum_{l\in N^x_{x\bar{y}}(g)}\sum_{k\in N^y_{x\bar{y}}(g)} I_{kl} = \sum_{k\in N^y_{x\bar{y}}(g)} \sigma\sigma^{d_{j'k}(g)}K = \sigma K I_{j'}\left(D^y_{x\bar{y}}(g)\right)$. By the same analogy, $\bar{I}_{x\to j''}\left(g - x\bar{y} + x\bar{j}''\right) = \sigma K I_{j''}\left(D^y_{x\bar{y}}(g)\right)$. Hence, $\bar{I}_{x\to j'}\left(g - x\bar{y} + x\bar{j}'\right) \geq \bar{I}_{x\to j''}\left(g - x\bar{y} + x\bar{j}''\right)$ if and only if $I_{j'}\left(D^y_{x\bar{y}}(g)\right) \geq I_{j''}\left(D^y_{x\bar{y}}(g)\right)$, which is what we intend to proof. For part (ii), simply recall from Remark 1 in De Jaegher and Kamphorst (2015) that if a link $ij \in g$ and g is Nash then j is a best-informed agent with respect to ij.

□

What drives this surprising result? Intuitively, both concepts revolve around a network position that causes minimum decay. Indeed, Lemma 1 in De Jaegher and Kamphorst (2015) shows that an agent whose network position is the 'in the middle' of the other agents tends to be a best informed agent because his position implies that each path through which information *arrives to him* is relatively short, resulting in him suffering less decay [15]. By the same analogy, being in the middle means that each path through which information *arrives to other agents* from him is also relatively short, resulting in the fact that information that reaches to other agents suffers relatively less decay. In other words, once an agent's position is optimal for receiving information for his own benefit, it also becomes optimal for transmitting information for the benefits of others. Example 2 below illustrates this intuition.

Example 2. *As a continuation of Example 1, consider again the network g with all other assumptions as in Example 1. Observe that in $D^{j_0}_{i_0 j_0}(g)$, which consists of j_0, j_1, j_2, the agent j_0 is a best-informed agent since j_0 is the middle agent in $D^{j_0}_{i_0 j_0}(g)$. At the same time, Example 1 shows that j_0 is also an efficient link receiver. Thus, j_0 is both an efficient link receiver and a best-informed agent in the network g.*

On one hand, Proposition 1 above tells us that in a Nash network every link receiver is an efficient link receiver. On the other hand, Remark 1 in the previous

[15] More generally, even if there is no agent whose position is in the middle, Lemma 1 in Charoensook (2020) shows that a "positionally optimal" agent always exists.

subsection states that every link sender and every link receiver are efficient in an efficient network. This allows us to establish the following relations between a Nash network and an efficient network.

Remark 3 (Partial efficiency of Nash network)**.** 1. *In the case of agent homogeneity, small decay and payoff as in Equation 2 , a Nash network and an efficient network share the following similarity: every link receiver is an efficient link receiver.*

2. *An efficient network, which is a star, and a Nash network have the following difference* [16]*. In a Nash network a link sender is not necessarily efficient, while it is so in an efficient network.*

Observe that Part (i) of this corollary implies that the strategic decision of every link sender to choose a link receiver for the best of his own interest does lead to a partially socially desirable outcome, since the identity of link receiver that maximizes the payoff of a link sender and the identity of link receiver who could efficiently transmit information to other agents are identical. Put differently, every Nash network is partially efficient. Observe further that Part (ii) of this corollary is driven by the assumption of agent homogeneity. That is, since link establishment cost is homogeneous, what matters is that total information has to be maximized for a network to be efficient. This in turn guarantees that in an efficient network every link sender has to be an efficient transmitter, while it is not so in the case of Nash network. If player heterogeneity is assumed, this line of reasoning certainly breaks down. Can, then, an efficient network be characterized? Proposition 2 below answers this question.

Proposition 2 (Characterization of efficient networks: player heterogeneity case)**.** *An efficient network g in the case of player heterogeneity, small decay and payoff as in Eq. 3 has at most one unique nonempty component, which contains a best-informed agent i^* that possesses the following properties: (a) every link sender has a link with a best-informed agent i^*, that is, for every i such that $n_i^S(g) \geq 1$ and $i \neq i^*$ it holds true that $ii^* \in g$ and (b) i^* is a largest sponsor* [17]*. Consequently, this unique nonempty component of efficient network is such that:*

(i) its diameter is at most 4.

(ii) the best-informed agent i^* is a unique multi-recipient agent such that every link he sponsors is a terminal link, and every other link that is not received by i^* is also a terminal link.

(iii) if its diameter is 4*, then this unique nonempty component of efficient network is a generalized interlinked center-sponsored star such that i^* is a bridge agent.*

[16]For the result that a star is a unique architecture of (non-empty) efficient network, see the second paragraph below Proposition 4.3 in Bala and Goyal (2000a)

[17]Note that a largest sponsor does not need to be a minimum cost player, which is assumed by imposing some restrictions as in Proposition 5 in Unlu (2018)

Remark 4. *By the definition of player heterogeneity link formation cost does not depends on the identity of link receivers. Hence, Point 1 in Remark 3 also holds for the case of player heterogeneity while Point 2 in Remark 3 no longer holds. For example, consider the network on the right in Figure 3 the agent j_0 is not an efficient link sender since it is not efficient as a transmitter with respect to the links j_0j_1 and j_0j_2.*

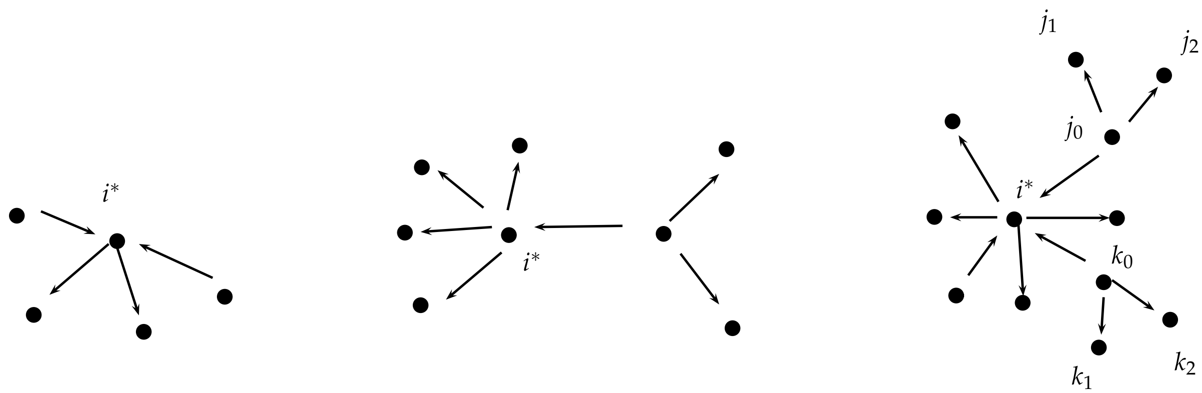


Figure 3: Three networks that are efficient according to Proposition 2. Observe that the network on the right is a generalized center sponsored star in which i^* is a bridge agent.

Figure 3 illustrates efficient networks according to Proposition 2. While the proof of this proposition and related lemmata are relegated to the appendix, I elaborate on the intuition of Lemma 7, which is a primary foundation of this proposition as follows. First, let a network g be a network such that there is a link sender i who does not have a link with the best-informed agent i^* in g. Let g' be a network modified from g by replacing an existing link ij with a new link ii^*, which means that i replaces a link to a non-best informed agent with a new link with the best-informed agent. In Lemma 7, I show that g' is an 'improved' network from g in the sense that g' has the same total link formation cost as that of g, while the total informational quantity is improved from that of g. Hence, the network g' dominates g. A repetition of this line of reasoning allow us to conclude that an efficient network has a diameter of at most 4 and posseses properties mentioned in Proposition 2. Example 3 and Example 4 illustrate this intuition.

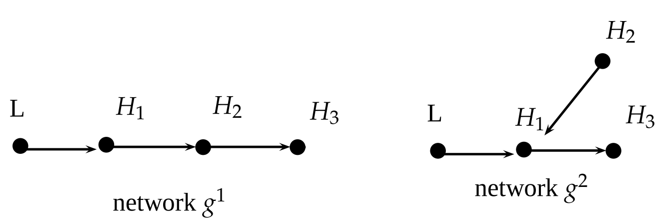


Figure 4: g^1 and g^2.

Example 3. *Consider the following example, which is slightly modified from Example 1 in Unlu (2018). Let* $n = 4$*. Let there be 1 minimum cost player L and 3 high cost players denoted by* H_1, H_2, H_3*. Let* $\sigma = 0.99$*,* $V = 100$*,* $c_L = 2$*,* $c_{H_1} = c_{H_2} = c_{H_3} = 3$*, with the payoff* $\pi_i(g) = 100\sum_j \sigma^{d_{ij}(g)} - \left(n_i^d c_i^2\right)$*. As in Unlu (2018), without decay we have that the line network* g^1 *and the star network* g^2 *in the above Figure 3 are efficient. Note that* $g^2 = g^1 - H_2H_3 - H_1H_2 + H_1H_3 + H_2H_1$ *so that* g^2 *and* g^1 *are ls-equivalent and* $\bar{C}(g^1) = \bar{C}(g^2)$*.*

However, if we assume the existence of small decay then clearly $\bar{I}(g^2) > \bar{I}(g^1)$*, which entails that* g^2 *is an improved network of* g^1*. Indeed,* g^2 *is an efficient network while* g^1 *is not. Note that in* g^2 H_1 *is the best-informed agent and link senders* H_2 *and L establish links with* H_1*, which reflects property (a) in Proposition 2.*

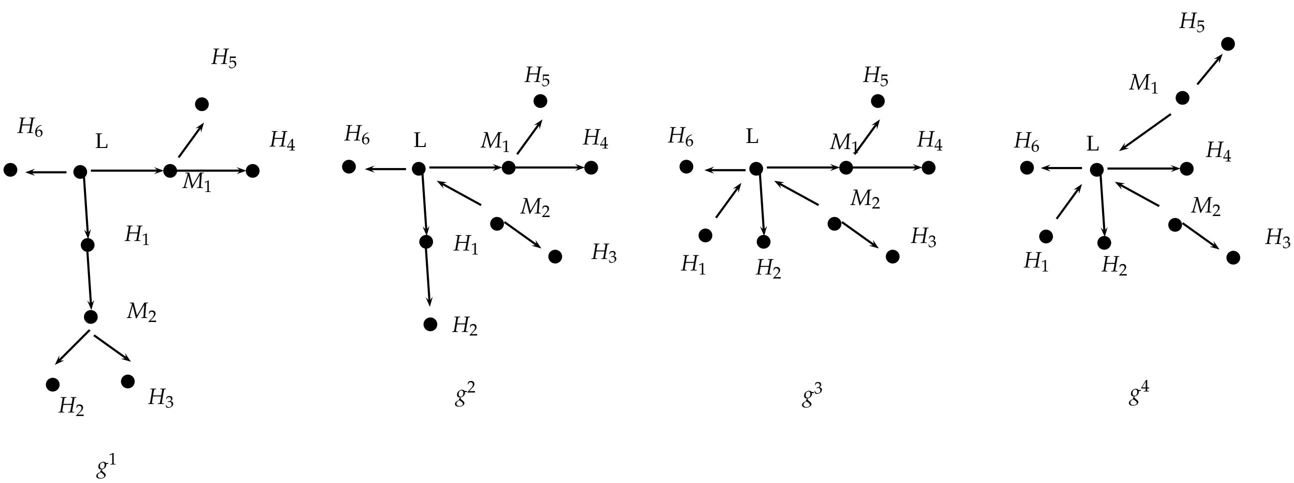


Figure 5: g^1 and g^2.

Example 4. *Consider the payoff as in Example 3 and let there be 9 agents consisting of 3 groups:* $\{L\}$*,* $\{M_1, M_2\}$*,* $\{H_1, H_2, H_3, H_4, H_5, H_6\}$ *and set* $c_L = 1.2$*,* $c_{M_1} = c_{M_2} = 1.5$*,* $c_{H_1} = c_{H_2} = c_{H_3} = c_{H_4} = c_{H_5} = 3$*. That is, we have 1 low cost agent, 2 intermediate-cost agents and 6 high cost agents. Without decay, networks* g^1*,* g^2*,* g^3*,* g^4 *in the above figures are efficient. Observe that* $g^2 = g^1 - M_2H_2 - H_1M_2 + M_2L + H_1H_2$*,* $g^3 = g^2 - LH_1 - H_1H_2 + LH_2 + H_1L$*,* $g^4 = g^3 - LM_1 - M_1H_4 + LH_4 + M_1L$*. Hence, these networks are ls-equivalent, i.e.,* $C(g^1) = C(g^2) = C(g^3) = C(g^4)$*, because L-agent sponsors 3 links, M-agents sponsor 2 links and H-agents sponsor just 1 link or no link in all these 4 networks. Similar to the previous example these networks have the same total informational quantity due to the assumption of no decay so that no network dominates another* [18]*.*

[18] Here we can (partially check) that these profiles are efficient. Say, if we increase number of links of L to 4 (from 3) but decreasing that of M to 1 (from 2), we have $16x1.2^2 - 9x1.2^2 = 10.08 > 4x1.5^2 - 1x1.5^2 = 6.75$. Also, if we increase number of links of L to 4 (from 3) but decreasing that of M to 1 (from 2), we have $16x1.2^2 - 9x1.2^2 = 10.08 > 4x1.5^2 - 1x1.5^2 = 6.75$. Thus, any change will accordingly increase the total cost.

However, if we begin to assume small decay then only g_4 is efficient. Indeed, $\bar{I}\left(g^1\right) < \bar{I}\left(g^2\right) < \bar{I}\left(g^3\right) < \bar{I}\left(g^4\right)$. Note that this fact exemplifies the intuition of Lemma 7 described above in the sense that any network that contains a link sender that has no link with the best informed agent can be modified into an improved network. Specifically L is the best-informed agent in each of these networks and g^1 is modified into g^2 by replacing M_2H_2 with M_2L, g^2 is modified into g^3 by replacing H_1H_2 with H_1L and g^3 is modified into g^4 by replacing M_1H_4 with M_1L. Such modifications allow the total information quantity to increase while maintaining the same level of total link formation cost. Note further that g^4 has properties that are precisely as described in Proposition 2. That is, $D\left(g^4\right) = 4$, the largest sponsor L sponsors links only to terminal agents and every other agent who is not a terminal agent sponsors a link to L and g^4 is a generalized interlinked center-sponsored star with L being the bridge agent.

I further elaborate on how Proposition 2 fills some gaps in the literature as follows. In Unlu (2018), a similar form of agent heterogeneity - yet without decay - is studied. Due to the absence of information decay, every network does have the same level of total information. Hence, an important aspect left unstudied in Unlu (2018) yet fulfilled by my Proposition 2 is the potential tradeoff between achieving higher total information and increasing total cost, which could arise when the diameter of a network is shortened. Indeed, the absence of this tradeoff in Unlu (2018) allows an efficient network to have a maximal diameter (a line). This shows that there is a substantial tension between stability and efficiency, since we know that with the same set of assumption a unique non-empty Nash network is either a star or a set of disconnected stars. To reconcile the existence of this tension, Unlu (2018) imposes several strong, nonintuitive restrictions either on the payoff or the cost structure (see, e.g., Condition 1 in Unlu (2018)). On the contrary, my Proposition 2 and Remark 4 show that by simply introducing a small degree of decay we do not need any further restriction to reconcile this tension. Also, due to the assumption of small decay large diameters networks are naturally eliminated from being efficient.

4 Conclusion

In this note, I show that the tension between efficiency and stability can be partially reconciled in the simple model of two-way flow network with nonrival information pioneered by Bala and Goyal (2000a) if small decay of information is assumed. Specifically, I show that every Nash network is partially efficient in the sense that every link receiver allows information to flow efficiently. I further show that this result extends to the case of player heterogeneity, which is a form of heterogeneity that is well studied in the literature. I also characterize the efficient networks under the assumption of player heterogeneity and small decay in fine details in Proposition 2. The assumptions of player heterogeneity and small decay allow for analysis of the potential tradeoff between achieving higher total information and incurring higher total information quantity, which could arise from shortening the diameter of a network. This characterization also resolves some technical difficulties in characterizing efficient networks when no decay is assumed as in Unlu (2018).

It is important to keep in mind, however, that this note follows the convention in the literature by assuming that the benefit of each agent is precisely the total information he receives in the network. Indeed, to the knowledge of the author there is no existing work in the literature of efficient networks that assumes a more general form of benefit function. Consequently, how a more general form of benefit function could impact the characteristic of efficient network becomes a future research question to explore.

5 Appendix: Proof of Proposition 2

The proof of Proposition 2 rests upon several lemmata, which I categorize into three groups follows. In Part 1, Lemma 1 states that if information decay is sufficiently small then an efficient network is minimal. In Part 2, Lemma 2 allows us to quantify total information quanitity of a minimally connected network using the fact that a removal of a link in minimal connectedness always splits the network into two disconnected components. I then make use of Lemma 2 to establish Lemma 3, which allows for the comparison of two networks that are partially similar in the sense that the two networks can contain subnetworks whose structures are identical. In Part 3, Lemma 7 shows that a minimally connnected network is not efficient if it has a link sender that does not establish a link with a best-informed agent. This Lemma, which is built upon Lemmata 4, 5 and 6, becomes the most important building block of Proposition 2.

5.1 Preliminary Lemmata Part 1: If Decay is Small, Then an Efficient Network Is Minimal

Lemma 1. *There exists a threshold level of decay σ_M such that for all $\sigma > \sigma_M$ every efficient network is minimal*

Proof. By Proposition 1 in Unlu (2018) in absence of decay every efficient network is minimal. Since the total payoff $W(g)$ is continuous in σ, it follows that such $\sigma_M < 1$ exists [19]. □

5.2 Preliminary Lemmata Part 2: Comparing total Information Quantities of Two Networks That Are Partially Similar.

Lemma 2. *Let g' and g' be two minimally connected networks. Let N' aand N'' be the sets of agents in g' and g'' respectively. Let $x \in N'$ and $y \in N''$. Define $g = g' \oplus_{x\bar{y}} g''$. Then,*

[19] This proof is analogous to Lemma 4 in De Jaegher and Kamphorst (2015)

1. $\sum_{i \in N'} I_i(g) = \bar{I}(g') + \sigma I_x(g') I_y(g'')$

2. $\sum_{i \in N''} I_i(g) = \bar{I}(g'') + \sigma I_x(g') I_y(g'')$

3. *and hence, as a corollary,* $\bar{I}(g) = \bar{I}(g') + \bar{I}(g'') + 2\sigma I_x(g') I_y(g'')$

Proof. First, consider $i \in N'$. Observe that in g' i receives information of agent in g'' via the agent x who, in turn, receives from the y with whom he is one-link away. This fact implies that $\sum_{l \in N''} I_{xl}(g) = \sigma I_y(g'')$ and $\sum_{l \in N''} I_{il}(g) = \sigma^{d_{ix}(g')} \sigma I_y(g'')$. Hence, we can express $\bar{I}_{y \to x}(g)$ - the total information that all agents in g' receives from g'' via the link $x\bar{y}$ - as:

$$\begin{aligned} \bar{I}_{y \to x}(g) &= \sum_{i \in N'} \left(\sum_{l \in N''} I_{il}(g) \right) \\ &= \sum_{i \in N'} \left(\sigma^{d_{ix}(g')} \sigma I_y(g'') \right) \\ &= \sigma I_x(g') I_y(g'') \end{aligned}$$

On the other hand, we know that the total information that all agents in g' exchange with each other is $\bar{I}(g')$. This fact and the above expression lead to

$$\begin{aligned} \sum_{i \in N'} I_i(g) &= \bar{I}(g') + \bar{I}_{y \to x}(g) \\ &= \bar{I}(g') + \sigma I_x(g') I_y(g'') \end{aligned}$$

This completes the first part of this prelemma. The second part of this lemma also follows the same analogy:

$$\sum_{i \in N''} I_i(g) = \bar{I}(g'') + \sigma I_y(g') I_x(g'')$$

Lastly, observe that part (iii) is simply a straightforward corollary of part (i) and (ii). This completes our proof.

□

Lemma 3. *Consider threee minimal networks g^1, g^2 and g' that are not connected to each other. Let N^1, N^2 and N' be the sets of agents in these three networks respectively. Let $x \in N^1$, $y \in N^2$, $z \in N'$. If $\bar{I}(g^1) \geq \bar{I}(g^2)$ and $I_x(g^1) \geq I_y(g^2)$ then it holds true that:*

$$\bar{I}\left(g^1 \oplus_{x\bar{z}} g'\right) \geq \bar{I}\left(g^2 \oplus_{y\bar{z}} g'\right)$$

and

$$I_x\left(g^1 \oplus_{x\bar{z}} g'\right) \geq I_y\left(g^2 \oplus_{y\bar{z}} g'\right)$$

Moreover, if $\bar{I}(g^1) > \bar{I}(g^2)$ or $I_x(g^1) > I_y(g^2)$ then both of the above inequalities become strict.

Proof. The proof is a straightforward application of Lemma 2, and hence is omitted.

□

5.3 Preliminary Lemmata Part 3: Any Minimally Connected Network That Has a Link Sender That Does Not Establish a Link with a Best-informed Agent in the Network Is Not Efficient.

Lemma 4. *In a minimally connected network g consider 3 (distinct) agents i, i', j' such that $j'i' \in g$, $i \in N^{i'}_{j'i'}(g)$ and $I_i(g) \geq I_{i'}(g)$. Define g' as $g' = g - j'i' + j'i$, we have: (i) $\bar{I}(g') > \bar{I}(g)$, (ii) g' and g are ls-equivalent and (iii) g' dominates g. Conseqeuently, there is no indirect link that points towards a best-informed agent in an efficient network.*

Proof. Let us prove (i). First, observe that if we remove $j'i'$ from g we have $I_i(D^{i'}_{j'i'}(g)) > I_{i'}(D^{i'}_{j'i'}(g))$ because i is farther away from j' than i' is and we assume that $I_i(g) \geq I_{i'}(g)$. This inequality and the fact that $g' = g - j'i' + j'i = D^{j'}_{j'i'}(g) \oplus_{j'i} D^{i'}_{j'i'}(g)$ and $g = D^{j'}_{j'i'}(g) \oplus_{j'i'} D^{i'}_{j'i'}(g)$ allow us to apply Lemma 3 to conclude that $\bar{I}(g') > \bar{I}(g)$.

Next, part (ii) and the rest of this lemma simply follow from the fact that g' is defined as $g - j'i' + j'i$.

□

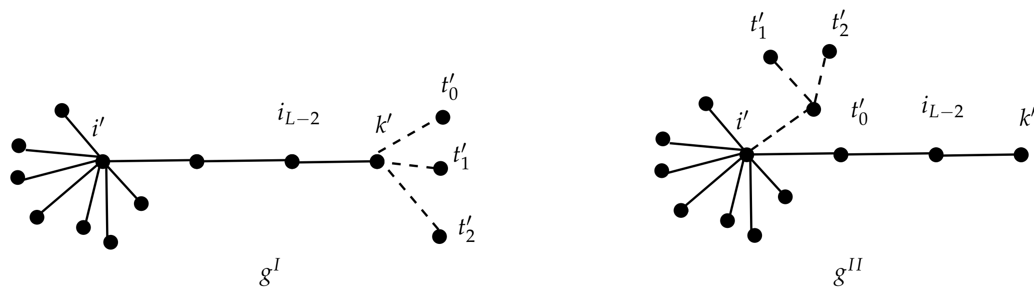


Figure 6: Examples of structures of networks g^I and g^{II} as in Lemma 5. Observe that in g^I the path between i' and t'_0 has more than two links, and the agent k' has links with 3 terminal agents. The dotted links are links whose positions are different in g^{II} and g^I. That is, $g^{II} = \left(g^I \backslash \{\overline{k't'_0}, \overline{k't'_1}, \overline{k't'_2}\}\right) \cup \{\overline{i't'_1}, \overline{t'_0t'_1}, \overline{t'_0t'_2}\}$

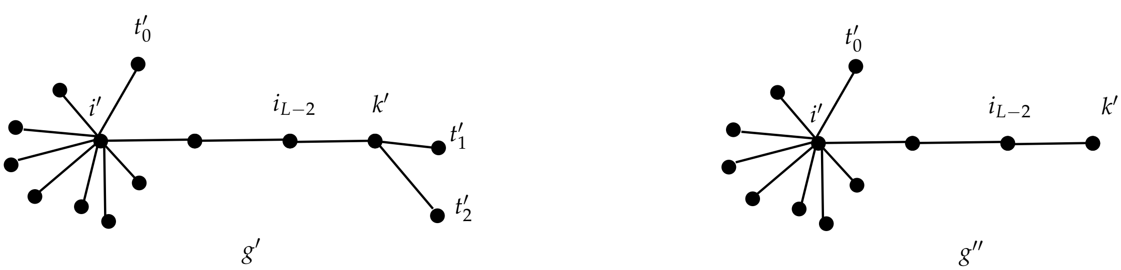


Figure 7: Examples of structures of networks Networks g', g'' in Lemma 5. Observe that the difference between the network g^I (see Figure 6) and g' is the location of agent t'_0, and g'' is simply the network g' with agents $t'_1, ..., t'_L$ removed

Lemma 5. *Let g^I be a network such that i' is the best informed agent and there exists a path of at least two links between i' and a terminal agent t'_0. Enumerate agents on this path as $i_1, ..., i_{L-2}, i_{L-1}, i_L$ where $i_1 = i', i_L = t'_0$ where $L \geq 2$. To ease notational cumbersomeness, further denote the agent i_{L-1} by k'. Let $N_{k'}\left(g^I\right) = \{i_{L-2}, t'_0, ..., t'_M\}$ be the set of all agents that are one-link away from k'. Define $g^{II} = \left(g^I \backslash \{\overline{k't'_0}, ..., \overline{k't'_M}\}\right) \cup \{\overline{i't'_0}, \overline{t'_0t'_1} + ...\overline{t'_0t'_M}\}$ (See Figure 6). Then it holds true that $\overline{I}\left(g^{II}\right) > \overline{I}\left(g^I\right)$.*

Proof. First, define $g' = g^I - \overline{k't'_0} + \overline{i't'_0}$ so that $g' = D^{k'}_{\overline{k't'_0}}(g^I) \oplus_{\overline{i't'_0}} \{t'_0\}$ (see Figure 6), where $\{t'_0\}$ is a network that consists of only one agent, t'_0. We divide the proof into two steps: (1) show that $\overline{I}\left(g'\right) > \overline{I}\left(g^I\right)$ and (2) show that $\overline{I}\left(g^{II}\right) \geq \overline{I}\left(g'\right)$ so that $\overline{I}\left(g^{II}\right) > \overline{I}\left(g'\right)$.

Step 1: $\overline{I}\left(g'\right) > \overline{I}\left(g^I\right)$. First, observe that the only difference between g' and g^I is the position of agent t'_0. In g' t'_0 is connected to i' while in g^I t'_0 is connected to agent k. Hence, a removal of the agent t'_0 from either g^I or g' results precisely in the same network. Let this network be $\tilde{g}$ and let $\{t'_0\}$ be the network that contains only one agent t'_0. That is, $g^I = \tilde{g} \oplus_{\overline{k't'_0}} \{t'_0\}$ and $g' = \tilde{g} \oplus_{\overline{i't'_0}} \{t'_0\}$. Observe that $\tilde{g}$ in i' is strictly better informed than k' because i' is assumed to be best informed in g^I and i' is farther from t'_0 than k' in g^I. Hence, we can apply Lemma 3 to conclude that $\overline{I}\left(g'\right) > \overline{I}\left(g^I\right)$.

Step 2: $\overline{I}\left(g^{II}\right) \geq \overline{I}\left(g'\right)$. First, define another network g'' as $g'' = \{xy \in g' | x, y \neq t'_1, ..., t'_M\}$. That is, g'' is the network g' modified by removing agents $t'_1, ..., t'_M$ (see Figure 7). Observe that $g'' + \overline{t'_0t'_1} + \overline{t'_0t'_2} + ... + \overline{t'_0t'_M} = g^{II}$ and $g'' + \overline{k't'_1} + \overline{k't'_2} + ... + \overline{k't'_M} = g'$. This allow us to further divide our proof into two substeps: (a) show that $I_{t'_0}\left(g''\right) \geq I_{k'}\left(g''\right)$, and (b) by applying multiple repetition of Lemma 3 to (a), conclude that $\overline{I}(g^{II}) = \overline{I}\left(g'' + \overline{t'_0t'_1} + \overline{t'_0t'_2} + ... + \overline{t'_0t'_M}\right) \geq \overline{I}\left(g'\right) = \overline{I}\left(g'' + \overline{k't'_1} + \overline{k't'_2} + ... + \overline{k't'_M}\right)$.

Substep (i): $I_{t'_0}\left(g''\right) \geq I_{k'}\left(g''\right)$. There are two cases: (a) in g^I the path between i' and t'_0 has precisely two links so that in g'' both t'_0 and k' are one-link away from i' (note that both are terminal agents) and (b) and in g^I the path between i' and t'_0 has more than two links.

For case (a), because both are terminal agents that are one-link away from i' it follows that $I_{t'_0}\left(g''\right) = I_{k'}\left(g''\right)$. For case (b), recall that we enumerate the sequence of agents in this path as $i', ..., i_{L-2}, k', t'_0$. Observe that i' is better informed than i_{L-2} in g'' since i' is also better informed than i_{L-2} in g' (which is due to the fact that i' is best-informed in g^I). Consequently, t'_0 receives information from a better-informed agent than k' does. Hence, $I_{t'_0}\left(g''\right) > I_{k'}\left(g''\right)$.

Substep (ii): if $I_{t'_0}\left(g''\right) \geq I_{k'}\left(g''\right)$ then $\overline{I}\left(g^{II}\right) \geq \overline{I}\left(g'\right)$. Applying Lemma 3 to the result of Step (i) that $I_{t'_0}\left(g''\right) \geq I_{k'}\left(g''\right)$ we have:

$$I_{t'_0}\left(g'' + \overline{t'_0t'_1}\right) \geq I_{k'}\left(g'' + \overline{k't'_1}\right)$$

and

$$\bar{I}\left(g'' + \overline{t'_0 t'_1}\right) \geq \bar{I}\left(g'' + \overline{k' t'_1}\right)$$

Indeed, multiple repetitions of the application of Lemma 3 lead to

$$I_{t'_0}\left(g'' + \overline{t'_0 t'_1} + \overline{t'_0 t'_2} + \ldots + \overline{t'_0 t'_M}\right) \geq I_{k'}\left(g'' + \overline{k' t'_1} + \overline{k' t'_2} + \ldots + \overline{k' t'_M}\right)$$

and

$$\bar{I}\left(g'' + \overline{t'_0 t'_1} + \overline{t'_0 t'_2} + \ldots + \overline{t'_0 t'_M}\right) \geq \bar{I}\left(g'' + \overline{k' t'_1} + \overline{k' t'_2} + \ldots + \overline{k' t'_M}\right)$$

Observe that, on the left-hand side of the above inequality, the network $g'' + \overline{t'_0 t'_1} + \overline{t'_0 t'_2} + \ldots + \overline{t'_0 t'_M}$ is nothing else but g^{II} while, on the right-hand side of the above inequality, the network $g'' + \overline{k' t'_1} + \overline{k' t'_2} + \ldots + \overline{k' t'_M}$ is nothing else but g'. Thus, we conclude that $\bar{I}\left(g^{II}\right) \geq \bar{I}\left(g'\right)$, which is what we intend to prove. □

Lemma 6. *Let g^1 be a network such that i^* is a best informed agent and there exists a path of at least two links between i^* and a terminal agent t_0. Enumerate agents in this path as $i^*, \ldots, j, k, t_0$. Let j accesses k and k accesses t_0. Define g^2 as $g^2 = g^1 - jk - kt_0 + jt_0 + ki^*$. Then it holds true that g^2 and g^1 are ls-equivalent and $\bar{I}\left(g^2\right) > \bar{I}\left(g^1\right)$.*

Proof. The proof that g^2 and g^1 are *ls*-equivalent is trivial and hence omitted. For $\bar{I}\left(g^2\right) > \bar{I}\left(g^1\right)$, simply observe that $\bar{g}^2$ and $\bar{g}^1$ share the same structure of information flow as $\bar{g}^{II}$ and $\bar{g}^I$ in Lemma 5 respectively. Thus, applying Lemma 5 it holds true that $\bar{I}\left(g^2\right) > \bar{I}\left(g^1\right)$. □

Lemma 7. *In a minimally connected network g with i^* being a best-informed agent, if there is an agent $i \neq i^*$ such that $ij \in g$ but $ii^* \notin g$, then there is an improved network of g denoted by g' such that $ii^* \in g'$.*

Proof. We split the proof into two cases: (i) g contains a link ij such that $j \neq i^*$ and ij points towards i^* and (ii) g contains no such a link.. For (i), define $g' = g - ij + ii^*$. Observe that g' and g are *ls*-equivalent. By Lemma 4 we also know that $\bar{I}\left(g'\right) > \bar{I}\left(g\right)$. Hence, g' is an improved network of g and g' dominates g. For (ii) since there is no such link ij as in case (i) we know that there is an indirect link jk that points away from i^* such that k is a terminal agent and j also receives a link from another agent i. Let $g'' = g^* - ij - jk + ik + ji^*$, which is *ls*-equivalent to g. By Lemma 6 $\bar{I}\left(g''\right) > \bar{I}\left(g\right)$. Similar to case (i) we conclude that g'' dominates g. This completes our proof. □

5.4 Proof of Proposition 2

Proof. I divide this proof into three parts as follows. Part 1 proves that an efficient network g in the case of player heterogeneity and small decay has at most one unique nonempty component. Part 2 proves properties (a), (i), (ii) and (iii) of this proposition. Part 3 proves property (b).

Part 1: an efficient network g has at most one unique nonempty component.

We prove by contradiction. Suppose there are multiple non-empty components. Denote two of them by g^1 and g^2. Let i_1^* and i_2^* be best-informed agents in these two components respectively. Without loss of generality let us assume that $I_{i_1^*}\left(g^1\right) \leq I_{i_2^*}\left(g^2\right)$. Onwards, we prove that there exists $ij \in g^1$ such that $\bar{I}\left(g - ij + ii_2^*\right) > \bar{I}\left(g\right)$ so that g is not efficient. We divide our proofs into two steps. First, we show that there exists $ij \in g^1$ such that $I_j\left(D_{ij}^j\left(g\right)\right) < I_{i_2^*}\left(g^2\right)$. Second, we show that if $I_j\left(D_{ij}^j\left(g\right)\right) < I_{i_2^*}\left(g^2\right)$ then $\bar{I}\left(g - ij + ii_2^*\right) > \bar{I}\left(g\right)$.

Step 1: there is $ij \in g^1$ such that $I_j\left(D_{ij}^j\right) < I_{i_2^*}\left(g^2\right)$. First, recall that for any $ij \in g^1$ it holds true that $g^1 = D_{ij}^i\left(g\right) \oplus_{ij} D_{ij}^j\left(g\right)$. Thus, $I_j\left(g^1\right) = \sum_{k \in N_{ij}^i(g)} I_{ij}(g) + \sum_{k \in N_{ij}^j(g)} I_{ij}(g)$. Hence, $\sum_{k \in N_{ij}^j(g)} I_{ij}(g) < I_j\left(g^1\right)$. Next, note that $\sum_{k \in N_{ij}^j(g)} I_{ij}(g) = I_j\left(D_{ij}^j\left(g\right)\right)$. Consequently, we conclude that $\sum_{k \in N_{ij}^j(g)} I_{ij}(g) = I_j\left(D_{ij}^j\left(g\right)\right) < I_j\left(g^1\right)$, which completes the proof of this first part.

Step 2: if $I_j\left(D_{ij}^j\right) < I_{i_2^*}\left(g^2\right)$ then $\bar{I}\left(g - ij + ii_2^*\right) > \bar{I}\left(g\right)$. Applying Prelemma 2, we express $\bar{I}\left(g^1\right)$ below:

$$\bar{I}\left(g^1\right) = \bar{I}\left(D_{ij}^i\left(g^1\right)\right) + \bar{I}\left(D_{ij}^j\left(g^1\right)\right) + 2\sigma I_i\left(D_{ij}^i\left(g^1\right)\right) I_j\left(D_{ij}^j\left(g^1\right)\right)$$

Thus:

$$\begin{aligned} \bar{I}\left(g\right) &= \bar{I}\left(g^1\right) + \bar{I}\left(g^2\right) + \bar{I}\left(g \backslash \{g^1 \cup g^2\}\right) \\ &= \Big[\bar{I}\left(D_{ij}^i\left(g^1\right)\right) + \bar{I}\left(D_{ij}^j\left(g^1\right)\right) \\ &\quad + 2\sigma I_i\left(D_{ij}^i\left(g^1\right)\right) I_j\left(D_{ij}^j\left(g^1\right)\right)\Big] + \bar{I}\left(g^2\right) + \bar{I}\left(g \backslash \{g^1 \cup g^2\}\right) \end{aligned} \tag{4}$$

Similarly, for $g - ij + ii_2^*$ we have:

$$\begin{aligned} \bar{I}\left(g - ij + ii_2^*\right) &= \bar{I}\left(D_{ij}^i \oplus_{ij} g^2\right) + \bar{I}\left(D_{ij}^j\left(g^1\right)\right) + \bar{I}\left(g \backslash \{g^1 \cup g^2\}\right) \\ &= \left[\bar{I}\left(D_{ij}^i\left(g^1\right)\right) + \bar{I}\left(g^2\right) + 2\sigma I_i\left(D_{ij}^i\left(g^1\right)\right) I_{i_2^*}\left(g^2\right)\right] \\ &\quad + \bar{I}\left(D_{ij}^j\left(g^1\right)\right) + \bar{I}\left(g \backslash \{g^1 \cup g^2\}\right) \end{aligned} \tag{5}$$

A comparison between the above two equations (Eq. 4 and 5) allow us to conclude that if $I_j\left(D^j_{ij}\right) < I_{i^*_2}\left(g^2\right)$, which is proven in Step 1, then $\bar{I}\left(g - ij + ii^*_2\right) > \bar{I}\left(g\right)$. This completes the proof of Step 2.

Part 2: Proofs of properties (a), (i), (ii) and (iii)

First, observe that property (a) is a straightforward corollary of Lemma 7. Next, to prove (i) first observe that by property (a) every path between a best-informed agent and a terminal agent has at most two links. Thus, if a best-informed agent is unique then every path betweeen two terminal agents has at most four links, which further results in properties (i). Thus, onwards it suffices to prove that a best-informed agent is unique.

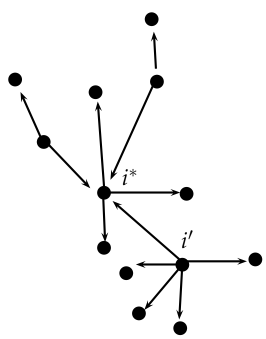
To prove by contradiction, let i^* and i^{**} be two best-informed agents. Let there be two paths that satisfy property (a) - a path t, j, i^* such that t is a terminal agent and j accesses both i^* and t, and a path t', j', i^{**} such that t' is a terminal agent and j' accesses both i^{**} and t'. By this construction we know that the links $j'i^{**}$ and ji^* constitute a path j', i^{**}, i^*, j. By Lemma 8 in De Jaegher and Kamphorst (2015) we further know that $i^* = i^{**}$. A contradiction. This completes the proof of property (i).

For property (ii), observe that due to the fact that $i^* = i^{**}$ we know that the path between the terminal agents t' and t'' is t', j', i^*, j'', t'' where t' and t'' receive links from j' and j'' respectively, which shows that every link not received by the best-informed agent i^* is a terminal link. This completes the proof of property (ii). Finally, observe that property (iii) is simply a corollary of properties (i) and (ii).

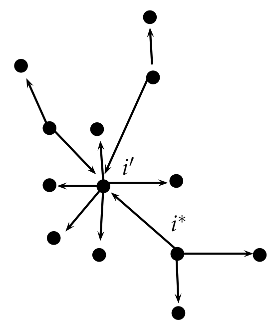
Part 3 (property (b)): i^* **is a largest sponsor**.

The proof is by contradiction. Suppose that an agent i' that accesses i^* is a largest sponsor. Let there be, beside i', L agents that access i^*. Enumerate these agents as $z_1, ..., z_L$. Next, let g^{*s} be a subnetwork of g^* such that $g^{*s} = \{xy \in g | x = i^* or\ i'\}$. Define $g^{'s}$ to be a network modified from g^{*s} as follows. If i^* sponsors no links $g^{'s} = g^{*s}$. If i^* sponsors at least one link to an agent, say j, then $g^{'s} = (g^{*s} \backslash \{i'i^*, i^*j\}) \cup \{i^*i', i'j\}$. Next, define $g' = \left(g^{'s} \cup_{k=1-L} D^{z_k}\left(g^*\right)\right) \cup_{k=1-L} \{z_l i'\}$. That is, g' is simply a network g^* with just one modification: i' becomes the bridge agent of the network instead of i^*. Observe that g^* can be expressed as $g^* = \left(g^{*s} \cup_{k=1-L} D^{z_k}\left(g^*\right)\right) \cup_{k=1-L} \{z_l i^*\}$, which necessitates that g^* and $g^{'s}$ are ls-equivalent. See Figure 8 for an illustration of networks g^*, g', g^{*s} and $g^{'s}$. Onwards we will prove that $\bar{I}\left(g'\right) > \bar{I}\left(g^*\right)$ so that g' dominates g^*. The proof is divided into three steps as follows.

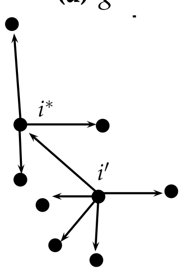
<u>Step 1: show that $I_{i'}\left(g^{'s}\right) > I_{i^*}\left(g^{*_s}\right)$</u>. For the case that i^* sponsors no link, recall that $g^{'s} = g^{*s}$. Observe that g^{*s} is nothing else but a center-sponsored star such that i' is the center who sponsors links to all other agents including i^*. Thus, $I_{i'}\left(g^{'s}\right) = I_{i'}\left(g^{*_s}\right) > I_{i^*}\left(g^{*_s}\right)$. For the case that i^* sponsors at least one link, we first



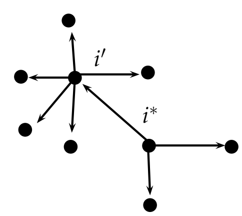
(a) g^*



(b) g'



(c) g^{*_s}



(d) g'_s

Figure 8: Networks as in the Part 2 of the proof. Note that in g^* i^* is not a largest sponsor while i' is $(n^s_{i^*}(g^*) = 3, n^s_{i'}(g^*) = 5)$.

simplify some notations as follows. Let $\nu_{i^*} = n^s_{i^*}(g^*)$ and $\nu_{i'} = n^s_{i'}(g^*)$. Observe that $\nu_{i^*} < \nu_{i'}$ since we prove by contradiction. Observe further that in g^{*s}, i^* has links with ν_{i^*} terminal agents and i' has links with $\nu_{i'} - 1$ terminal agents, while in $g^{'s}$ i^* has links with $\nu_{i^*} - 1$ terminal agents and i' has links with $\nu_{i'}$ terminal agents since we define $g^{'s} = (g^{*s} \backslash \{i'i^*, i^*j\}) \cup \{i^*i', i'j\}$ in the above paragraph. This leads to:

$$\begin{aligned} I_{i'}\left(g^{'s}\right) &= 1 + \sigma\left(\nu_{i'} + 1\right) + \sigma^2\left(\nu_{i^*} - 1\right) \\ I_{i*}\left(g^{*s}\right) &= 1 + \sigma\left(\nu_{i^*} + 1\right) + \sigma^2\left(\nu_{i'} - 1\right) \end{aligned}$$

since we assume that $\nu_{i'} > \nu_{i^*}$ we conclude that

$$I_{i'}\left(g^{'s}\right) > I_{i^*}\left(g^{*s}\right)$$

Step 2: show that $\bar{I}\left(g^{'s}\right) > \bar{I}\left(g^{*s}\right)$. First, recall that $i'i^* \in g^{*s}$ so that $g^{*s} = D^{i^*}_{i'i^*}\left(g^{*s}\right) \oplus_{i'i^*} D^{i'}_{i'i^*}\left(g^{*s}\right)$. Next, observe that $D^{i^*}_{i'i^*}\left(g^{*s}\right)$ and $D^{i'}_{i'i^*}\left(g^{*s}\right)$ are nothing else but stars with i^* and i' being the centers and with ν_{i^*} terminal agents and $\nu_{i'} - 1$ terminal agents respectively. Thus, by Lemma 2 we have:

$$\begin{aligned} \bar{I}\left(g^{*s}\right) &= \bar{I}\left(D^{i^*}_{i'i^*}\left(g^{*s}\right)\right) + \bar{I}\left(D^{i'}_{i'i^*}\left(g^{*s}\right)\right) + 2I_{i^*}\left(D^{i^*}_{i'i^*}\left(g^{*s}\right)\right) I_{i'}\left(D^{i'}_{i'i^*}\left(g^{*s}\right)\right) \\ &= \left\{\left[1 + \sigma\left(\nu_{i^*}\right)\right] + \left[\left(\nu_{i^*}\right)\left(1 + \sigma + \left(\nu_{i^*} - 1\right)\sigma^2\right)\right]\right\} + \left\{\left[1 + \sigma\left(\nu_{i'} - 1\right)\right]\right. \\ &\quad \left. + \left[\left(\nu_{i'} - 1\right)\left(1 + \sigma + \left(\nu_{i'} - 2\right)\sigma^2\right)\right]\right\} + \left\{2\sigma[1 + \sigma\left(\nu_{i^*}\right)][1 + \sigma\left(\nu_{i'} - 1\right)]\right\} \end{aligned}$$

Similarly, for $\bar{I}\left(g^{'s}\right)$. Recall that in g' i' has links with $\nu_{i'}$ terminal agents and i^* has links with $\nu_{i^*} - 1$ terminal agents and $g^{'s} = D^{i^*}_{i^*i'}\left(g^{'s}\right) \oplus_{i^*i'} D^{i'}_{i^*i'}\left(g^{'s}\right)$, we have:

$$\begin{aligned} \bar{I}\left(g^{'s}\right) &= \bar{I}\left(D^{i^*}_{i^*i'}\left(g^{'s}\right)\right) + \bar{I}\left(D^{i'}_{i^*i'}\left(g^{'s}\right)\right) + 2I_{i^*}\left(D^{i^*}_{i^*i'}\left(g^{'s}\right)\right) I_{i'}\left(D^{i'}_{i^*i'}\left(g^{'s}\right)\right) \\ &= \left\{\left[1 + \sigma\left(\nu_{i^*} - 1\right)\right] + \left[\left(\nu_{i^*} - 1\right)\left(1 + \sigma + \left(\nu_{i^*} - 2\right)\sigma^2\right)\right]\right\} + \left\{\left[1 + \sigma\left(\nu_{i'}\right)\right]\right. \\ &\quad \left. + \left[\left(\nu_{i'}\right)\left(1 + \sigma + \left(\nu_{i'} - 1\right)\sigma^2\right)\right]\right\} + \left\{2\sigma[1 + \sigma\left(\nu_{i^*} - 1\right)][1 + \sigma\left(\nu_{i'}\right)]\right\} \end{aligned}$$

Since we assume that $\nu_{i'} > \nu_{i^*}$, a comparion between the above two expressions allows us to conclude that $\bar{I}\left(g^{'s}\right) \geq \bar{I}\left(g^{*s}\right)$.

Step 3: show that $\bar{I}\left(g^{'}\right) > \bar{I}\left(g^{*}\right)$. Let $g^{*1} = D^{z_1}\left(g^*\right) \oplus_{z_1 i^*} g^{*s}$ and $g^{'1} = D^{z_1}\left(g^*\right) \oplus_{z_1 i'} g^{'s}$. Recall from the previous step that $I_{i'}\left(g^{'s}\right) > I_{i^*}\left(g^{*s}\right)$ and $\bar{I}\left(g^{'s}\right) \geq \bar{I}\left(g^{*s}\right)$.

Thus, by applying Lemma 3 we conclude that $\bar{I}\left(g^{'1}\right) > \bar{I}\left(g^{*_1}\right)$ and $I_{i'}\left(g^{'1}\right) > I_{i^*}\left(g^{*_1}\right)$. Similarly, define $g^{*_2} = D^{z_2}\left(g^*\right) \oplus_{z_2 i^*} g^{*_1}$ and $g^{'2} = D^{z_k}\left(g^*\right) \oplus_{z_2 i'} g^{'1}$. By the same analogy as the penultimate sentence we can apply Lemma 3 to conclude that $\bar{I}\left(g^{'2}\right) > \bar{I}\left(g^{*_2}\right)$ and $I_{i'}\left(g^{'2}\right) > I_{i^*}\left(g^{*_2}\right)$. Thus, by repeating this analogy L times we have $\bar{I}\left(g^{'L}\right) > \bar{I}\left(g^{*_L}\right)$ and $I_{i'}\left(g^{'L}\right) > I_{i^*}\left(g^{*_L}\right)$ where $g^{*_L} = D^{z_L}\left(g^*\right) \oplus_{z_L i^*} g^{*_{L-1}}$ and $g^{'L} = D^{z_L}\left(g^*\right) \oplus_{z_L i'} g^{'L-1}$. But then observe that $g^{*_L} = g^*$ and $g^{'L} = g^{'}$ (recall how g^* and $g^{'}$ are defined in the first paragraph of this proof). Consequently, we conclude that $\bar{I}\left(g'\right) > \bar{I}\left(g^*\right)$.

□

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