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Trust and Manipulation in Social Networks

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Summary

We investigate the role of manipulation in a model of opinion formation. Agents repeatedly communicate with their neighbors in the social network, can exert effort to manipulate the trust of others, and update their opinions about some common issue by taking weighted averages of neighbors' opinions. The incentives to manipulate are given by the agents' preferences. We show that manipulation can modify the trust structure and lead to a connected society. Manipulation fosters opinion leadership, but the manipulated agent may even gain influence on the long-run opinions. Finally, we investigate the tension between information aggregation and spread of misinformation.

Keywords: Social networks, Trust, Manipulation, Opinion leadership, Consensus, Wisdom of Crowds

JEL Classification: D83, D85, Z13

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Opinion Dynamics and Wisdom under Conformity

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Abstract

We study a dynamic model of opinion formation in social networks. In our model, boundedly rational agents update opinions by averaging over their neighbors' expressed opinions, but may misrepresent their own opinion by conforming or counter-conforming with their neighbors. We show that an agent's social influence on the long-run group opinion is increasing in network centrality and decreasing in conformity. Concerning efficiency of information aggregation or “wisdom” of the society, it turns out that misrepresentation of opinions need not undermine wisdom, but may even enhance it. Given the network, we provide the optimal distribution of conformity levels in the society and show which agents should be more conforming in order to increase wisdom.

Keywords: opinion leadership, wisdom of crowds, consensus, social networks, conformity, eigenvector centrality

JEL: C72, D83, D85, Z13

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1 Introduction

Opinions crucially shape individual behavior and affect economic decisions and outcomes.¹ For instance, opinions on political issues set the political course, opinions about a product’s quality and the integrity of its producer influence demand, and opinions about an economy’s growth determine investment decisions. The formation and evolution of opinions are often carried by day-to-day interactions of individuals, i.e. the opinions are determined by exchange in a social network.

We model the formation of opinions through communication in a given social network such that individuals are influenced by the opinions stated by others: individuals update their opinion in a naïve way by taking a weighted average of others’ stated opinions (as in the literature on naïve learning, see e.g. DeGroot, 1974; DeMarzo et al., 2003; Golub and Jackson, 2010; Acemoglu et al., 2010). However, influence often goes beyond this simple updating of opinions. When asked for a personal opinion, people usually do not straightforwardly state what they truly think, *rather they are tempted to misrepresent their opinion to conform to their friends since disagreement entails uncomfortable feelings* (see Zafar, 2011, for empirical evidence). In this paper, we consequently allow that not only the own opinion is influenced by what others say, but also the statement itself. In other words, some individuals tweak their stated opinions to conform to what their social contacts say.

In such a framework, we study the dynamics of opinions and particularly focus on the long-run distribution of opinions in the society. We show that under mild conditions dynamics converge and subgroups of the society reach a consensus. Moreover, we obtain a closed-form solution for long-run influence (opinion leadership): an individual’s influence on consensus is increasing in her network centrality (as in DeMarzo et al., 2003), but decreasing in her degree of conformity. This result, hence, explains the empirical finding that opinion leaders are often characterized by low conformity.² When interpreting initial opinions as signals about some true state of nature, the quality of information aggregation (wisdom) can be assessed by the precision of the consensus belief. We show in this paper that information does not necessarily get distorted when individuals are conforming. In fact, the society may be quite wise compared to the case where nobody misrepresents. The reason is that opinion leaders (as characterized before) are not necessarily well-informed, i.e. may not receive the best signal. To avoid that these powerful agents mislead the society and to benefit the quality of information aggregation, these opinion leaders should

¹Under the term *opinions* we subsume also beliefs, judgments, and estimations – depending on the application.

²A personality trait that has been found to discriminate opinion leaders from followers is called ‘public individuation’ (Chan and Misra, 1990). It measures by a list of questions the extent to which “people choose to act differently than others” (Maslach et al., 1985).

conform more to the society. Depending on the network structure and the quality of the signals, we characterize the set of optimal distributions of conformity to maximize wisdom.

We allow for conformity in an opinion formation framework since there is substantial empirical evidence that individuals conform to the actions of others when these actions are observable (as stated opinions are). For instance in the famous study by Asch (1955), subjects wrongly judged the length of a line after other participants of the experiment (conceived as neutral by the subjects, but being collaborators) had placed the same wrong judgment. Follow-up studies revealed that this effect is weaker if the subjects do not have to report their judgments publicly (Deutsch and Gerard, 1955). In the study by Asch (1955), subjects were asked for the reasons of their wrong judgment. Some said they were convinced of the wrong answer by the collaborators; others said that they knew that their answer was wrong, but felt uncomfortable by not conforming to what the collaborators said (see Asch, 1955, p.21). Deutsch and Gerard (1955), hence, distinguish two forms of social influence that can be observed in this study. While *informational social influence* describes the updating of (true) opinions according to what others have said, *normative social influence* describes the behavior of stating an opinion that fits the group norm.³

Normative social influence is also documented with respect to other publicly observable behavior. In an experiment on charitable giving, Zafar (2011) shows that individuals adjust more to the contributions of their neighbors (and hence conform more by reducing respectively increasing their contribution), the more their donations are observable, supporting the findings by Asch (1955) and Deutsch and Gerard (1955). Moreover, subjects in Zafar’s experiment mainly conform to the actions of participants who are their friends outside the lab. Hence, normative social influence is determined by the social network itself. Zafar (2011) concludes that individuals experience “a utility gain by simply making the same choice as [their] reference group” (Zafar, 2011, p. 774). Incentives to conform can be derived from desires for social status (Bernheim, 1994) and are embodied in a utility component that depends on the difference of the behavior of the focal actor and the behavior of some peer group (Jones, 1984).

While normative social influence affects the choice of stated opinions, informational social influence embodies the updating of the true opinions. We assume that individuals update their true opinions naïvely rather than sophisticatedly since empirical evidence strongly suggests that individuals in these settings behave boundedly rational (Corazzini et al., 2012; Grimm and Mengel, 2013; Battiston and Stanca, 2014). If individuals were

³Deutsch and Gerard (1955, p. 629) further explain: “Commonly these two types of influence are found together. However, it is possible to conform behaviorally with the expectations of others and say things which one disbelieves but which agree with the beliefs of others. Also, it is possible that one will accept an opponent’s beliefs as evidence about reality even though one has no motivation to agree with him, per se.”

fully rational, they would perfectly account for repetition of information (for some references on Bayesian learning in opinion formation, see Gale and Kariv, 2003; Acemoglu et al., 2011; Mueller-Frank, 2013). This, however, requires knowledge of the social network (personal relationships, individual trust in one another). Moreover, under Bayesian learning, the social network plays no role for the long-run outcome since individuals are able to derive the initial signals and thus to extract all information perfectly,⁴ which is a rather unrealistic assumption. In fact, evidence from laboratory experiments shows that even in small social networks (of only four people) where the network is made common knowledge, people fail to properly account for repetitions of information (Corazzini et al., 2012; Battiston and Stanca, 2014). Making the network structure more complex, Grimm and Mengel (2013) also confirm that learning in the lab is very well approximated by the naïve learning approach.

Hence, we model informational social influence by assuming that individuals learn naïvely from what others say (see also DeMarzo et al., 2003; Golub and Jackson, 2010; Acemoglu et al., 2010). In view of the substantial empirical evidence, we enrich the naïve learning model by studying the effects of individuals who have a desire to adjust their behavior (i.e. their stated opinion) to the behavior of their friends (i.e. their friends' stated opinions). In the words of psychology, this corresponds to modeling normative social influence. Remarkably, this type of influence has not been studied in a theoretical model of opinion dynamics despite the large empirical evidence.⁵ The main conceptual contribution of this work is hence to fill this gap by studying a model incorporating both informational and normative social influence. We focus on two motives for the misrepresentation of opinions: conformity and counter-conformity, while we also allow for honest agents.⁶ The desire to relate own stated opinions to the stated opinions of friends is given by an additional utility component parameterized by a preference parameter which we call the agents' degree of conformity. If positive, agents are of conforming type and state an opinion which is a convex combination of their own true opinion and other agents' stated opinions. If negative, an agent is counter-conforming and will state a more extreme opinion and if zero, an agent is honest, i.e. behaving like agents in the standard DeGroot

⁴Indeed, among equally informed agents with a strongly connected communication structure that is common knowledge, Bayesian updating leads to convergence of each opinion to the average of the initial opinions (DeMarzo et al., 2003, theorem 3).

⁵Meanwhile, the concepts of informational and normative social influence have become a cornerstone in analyzing social influence, e.g. Ariely and Levav (2000, p. 279) call it the “primary paradigm”. However, this paradigm did not explicitly enter economic models. The terms ‘social influence’ and ‘conformity’ do usually not clarify whether social or normative influence is at work. We will be more explicit on this distinction and only refer to conformity as a form of normative social influence. In terms of this paradigm, the DeGroot model of opinion formation and its variations are models of informational social influence, but not of normative social influence.

⁶This is consistent with the psychological theory where identification, non-identification and disidentification lead to these three types of normative social influence (Hogg and Abrams, 1988).

model.

When opinions are exchanged and updated repeatedly, we show that if agents are honest or of conforming type, the dynamics converge to a steady state.⁷ In contrast, counter-conformity may lead to divergence of opinions which we exemplify in the case of only two agents. This type of divergence is caused by agents stating more and more extreme opinions implying cycling dynamics, a feature we do not observe in the DeGroot model.

If agents in a subgroup find a consensus, the immediate question emerges how much this consensus is influenced by each agent’s initial opinion.⁸ Each individual’s influence on the long-run, hence, defines a measure of opinion leadership. As one of the main results, we show how opinion leadership or *power* is determined not only by each individual’s position in the network, given by eigenvector centrality (Bonacich, 1972; Friedkin, 1991), but also by the distribution of conformity in the society. Comparative statics reveal that an agent’s power is decreasing in own level of conformity, increasing in other agents’ level of conformity and increasing in own network centrality.

Finally, we consider a context where there is a true state of nature and the individuals’ initial opinions are unbiased noisy signals which may differ with respect to signal precision (the inverse of the variance). The question is *how the misrepresentation of opinions affects the accuracy of information aggregation (the society’s wisdom)*. A negative effect might be expected since stated opinions may become even less reliable signals about the truth. Our results show that this conjecture does not hold in general. First, if the society is homogeneous with respect to conformity, then information aggregation is neither worse nor better than in the DeGroot model (i.e. when all individuals are honest). Moreover, heterogeneous levels of conformity foster wisdom if they balance the power of agents with their signal precision, while an unbalanced distribution can lead to lower wisdom. Using comparative statics we observe that for the goal of higher accuracy of the consensus opinion, it would be helpful if people with a low signal precision (relative to their power) were more conforming, while people with a high signal precision (relative to their power) should be less conforming, or in more poetic words: “The whole problem with the world is that fools and fanatics are always so certain of themselves, but wiser people so full of doubts.”⁹

⁷As in the classic DeGroot model, steady states feature consensus in closed and strongly connected subgroups.

⁸This research question is also motivated by empirical research on identifying opinion leaders, which started with Katz and Lazarsfeld (2005).

⁹Credit for this quote is often given to Bertrand Russell although the origin of the quote is actually unknown. It is at least confirmed that Russell made a similar statement in his essay “The Triumph of Stupidity” (10 May 1933), which can be found on pp. 203-204 in the collection of essays “Mortals and Others”.

Related Models There is a growing body of literature that studies naïve learning in social networks. DeMarzo et al. (2003) introduce this approach into the economics literature arguing that people are often unable to properly account for repetition of information. The underlying assumption of a “persuasion bias” is helpful to understand different empirical phenomena such as the importance of airtime in political discussions and it has also found empirical support in the laboratory (Corazzini et al., 2012; Grimm and Mengel, 2013; Battiston and Stanca, 2014). Among naïve agents the social network becomes vital in the sense that not only accuracy of information but also network centrality determines an agent’s influence on her group (DeMarzo et al., 2003). This form of social influence makes naïve agents prone to be misled by powerful actors such as community leaders or lobbyists (Acemoglu et al., 2010). On the other hand, dispersed pieces of information can also be efficiently aggregated among naïve agents if the influence of each individual is vanishingly small (Golub and Jackson, 2010). The crucial question is hence under which conditions exchange of opinions among naïve agents leads to efficient information aggregation which is also called wisdom (Golub and Jackson, 2010). Our model takes the examination of the questions of power and wisdom to a further level since it incorporates not only the social network structure but also individual degrees of conformity.

The modeling approach of the above literature roots in the pioneer work of French (1956), Harary (1959), DeGroot (1974) and Friedkin and Johnsen (1990).¹⁰ One variation of the naïve learning approach is to let agents only be affected by opinions that are not too different from the own opinion (Hegselmann and Krause, 2002). Moreover, DeMarzo et al. (2003) allow the self-confidence to vary over time, while Lorenz (2005) allows the whole learning matrix to vary and identifies general conditions for convergence. Under some conditions, convergence to consensus is also robust if updating is noisy, as Mueller-Frank (2011) shows. There are also studies which extend the model by DeGroot (1974) to allow for adaption of learning weights, e.g. in Pan (2010) the influence weights are updated over time and Flache and Torenvlied (2004) study a variation of the classic model where actors anticipate the difference between own opinion and group decision (“frustration”) and adapt learning weights (“salience”) accordingly. The case where agents are able to manipulate learning weights of others is studied in Foerster et al. (2013). The focus of many of these models is to provide conditions for convergence, or determine opinion leadership. We contribute to this literature by allowing agents to misrepresent their opinion and study the effect on convergence conditions and opinion leadership. In a context of cultural transmission of traits, Buechel et al. (2011) introduce strategic interaction for the DeGroot model in an OLG framework. While this resembles counter-conforming misrepresentation of opinions, their model differs with respect to the optimization problem of individuals, the updating rule, and the resulting dynamics.

¹⁰We adopt their assumptions on naïve learning to model informational social influence.

Besides these highly related works, there are several contributions to somewhat similar research questions, but with respect to different settings. While their discussion is beyond the scope of this paper, we refer the reader to the following few prominent examples: other models of social learning (Bikhchandani et al., 1992; Ellison and Fudenberg, 1993, 1995; Bala and Goyal, 1998, 2001), cooperative models of social influence (Grabisch and Rusinowska, 2010, 2011), a model of strategic influence (Galeotti and Goyal, 2009), a model on rumors (Merlone and Radi, 2014) and a framework which contains strategic misrepresentation of opinions under Bayesian learning as a special case (Rosenberg et al., 2009). Most of these models investigate social influence on a discrete choice of actions, such as the choice of one out of two technologies, as opposed to continuous opinions.

The remainder of this paper is organized as follows. In Section 2 we introduce the model. Before we present the main results (in Section 4), we discuss the two-player case (Section 3). Section 5 addresses the wisdom of the society and in Section 6 we conclude, while proofs are relegated to the appendix.

2 Model

2.1 Informational Social Influence

There is a set of agents/players $\mathcal{N} = \{1, 2, \dots, n\}$ who interact with each other. A learning structure is given by a $n \times n$ row stochastic matrix G , i.e. $g_{ij} \geq 0$ for all $i, j \in \mathcal{N}$ and $\sum_{j=1}^n g_{ij} = 1$ for all $i \in \mathcal{N}$. This learning matrix represents the extent to which agents listen to other agents and it can be interpreted as a weighted and directed social network. We say that there is a directed path from i to j in this network if there exists $i_0, \dots, i_k \in \mathcal{N}$ such that $i_0 = i$ and $i_k = j$ and $g_{i_l i_{l+1}} > 0$ for all $l = 0, \dots, k-1$, which is equivalent to $(G^k)_{ij} > 0$.¹¹ Moreover, we assume that $g_{ii} < 1$ for all i to assure that all agents update their opinion.

We study a dynamic model where time is discrete $t = 0, 1, 2, \dots$ and initially each agent has a predefined opinion $x_i(0)$ concerning some topic. The opinions of all agents at time t are collected in $x(t) \in \mathbb{R}^n$. In every period, agents talk to each other and finally update their opinions according to the matrix G . In the classical DeGroot model agents exchange opinions such that the opinions in period $t+1$ are formed by $x(t+1) = Gx(t) = G^{t+1}x(0)$ (DeGroot, 1974). The motivation for such a model is that agents always report their true opinions and suffer from persuasion bias when the next period's opinion is formed as a weighted average of own and others' opinions according to the social network G .

¹¹We follow the convention of Jackson (2008) and DeMarzo et al. (2003) that a directed link from agent i to agent j indicates that i listens to j , i.e. $g_{ij} > 0$, while the opposite convention is used by Corazzini et al. (2012).

Concerning the assumption of honesty in opinion formation, DeMarzo et al. (2003) note:

“For simplicity, we assume that agents report their beliefs truthfully.”¹²

We relax this assumption: an agent $i \in \mathcal{N}$ expresses some opinion $s_i(t) \in \mathbb{R}$ which need not coincide with her true opinion $x_i(t)$.¹³

A central assumption of our approach is that an agent cannot observe the true opinions of the others but only their stated opinions. Since each agent knows her own true opinion $x_i(t)$, we get that agent i 's next period's opinion is formed by $x_i(t+1) = g_{ii}x_i(t) + \sum_{j \neq i} g_{ij}s_j(t)$, where the weights g_{ij} are the individual learning weights as in the classical model by DeGroot (1974). This holds for all agents $i \in \mathcal{N}$ and, thus, the updating process becomes

$$x(t+1) = Dx(t) + (G - D)s(t), \quad (1)$$

where D is the $n \times n$ diagonal matrix containing the diagonal of G .

2.2 Normative Social Influence

Misrepresenting the own opinion (i.e. being dishonest) might cause discomfort (e.g. Festinger, 1957). However, there are various motives to misrepresent the own opinion. Not only strategic considerations of persuasion play a role, but also personality traits or emotional motives. There is ample evidence that many people feel discomfort from stating an opinion that is different from their peer group's opinion (e.g. Deutsch and Gerard, 1955). While certainly many people feel this type of normative social influence, this need not be true for all people – there are even some who prefer to state an opinion that is far away from what others say.¹⁴ We focus on these two motives for the misrepresentation of opinions: conformity and counter-conformity.

To formalize these ideas, consider an agent i who is confronted with some group opinion q_i , while her own opinion on this topic is x_i . In the spirit of the model of Bernheim (1994) we consider a utility function that depends on an intrinsic part – this will be the incentive to be honest – and a social part – this will be the incentive to conform/counter-conform. Additionally, we assume that utility of an agent is additively separable into these two parts and that for each part disutility takes a quadratic form.

¹²DeMarzo et al. (2003, p. 3, footnote 9).

¹³The incentive to state an opinion different from true opinion will be based on preferences for conformity or counter-conformity (cf. Subsection 2.2). Moreover, agents adapt their stated opinions faster than true opinions such that $s(t)$ is given by Proposition 1.

¹⁴For instance, Hornsey et al. (2003) conducted a laboratory experiment where subjects reported their willingness to privately or publicly express and support their opinion. For subjects with a strong moral basis on the topic, the treatment of suggesting that a majority of the other subjects disagreed slightly increased the willingness to publicly express the opinion.

Thus, the utility of agent i depends on the distance of true opinion x_i to stated opinion s_i as well as on the distance of stated opinion s_i to group opinion q_i in the following way:

$$u_i(s_i|x_i) := -(1 - \delta_i)(s_i - x_i)^2 - \delta_i(s_i - q_i)^2, \quad (2)$$

where $\delta_i \in (-1, +1)$ displays the relative importance of the preference for conformity in relation to the preference for honesty. The preference peak (or “bliss point,” Bernheim, 1994) for such an agent is given by $s_i = (1 - \delta_i)x_i(t) + \delta_i q_i(t)$. This assumption is illustrated in Figure 1. For $\delta_i \in (0, 1)$ the agent faces a trade-off between conforming and being honest such that her preference peak lies within the interval (x_i, q_i) . For $\delta_i \in (-1, 0)$, a similar trade-off can be seen between counter-conforming and being honest. In that case the preference peak lies within the interval $(x_i - (q_i - x_i), x_i)$. We assume that $\delta_i > -1$ to restrict counter-conformity to a certain bound which seems weak enough to cover all reasonable cases, but keeps the analysis tractable.

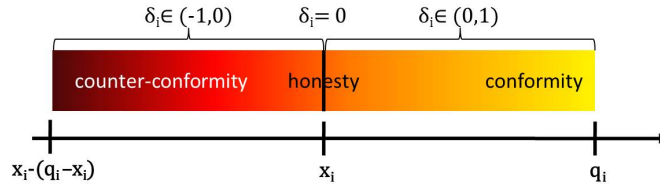


Figure 1: Preferences for conformity, counter-conformity, and honesty.

A stylized fact on normative social influence is that people are heterogeneous in the way and their degree of being influenced. The degree of conformity can hence be considered a personality trait, but it might also depend on the topic under discussion. Let Δ denote the $n \times n$ diagonal matrix with entries $\delta_i \in (-1, 1)$ on the diagonal representing the levels of conformity in the society.

Now, we want to determine each agents' stated opinion without assuming that the network structure and the individual types are common knowledge and without assuming that agents are sophisticated in anticipating the consequences of their behavior. For this purpose we consider an adaption process of stated opinions which takes place within a time period t , while true opinions are updated from one period to the next.¹⁵ Thus, suppose that within each period $t \in \mathbb{N}$, there is a fast time scale $\tau \in \mathbb{N}$ such that at each time step τ one or more agents speak. The (possibly random) set of agents who are selected to state their opinions at time step τ (of period t) is denoted by $A^\tau(t)$. Let $s^\tau(t)$ be the vector of stated opinions. Agents who are not selected to revise keep the stated opinion of the previous time step, i.e. $s_i^\tau(t) = s_i^{\tau-1}(t)$ if $i \in \mathcal{N} \setminus A^\tau(t)$. Agents, who are

¹⁵An interpretation for this assumption is that each period is a discussion round within which stated opinions are adjusted, while learning takes place between discussion rounds.

selected to speak and thereby revise their stated opinion, observe last time step's stated opinions of their neighbors. These are perceived as a reference opinion $q_i^{\tau-1}(t)$, which is the average of the stated opinions with weights according to the listening matrix G , i.e.

$$q_i^\tau(t) = \sum_{j \neq i} \frac{g_{ij}}{1 - g_{ii}} s_j^\tau(t). \quad (3)$$

In line with our assumption that agents are naïve when updating, we also assume that agents are boundedly rational when revising their stated opinions. Upon revision opportunity, i.e. $i \in A^\tau(t)$, an agent i myopically chooses a stated opinion which maximizes her current utility given by (2), i.e.

$$s_i^\tau(t) = (1 - \delta_i)x_i(t) + \delta_i q_i^{\tau-1}(t), \quad (4)$$

for any true opinion $x_i(t)$ and any reference opinion $q_i^{\tau-1}(t)$.¹⁶ Hence, the stated opinion given by myopic best response differs from the true opinion proportionally to the difference of reference opinion and true opinion, and the proportion is determined by the preference parameter δ_i . The parameter δ_i can thus be directly interpreted as the degree of conformity of agent i 's behavior (cf. Figure 1). A conforming agent, characterized by $\delta_i \in (0, 1)$, states an opinion between the true opinion $x_i(t)$ and perceived opinion $q_i^{\tau-1}(t)$. A counter-conforming agent, characterized by $\delta_i \in (-1, 0)$, states an opinion that is more extreme than the true opinion $x_i(t)$ (with respect to the perceived opinion $q_i^{\tau-1}(t)$). Finally, an honest agent, characterized by $\delta_i = 0$, straight-forwardly states the true opinion, i.e. $s_i^\tau(t) = x_i(t)$ for all $\tau \in \mathbb{N}$.

To ensure that every agent takes part in opinion exchange in period t , we assume that for each agent i , the set $\{\tau \in \mathbb{N} : i \in A^\tau(t)\}$ is (almost surely) infinite, reflecting the idea that no agent will stay forever with a stated opinion that is not in line with her preferences. This assumption is satisfied if, e.g., at each time step τ agents are randomly selected to speak according to some probability distribution with full support on \mathcal{N} .

It turns out that such a myopic best reply process within period $t \in \mathbb{N}$ inevitably leads to one specific profile of stated opinions $s(t)$ which only depends on the network G and the conformity parameters Δ , but not on the starting stated opinions $s^0(t)$.

Proposition 1. *Given the assumptions above, the within-period dynamics $s^\tau(t)$ converge for $\tau \rightarrow \infty$ to*

$$s(t) := [I - \Delta(I - D)^{-1}(G - D)]^{-1}(I - \Delta)x(t). \quad (5)$$

The proof of Proposition 1 as well as all proofs of the following propositions are relegated to an appendix. Proposition 1 shows that agents who revise opinions by conforming

¹⁶Myopic maximizing is a common assumption in such models (see, e.g. Corazzini et al., 2012).

or counter-conforming to what their neighbors last said, finally state (or *express*) the opinions given by (5).

It is worth noting that considering the action sets $S_i(t) = \mathbb{R}$ and utility functions $u_i(s_i(t)|x_i(t))$ given by (2) implies that $s(t)$ obtained by Proposition 1 is the unique Nash equilibrium of the normal form game $(\mathcal{N}, S(t), u(\cdot|x(t)))$ for each $t \in \mathbb{N}$. Note that the process that leads into this Nash equilibrium within period t neither requires complete information (e.g. on the network structure G), nor high degrees of rationality, nor some sort of common knowledge.

2.3 Model Summary

In our model each period $t \in \mathbb{N}$ can be viewed as a discussion round within which agents express opinions and then learn from one discussion round to the next. Proposition 1 determines which opinions are finally stated in a given period as a function of the true opinions $x(t)$. These stated opinions $s(t)$ determine the vector of reference opinions $q(t)$ by (3) and are then a crucial ingredient of the updating process.¹⁷ Since opinions of period $t + 1$ are formed by (1) and the stated opinions of each period can be calculated as in Proposition 1, we conclude that the opinion profile in period $t + 1$ depends on the opinion profile in period t in the following way:

$$x(t + 1) = Mx(t), \tag{6}$$

where $M := \left[D + (G - D)[I - \Delta(I - D)^{-1}(G - D)]^{-1}(I - \Delta) \right]$. Note that the transformation from $x(t)$ to $x(t + 1)$, i.e. the matrix M , is independent of $x(t)$. Thus, the opinion dynamics are fully described by the power series M^t , since $x(t + 1) = Mx(t) = M^2x(t - 1) = \dots = M^{t+1}x(0)$.¹⁸ The relation to the classical DeGroot model becomes apparent in this expression when recalling $x(t + 1) = Gx(t) = G^{t+1}x(0)$. In that light the misrepresentation of opinions leads to a transformation of the matrix G into the matrix M . If every agent is honest, i.e. $\delta_i = 0$ for any $i \in \mathcal{N}$, then $M = G$ and, hence, we are back in the standard case of DeGroot (1974).

Before we analyze this model in full generality in Section 4, we derive and illustrate its properties for the case of two agents in Section 3.

¹⁷Since one interpretation for $q_i(t)$ is that this is the society's opinion at time t as perceived by agent i , we also call it *i's perceived opinion*.

¹⁸The simple linear structure is of course implied by our assumption of quadratic utility.

3 Two-Agent Case

In this case, closed form solutions are easy to obtain and, still, it is possible to observe several important properties of the opinion dynamics.

Let $n = 2$. Then we can write G as

$$G = \begin{pmatrix} 1 - g_{12} & g_{12} \\ g_{21} & 1 - g_{21} \end{pmatrix}$$

with $g_{12}, g_{21} \in (0, 1)$. With only two agents, the relevant group average for one agent is simply the stated opinion of the other agent, i.e. $q_1(t) = s_2(t)$ and $q_2(t) = s_1(t)$. Plugging in the variables for G into (6) yields

$$M = \begin{pmatrix} 1 - m_{12} & m_{12} \\ m_{21} & 1 - m_{21} \end{pmatrix} = \begin{pmatrix} 1 - g_{12} \frac{1 - \delta_2}{1 - \delta_1 \delta_2} & g_{12} \frac{1 - \delta_2}{1 - \delta_1 \delta_2} \\ g_{21} \frac{1 - \delta_1}{1 - \delta_1 \delta_2} & 1 - g_{21} \frac{1 - \delta_1}{1 - \delta_1 \delta_2} \end{pmatrix}.$$

Since $x(t+1) = Mx(t)$, an entry m_{ij} gives the importance of Player j on the one-period opinion change of Player i . From $\frac{\partial m_{12}}{\partial \delta_2} = -g_{12} \frac{1 - \delta_1}{(1 - \delta_1 \delta_2)^2}$, we see the following comparative static effect: higher conformity of Player 2 reduces her one-period influence on Player 1 (m_{12}), which vanishes ($m_{12} \rightarrow 0$) when Player 2's conformity approaches 1. Thus, in the short run, conformity results in a reduction of influence. To investigate long-run effects, we examine the power series M^t since $x(t) = M^t x(0)$. By induction one can easily see that M^t can be rewritten as follows:

$$M^t = \frac{1}{m_{12} + m_{21}} \begin{pmatrix} m_{21} + m_{12}(1 - m_{12} - m_{21})^t & m_{12} - m_{12}(1 - m_{12} - m_{21})^t \\ m_{21} - m_{21}(1 - m_{12} - m_{21})^t & m_{12} + m_{21}(1 - m_{12} - m_{21})^t \end{pmatrix}. \quad (7)$$

From (7), we observe that the decisive quantity for the (speed of) convergence of M^t is

$$\lambda := 1 - m_{12} - m_{21} = 1 - \frac{g_{12}(1 - \delta_2) + g_{21}(1 - \delta_1)}{1 - \delta_1 \delta_2} < 1,$$

which is the second (largest) eigenvalue of M (the other eigenvalue of M is always 1). In particular, M^t converges if $|\lambda| < 1$ and, moreover, the smaller $|\lambda|$, the higher the speed of convergence. Before discussing the issue of convergence in more detail, let us have a brief look at the limit of M^t in case of convergence: with the help of (7), we have

$$M^\infty = \lim_{t \rightarrow \infty} M^t = \begin{pmatrix} \frac{m_{21}}{m_{12} + m_{21}} & \frac{m_{12}}{m_{12} + m_{21}} \\ \frac{m_{21}}{m_{12} + m_{21}} & \frac{m_{12}}{m_{12} + m_{21}} \end{pmatrix}$$

such that, in the long run, the two agents will reach a consensus because $x(\infty) = M^\infty x(0)$. Player 1's and Player 2's initial opinions enter this consensus opinion with weights $\frac{m_{21}}{m_{12}+m_{21}}$ and $\frac{m_{12}}{m_{12}+m_{21}}$, respectively. Since $\frac{m_{12}}{m_{12}+m_{21}} = \frac{g_{12}(1-\delta_2)}{g_{12}(1-\delta_2)+g_{21}(1-\delta_1)} = 1 - \frac{g_{21}(1-\delta_1)}{g_{12}(1-\delta_2)+g_{21}(1-\delta_1)}$, Player 2's influence in the long run is decreasing in δ_2 . Therefore, increasing conformity not only decreases the short-run importance of an agent, but also the long-term impact of this agent's initial opinion.

To study the effect of conformity/counter-conformity on convergence, we will first consider the special case $\delta_1 = \delta_2 =: \delta$ which simplifies λ to

$$\lambda = 1 - \frac{1}{1+\delta}(g_{12} + g_{21}).^{19} \quad (8)$$

Since $\lambda < 1$, the decisive thresholds for λ are $\lambda = 0$ and $\lambda = -1$: for $\lambda = 0$, convergence will be fastest (one-step convergence due to $M = M^2 = \dots = M^\infty$), while $\lambda = -1$ marks the case of cycling M^t (M^t will alternate between $M^1 = M^3 = \dots$ and $M^2 = M^4 = \dots$). Figure 2 exemplifies the corresponding across-period dynamics for $G = \begin{pmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{pmatrix}$ and initial opinions $x(0) = (0, 100)'$. For better readability, we abstract from within-period dynamics and simply connect the opinions at time t and $t+1$ by straight lines in this and the following figures. Notice, in particular, that the speed of convergence of true opinions $x(t)$ is not monotone in δ : when δ decreases from 0.5 to -0.4 , speed increases and eventually reaches one-step convergence; however, further reducing δ first leads to slower, alternating dynamics, cycling, and finally divergent behavior.²⁰ It might be surprising that higher levels of conformity can decrease the speed of convergence. The intuition for this effect can be gained by comparing cases (a) and (b). Under conformity, i.e. in case (a), stated opinions $s(t)$ are closer to each other in the first time periods such that agents' true opinions $x(t)$ are less swayed to the center compared with case (b) where agents are honest.²¹

If we relax the assumption of equal conformity ($\delta_1 = \delta_2$), the necessary and sufficient condition for convergence of M^t ($\lambda > -1$) is equivalent to

$$g_{12} \frac{1-\delta_2}{1-\delta_1\delta_2} + g_{21} \frac{1-\delta_1}{1-\delta_1\delta_2} < 2. \quad (9)$$

To interpret this condition in terms of individual conformity parameters, let us distin-

¹⁹ λ and $|\lambda|$ as a function of δ are depicted in part (0) of Figure 2.

²⁰Another aspect that can be observed in Figure 2 is that, under convergence, i.e. in cases (a)-(e), the dynamics converge to the same limit independently of δ . We will show later on that this observation is not a coincidence and that it is induced by setting $\delta_1 = \delta_2 = \delta$.

²¹Recall that agents know their own true opinion and are thus resistant against their own misrepresentation.

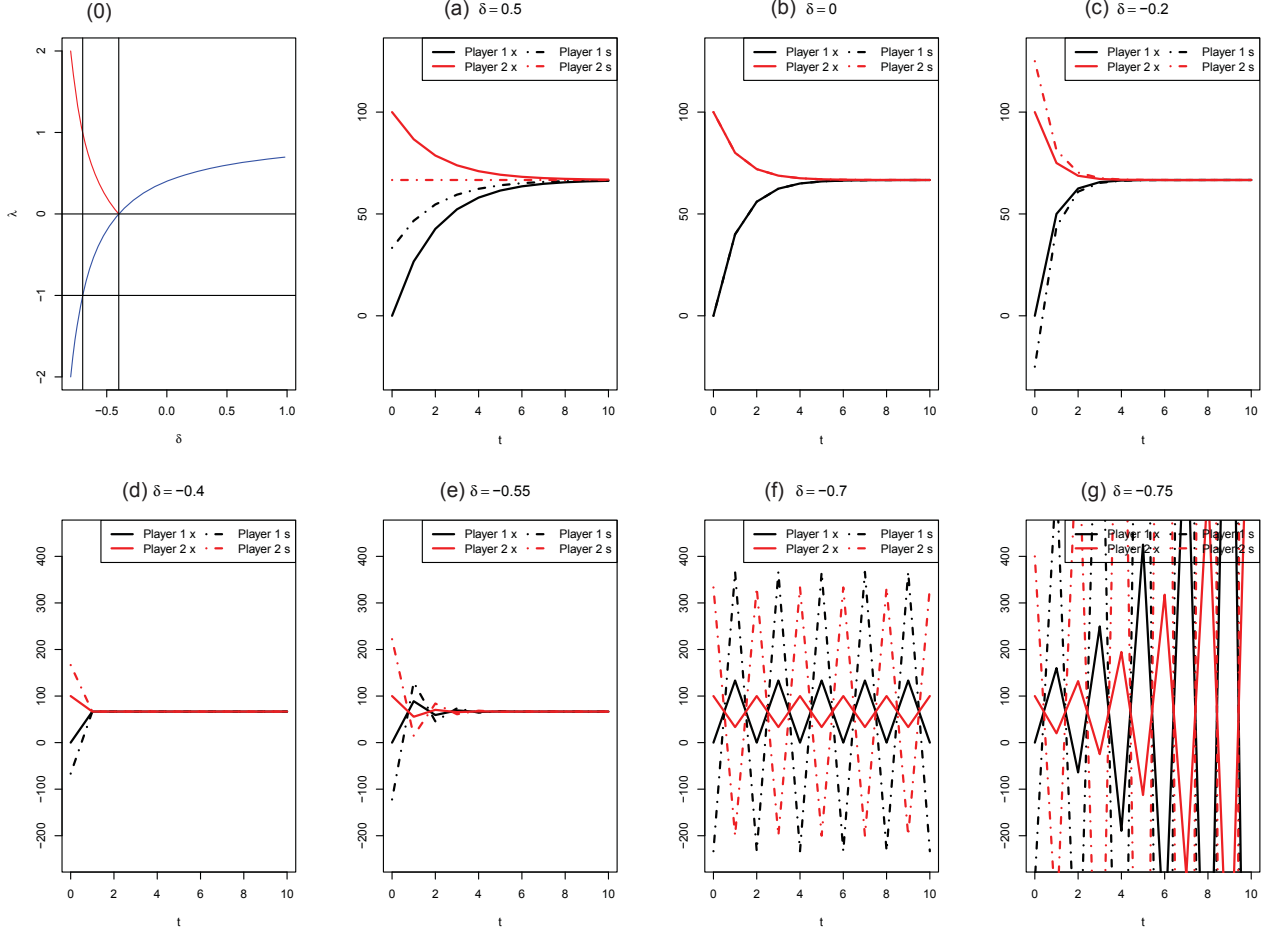


Figure 2: Seven cases of two-agent dynamics for $\delta_1 = \delta_2 = \delta$. Solid lines represent true opinions and dashed lines display stated opinions. (0) Shape of λ . (a) $\delta > 0$, conformity. (b) $\delta = 0$, honesty. (c) $-0.4 < \delta < 0$, smooth convergence under counter-conformity. (d) $\delta = -0.4$, one-step convergence. (e) $\delta < -0.4$, alternating dynamics with convergence. (f) $\delta = -0.7$, alternating dynamics ($\lambda = -1$). (g) $\delta < -0.7$, divergence.

guish two cases:²²

(i) If $\delta_2 \leq \frac{2g_{21}+g_{12}-2}{2+g_{12}}$, then M^t converges if and only if $\delta_1 > \frac{g_{12}(1-\delta_2)+g_{21}-2}{g_{21}-2\delta_2}$.

(ii) If $\delta_2 > \frac{2g_{21}+g_{12}-2}{2+g_{12}}$, then M^t converges for any $\delta_1 \in (-1, +1)$.

Thus, if Player 2 has a relatively low degree of conformity (case (i)), then Player 1 must be sufficiently conforming in order to assure convergence. However, if Player 2's conformity is above some threshold, then we will have convergence for any conformity level of Player 1. In fact, $\delta_2 > \frac{1}{3}$ is sufficient for (ii) to hold. Since similar arguments can be made by exchanging the players' labels, in the two-agent case we always have convergence if there

²²It can be checked that the threshold which defines the two cases is always in $(-1, \frac{1}{3})$. Additionally, given that (i) holds, the threshold for δ_1 is below 1.

is an agent with $\delta_i > \frac{1}{3}$. Thus, a sufficiently conforming agent will reach consensus with any other agent.

4 Opinion Dynamics

To study the dynamics of opinions for an arbitrary number of agents, we first elaborate on conditions of convergence and then determine where opinions converge to.

4.1 Convergence

By convergence, we mean that opinions settle down in a steady state, but not necessarily that a consensus in the society is reached. In the standard DeGroot model, convergence of opinions is obtained under very mild conditions, which basically exclude cycling dynamics (Golub and Jackson, 2010). In our more general model, opinions may not only converge or cycle, but also diverge, as shown in the case of two agents with the same level of (counter-)conformity (cf. Figure 2). The two-agent case nurtures the intuition that among conforming agents opinions always approach each other, while among counter-conforming agents opinions may alternate and eventually diverge. Mathematically, convergence of opinions is driven by convergence of M^t .²³ Counter-conforming agents can lead to negative entries of matrix M which may but need not make M^t divergent.²⁴ Reversely, honest and conforming agents do not induce negative entries of M such that convergence can be guaranteed by standard results. This yields the following simple condition that ensures convergence of opinion dynamics for any vector of starting opinions $x(0)$.

Proposition 2 (First Convergence Result). *M^t converges for $t \rightarrow \infty$ if for all $i \in \mathcal{N}$ we have $g_{ii} > 0$ and $\delta_i \geq 0$.*

The condition presented here is fairly weak. If we exclude counter-conformity ($\delta_i \geq 0$), and every individual has at least some self-confidence, then the opinion dynamics converge. The assumption of positive self-confidence thereby only serves to assure aperiodicity of matrix M which could also be generated by weaker assumptions. Although all cases of conformity are covered by Proposition 2, it is important to emphasize that conformity is not necessary for convergence. Examples of convergence which include counter-conforming agents were already given in the two-agent case (Section 3). In order to analyze necessary

²³Indeed, since $x(t) = M^t x(0)$, true opinions $x(t)$ converge for arbitrary starting opinions $x(0)$ if and only if M^t converges. Moreover, it is easy to show that stated opinions $s(t)$, as well as perceived opinions $q(t)$, converge if and only if true opinions converge (cf. Lemma A.2 in Appendix A.2).

²⁴Negative entries of M are not only remarkable because of the different dynamics they induce, but also because of their interpretation as a negative relation between two agents: Although only positive weights are put on each other's opinions, an agent may negatively incorporate a peer's opinion due to counter-conformity.

and sufficient conditions for convergence, the block structure of matrix M has to be inspected. As we will see later, the block structure also determines which subgroups of the society reach a consensus in the long run. In the standard model, agents within a closed and strongly connected group, which corresponds to a block in the matrix G , reach a consensus (e.g. DeMarzo et al., 2003; Golub and Jackson, 2010). Accordingly, we partition the set of agents \mathcal{N} with respect to the paths in the network as follows.

Definition 1. Let $\Pi(\mathcal{N}, G) = \{\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_K, \mathcal{R}\}$ be a partition of \mathcal{N} into $K(\geq 1)$ groups and the (possibly empty) rest of the world \mathcal{R} such that:

- Each group \mathcal{C}_k is strongly connected, i.e. for all $i, j \in \mathcal{C}_k$ there exists $l \in \mathbb{N}$ such that $(G^l)_{ij} > 0$.
- Each group \mathcal{C}_k is closed, i.e. for all $i \in \mathcal{C}_k$, $G_{ij} > 0$ implies $j \in \mathcal{C}_k$.
- The (possibly empty) rest of the world consists of the agents who do not belong to any closed and strongly connected set, i.e. $\mathcal{R} = \mathcal{N} \setminus \bigcup_{k=1}^K \mathcal{C}_k$.

With a suitable renumeration, the matrix G is organized into blocks which correspond to the groups of the partition $\Pi(\mathcal{N}, G)$:

$$G = \begin{pmatrix} G_{11} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & G_{KK} & 0 \\ G_{\mathcal{R}1} & \cdots & \cdots & G_{\mathcal{R}K} & G_{\mathcal{R}\mathcal{R}} \end{pmatrix} \quad (10)$$

with $G_{kk} = G|_{\mathcal{C}_k}$, $G_{\mathcal{R}\mathcal{R}} = G|_{\mathcal{R}}$, and $G_{\mathcal{R}k}$ consisting of the rows of G belonging to \mathcal{R} and the columns of G belonging to \mathcal{C}_k .

Intuitively, convergence of opinions requires that dynamics in each part of the network settle down. Formally, Proposition A.1 in Appendix A.2 explicitly determines M – and in fact M^t , for all $t \in \mathbb{N}$ – showing that the structure of the society extends from the standard model to our more general set-up.²⁵ This means that the opinion dynamics of each group \mathcal{C}_k can be studied independently, while only for agents in \mathcal{R} multiple groups may matter. Thus, for convergence of opinions it is necessary that dynamics within each group converge, as it is the case in the classic model. In contrast to the classic model, however, this is not sufficient, as the following example shows.

²⁵This result is not self-evident. It crucially depends on the definition of the reference opinion $q_i^r(t)$.

Example 1. Suppose there are four agents such that $G = \begin{bmatrix} 0.7 & 0.3 & 0 & 0 \\ 0.3 & 0.7 & 0 & 0 \\ 0.085 & 0.085 & 0.49 & 0.34 \\ 0.085 & 0.085 & 0.34 & 0.49 \end{bmatrix}$. Players 1 and 2 form a closed and strongly connected group \mathcal{C}_1 , while Players 3 and 4 from the rest of the world \mathcal{R} . Let the conformity parameter δ be given by $\delta = (0, 0, \delta_{\text{ROTW}}, \delta_{\text{ROTW}})$. Figure 3 shows the opinion dynamics for the cases $\delta_{\text{ROTW}} = -0.75$ and $\delta_{\text{ROTW}} = -0.9$. While convergence within the closed and strongly connected group is guaranteed, the rest of the world (ROTW) may cause divergence of M^t for $t \rightarrow \infty$.²⁶

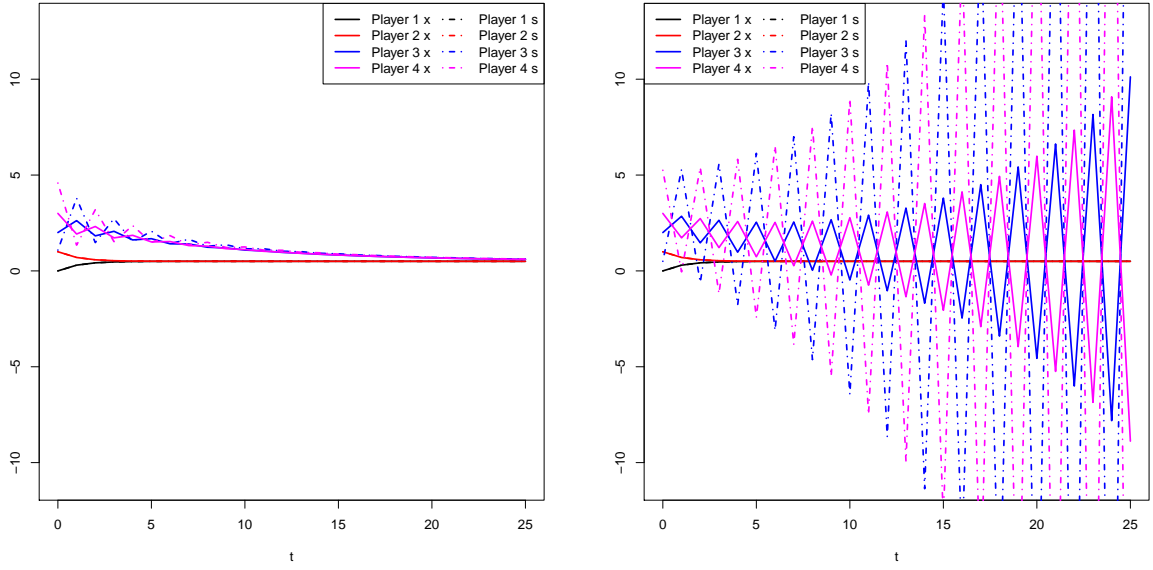


Figure 3: The opinion dynamics of Example 1 for (a) $\delta_{\text{ROTW}} = -0.75$ and (b) $\delta_{\text{ROTW}} = -0.9$.

Thus, convergence of opinions in all closed and strongly connected groups is not sufficient for convergence of opinions in the society. In Proposition 3, we identify the additional condition on the rest of the world such that M^t converges.

Proposition 3 (Second Convergence Result). *Let the block structure of M be given as in (10). M^t converges for $t \rightarrow \infty$ if and only if M_{kk}^t converges for all $k = 1, \dots, K$ and $M_{\mathcal{R}\mathcal{R}}^t$ converges to 0.*

Proposition 3 presents a necessary and sufficient condition for convergence of M in terms of the block structure. In Example 1 the condition that $M_{\mathcal{R}\mathcal{R}}$ converges to 0 fails

²⁶Notice that, for the latter case, M not only has negative entries but also entries larger than unity:

$$M = \begin{pmatrix} 0.7 & 0.3 & 0 & 0 \\ 0.3 & 0.7 & 0 & 0 \\ 0.053125 & 0.053125 & -0.115625 & 1.009375 \\ 0.053125 & 0.053125 & 1.009375 & -0.115625 \end{pmatrix}.$$

since strong counter-conformity of two agents leads to eigenvalues with high absolute value to the extent that $|\lambda_{\mathcal{R}\mathcal{R}}| > 1$, for some eigenvalue of $M_{\mathcal{R}\mathcal{R}}$. A similar violation of the necessary condition for convergence occurs if counter-conformity of agents in the closed and strongly connected groups is too strong.

While tight convergence conditions for each block of matrix M are known, it is difficult to trace these conditions back to the model fundamentals, which are the network G and the distribution of conformity Δ . Already in the case of two agents, such conditions are relatively complex (cf. (9)). For every numerical example, however, it is an easy computational exercise to determine M and M^t and thereby establish the dynamic properties including whether opinions converge or not. Therefore, we now assume for the remainder, that the power series M^t converges. Notice that this does not preclude the presence of counter-conforming agents.

4.2 Long-run Opinions

Having established convergence, we now address where opinions converge to (in the long run) when starting with some opinion profile $x(0)$. It turns out that true, stated, and perceived opinions always converge to the same limit, i.e. $x(\infty) = s(\infty) = q(\infty)$, as we formally show by Lemma A.2 in Appendix A.2. Therefore, we can restrict our analysis of the long run to the dynamics of true opinions. We are particularly interested in the influence of each agent's initial opinion on the long-run opinion given her position in the network G and her degree of conformity δ_i . The following result characterizes the long-run opinions explicitly (conditional on convergence). We present it first in a formal way and turn to its interpretation with respect to opinion leadership in Section 4.3.

Theorem 1. *Let G and M be organized as in (10). We denote by $w, v \in \mathbb{R}^n$ the vectors that fulfill the following: for each closed and strongly connected group $\mathcal{C}_k \in \Pi(\mathcal{N}, G)$, $w|_{\mathcal{C}_k}$ is the left unit eigenvector of G_{kk} with $\sum_{i \in \mathcal{C}_k} w_i = 1$, while $v|_{\mathcal{C}_k}$ is left unit eigenvector of M_{kk} with $\sum_{i \in \mathcal{C}_k} w_i = 1$. If M^t converges for $t \rightarrow \infty$ to some matrix M^∞ , then the following holds:*

$$M^\infty = \begin{pmatrix} M_{11}^\infty & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & M_{KK}^\infty & 0 \\ M_{\mathcal{R}1}^\infty & \cdots & \cdots & M_{\mathcal{R}K}^\infty & 0 \end{pmatrix}$$

with

$$M_{kk}^\infty = \mathbb{1}_{|\mathcal{C}_k} v'_{|\mathcal{C}_k} = \mathbb{1}_{|\mathcal{C}_k} w'_{|\mathcal{C}_k} \frac{I - \Delta_{kk}}{\mathbb{1}'_{|\mathcal{C}_k} (I - \Delta_{kk}) w_{|\mathcal{C}_k}}, \quad (11)$$

and

$$M_{\mathcal{R}k}^\infty = (I - G_{\mathcal{R}\mathcal{R}})^{-1} G_{\mathcal{R}k} M_{kk}^\infty \quad (12)$$

for all $k = 1, \dots, K$.

Theorem 1 fully characterizes the long-run dynamics of (true) opinions given convergence since $x(\infty) = M^\infty x(0)$.²⁷ For the interpretation of the result, we distinguish between the closed and strongly connected groups \mathcal{C}_k and the rest of the world \mathcal{R} .

We can first observe that the long-run opinions may differ across groups, but each closed and strongly connected group \mathcal{C}_k reaches a consensus $c_k \in \mathbb{R}$ as each block M_{kk}^t of M^t converges to a matrix of rank 1. Each row of M_{kk}^∞ is given by the left-hand unit eigenvector $v'_{|\mathcal{C}_k}$, implying

$$c_k := x_i(\infty) = x_j(\infty) = v'_{|\mathcal{C}_k} x(0)_{|\mathcal{C}_k} \quad (13)$$

for all agents i, j in group \mathcal{C}_k . The left-hand normalized unit eigenvector $v'_{|\mathcal{C}_k}$ thus displays the extent to which the initial opinion of each agent i matters for consensus within group \mathcal{C}_k . Moreover, $v'_{|\mathcal{C}_k}$ is a function of $w'_{|\mathcal{C}_K}$, the left-hand unit eigenvector of G_{kk} , and the conformity parameters within the group, Δ_{kk} . (We delay the interpretation of this result and its comparative statics to the next subsection.)

The long-run opinion of an agent in \mathcal{R} is simply some weighted average of the long-run opinions c_1, \dots, c_K within the groups $1, \dots, K$.²⁸ To see this, consider the matrix

$$\Gamma := (I - G_{\mathcal{R}\mathcal{R}})^{-1} (G_{\mathcal{R}1} \mathbb{1}_{|\mathcal{C}_1}, \dots, G_{\mathcal{R}K} \mathbb{1}_{|\mathcal{C}_K}),$$

which is easily seen to be row-stochastic. Γ enables translating (12) into

$$x(\infty)_{|\mathcal{R}} = \Gamma c \quad (14)$$

combining the long-run opinions of the closed and strongly connected groups denoted by the K -dimensional vector $c = (c_1, \dots, c_K)'$. Thus, the initial opinion of some agent in the ROTW does not affect the long-run opinion profile $x(\infty)$ since the ROTW agents end up with a weighted average of the consensus opinions of the closed and strongly connected groups, which in turn are dependent on the initial opinions within those groups. Moreover, the weights of averaging depend on G but not on the conformity parameters δ_i for $i \in \mathcal{R}$. Consequently, the long-run opinion of an agent in the ROTW neither depends on an initial opinion nor on the conformity parameter of any agent within the ROTW (including herself). Thus, the only way conformity of agents in the rest of the world

²⁷The dynamics collapse to the well-known DeGroot dynamics if every agent i is honest, i.e. Δ is a matrix of zeros.

²⁸This result is fully analogous to theorem 10 in DeMarzo et al. (2003).

can affect long-run opinions is to induce divergence such as in Example 1. Since each agent in the ROTW may average differently between consent opinions of the closed and strongly connected groups, the agents in the ROTW need not reach a consensus if there is more than just one closed and strongly connected group. The important contribution of Theorem 1 lies in the characterization of v as a function of w and Δ , as we will discuss next.

4.3 Opinion Leadership

To simplify the discussion, let us now restrict attention to one closed and strongly connected group by assuming that there is only one such group, i.e. $\Pi(\mathcal{N}, G) = \mathcal{N}$. For this purpose it is sufficient to assume that G is strongly connected or, equivalently, that $\text{rk}(I - G) = n - 1$, where rk yields the rank of a matrix.

From (13), we get that $x(\infty) = \mathbb{1}v'x(0)$ and hence $x_j(\infty) = v'x(0) = \sum_{i \in \mathcal{N}} v_i x_i(0)$. Thus, an entry v_i of v determines the weight of the initial opinion of agent i on the long-run consensus opinion. This is a very intuitive formalization of opinion leadership: v measures the *power* of each agent.

Note that for $\delta_i = 0$ for all $i \in \mathcal{N}$, (11) yields $v = w$, i.e. opinion leadership is fully determined by the left-hand unit eigenvector of G . w is a well-studied object in network science: it is known as eigenvector centrality (Bonacich, 1972; Friedkin, 1991).²⁹ Relaxing the assumption that every agent is honest, the following corollary of Theorem 1 shows how opinion leadership is not only determined by eigenvector centrality, but also by the degree of conformity.

Corollary 1. *Let $\text{rk}(I - G) = n - 1$. Let w and v be the normalized left-hand unit eigenvectors of G and M , respectively. Then we have for any $i \in \mathcal{N}$*

$$v_i = \frac{(1 - \delta_i)w_i}{\sum_{j \in \mathcal{N}} (1 - \delta_j)w_j}. \quad (15)$$

Moreover,

$$\frac{\partial v_i}{\partial \delta_k} = \frac{w_k}{\sum_{j=1}^n w_j(1 - \delta_j)} \left(\frac{w_i(1 - \delta_i)}{\sum_{j=1}^n w_j(1 - \delta_j)} - \mathbb{1}_{i=k} \right) = \frac{w_k}{\sum_{j=1}^n w_j(1 - \delta_j)} (v_i - \mathbb{1}_{i=k}). \quad (16)$$

As it becomes apparent from (15) opinion leadership (power) v_i of some agent i is determined by the *combination* of her network centrality in G (w_i) and the individual

²⁹This index of centrality in a social network is recurrently defined via the rows of G' (i.e. via the columns of G): An agent's centrality is the weighted sum of centralities of the agents who listen to her.

conformity δ_i divided by the sum of these values over all agents. Thus, there is a complementary relationship between network centrality and $1 - \delta_i$: power becomes minimal ($v_i \rightarrow 0$) if either i 's network centrality approaches zero or if i is fully conform ($\delta_i \rightarrow 1$).

Taking the network G as given, we can observe the comparative statics with respect to δ_i . From (16) we get for all $i \in \mathcal{N}$ that opinion leadership is decreasing in own conformity δ_i and increasing in other agents' conformity δ_k , $k \neq i$, since $w_j \in [0, 1]$ and $1 - \delta_j \geq 0$ for all $j \in \mathcal{N}$. Thus, low own conformity fosters opinion leadership. The same is true if other agents are more conforming. We also may use (16) to examine which agent's power changes most in response to a marginal increase in her own conformity. From (16), we calculate that

$$\left| \frac{\partial v_i}{\partial \delta_i} \right| < \left| \frac{\partial v_j}{\partial \delta_j} \right| \Leftrightarrow w_j^2(1 - \delta_j) - w_i^2(1 - \delta_i) < (w_j - w_i) \sum_{k=1}^n w_k(1 - \delta_k). \quad (17)$$

Thus, if two agents have the same network centrality $w_i = w_j$, then by (17), $\left| \frac{\partial v_i}{\partial \delta_i} \right| < \left| \frac{\partial v_j}{\partial \delta_j} \right|$ if and only if $\delta_i < \delta_j$. In other words, the agent with the already higher degree of conformity and thus lower power loses even more power in response to a marginal increase in conformity compared with an agent with low conformity. Holding $\delta_i = \delta_j$, we get $\left| \frac{\partial v_i}{\partial \delta_i} \right| < \left| \frac{\partial v_j}{\partial \delta_j} \right|$ if and only if $w_i < w_j$, which implies that for two agents with equal conformity the agent with the higher network centrality loses more power when increasing own conformity.

We can also use Corollary 1 to compare opinion leadership in our model, v , with opinion leadership in the classic DeGroot model, w , (i.e. with the special case of our model where every agent i is honest, $\delta_i = 0$). For this purpose consider first a society where all agents are characterized by the same trait, i.e. $\delta_j = \bar{\delta}$ for all $j \in \mathcal{N}$. Then (15) yields $v = w$: opinion leadership is not affected by conformity when all agents are characterized by the same level of conformity. More generally, we have $v_i \geq w_i$ if and only if $\delta_i \leq \sum_{j \neq i} \frac{w_j}{\sum_{k \neq i} w_k} \delta_j$, i.e. an agent's power in our model compared to the classic DeGroot model is fostered if δ_i is below some average of the others' conformity parameters.

This is illustrated in Figure 4 which depicts power v_i as a function of conformity δ_i in a variation of Example 1. We fixed $\delta_1 = -0.7$ and study the comparative-static effect of Player 2's conformity level δ_2 on her power v_2 . If Player 2 is honest, her initial opinion's impact on the long-run consensus is approximately 0.37, it completely vanishes for conformity level δ_2 approaching 1, while counter-conformity allows Player 2 to "become" more important in a comparative-static sense. One can show generally that the power gain by counter-conforming is bounded by $v_i(\delta_i) \leq (2 - w_i)v_i(0)$. In this example, it is further bounded by the fact that too strong counter-conformity ($\delta_2 \leq -0.7$) leads to divergence of opinions.

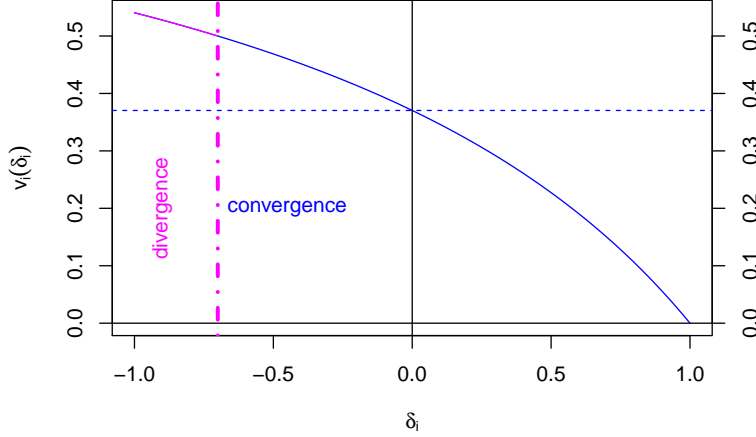


Figure 4: Power as a function of own conformity level.

5 Wisdom

The discussion so far applies to any continuous opinion including those for which no true value can be determined. In some applications, however, agents' opinions are more or less accurate with respect to some objective truth. As in the discrete context of Condorcet's Jury theorem, the question whether agents aggregate information in an efficient way is also of interest in the context of continuous opinions (Golub and Jackson, 2010; Acemoglu et al., 2010).

Therefore we assume that there is some true value $\mu \in \mathbb{R}$ and that all agents of the society receive independent unbiased signals about μ with individual precision (i.e. inverse of the variance) which constitute the agents' initial opinions. Formally, for all $i \in \mathcal{N}$, agent i 's initial opinion $x_i(0)$ is a random variable with expected value μ and some individual variance σ_i^2 , and all $x_i(0)$ are uncorrelated random variables. Assuming that opinion dynamics converge, a natural question to ask is how close the different steady state opinions will be to the true, but to the agents unknown, value μ .³⁰ To measure this difference between μ and an estimate $\hat{\mu}$, we use the mean squared error (MSE), which is defined as $E((\hat{\mu} - \mu)^2)$.³¹ The MSE can be decomposed into the squared bias $(E(\hat{\mu} - \mu))^2$ and the estimator's variance $\text{Var}(\hat{\mu})$:

$$E((\hat{\mu} - \mu)^2) = (E(\hat{\mu} - \mu))^2 + \text{Var}(\hat{\mu}).$$

³⁰Recall that in a steady state true opinions and stated opinions coincide and there is consensus within groups.

³¹The mean squared error as a measure of wisdom has also been used by Rauhut and Lorenz (2010).

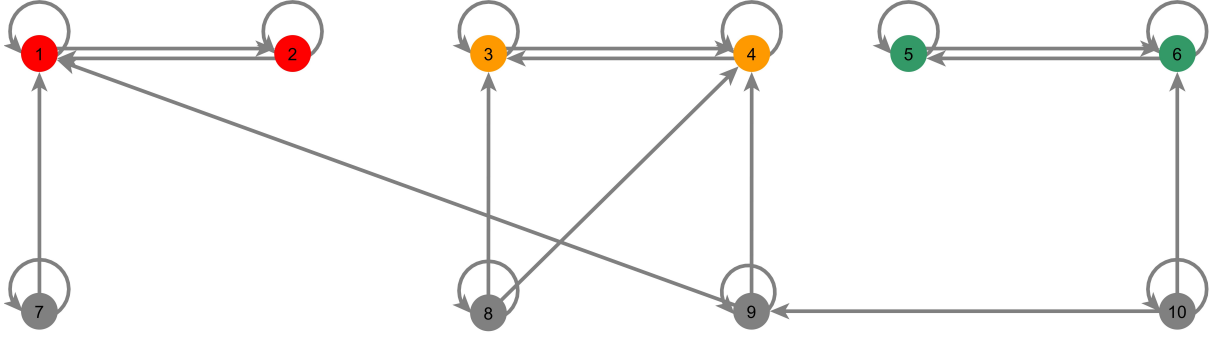


Figure 5: Network for wisdom example. This society consists of three closed and strongly connected groups of size two and four agents in the rest of the world.

As $x(\infty) = M^\infty x(0)$ and $M^\infty \mathbf{1} = \mathbf{1}$, it is obvious that $E(x(\infty)) = \mu \mathbf{1}$, i.e. all agents' long-run opinions are unbiased estimates for μ . Denoting by Σ the covariance matrix of $x(0)$, the corresponding MSEs are therefore given by the entries on the diagonal of $M^\infty \Sigma (M^\infty)'$. To study the effects of conformity on wisdom, we begin with an illustrative example.

5.1 Wisdom: an Example

Let $n = 10$, $(\sigma_1^2, \dots, \sigma_{10}^2) = (6, 4, 8, 7, 6, 3, 10, 12, 14, 16)$, and

$$G = \begin{pmatrix} 0.9 & 0.1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.4 & 0.6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.3 & 0.7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.7 & 0.3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.3 & 0.7 & 0 & 0 & 0 & 0 \\ 0.1 & 0 & 0 & 0 & 0 & 0 & 0.9 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0.3 & 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0.1 & 0 & 0 & 0.1 & 0 & 0 & 0 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0.2 & 0.6 \end{pmatrix}.$$

In this situation, we have $K = 3$ closed and strongly connected groups, $\mathcal{C}_1 = \{1, 2\}$, $\mathcal{C}_2 = \{3, 4\}$, and $\mathcal{C}_3 = \{5, 6\}$, while Players 7 to 10 form the rest of the world, as also illustrated in Figure 5. If all agents report their opinions truthfully ($\Delta = 0$), we find the MSEs equal to $(4, 4, 4, 4, 2.25, 2.25, 4, 4, 2, 1.0625)$. There are several notable features of this observation. First, due to the fact that their long-run opinions are equal, all agents within a given closed and strongly connected group share the same level of wisdom. Comparing the first two groups, we note that the MSEs of these two groups are

4 each, although the first group enjoys significantly better initial signals (of variances 6 and 4), while the second group seems to combine their less precise signals (of variances 8 and 7) much more effectively. It is also remarkable that Player 2, by communicating with Player 1, ends up with exactly the same MSE of 4 that she would reach if she used only her own signal. With respect to the rest of the world, notice that these agents typically have different MSEs. Furthermore, Players 7 and 8 each end up with the same MSE as the first two groups, while Players 9 and 10 achieve MSEs better than all members of the closed and strongly connected groups.

Now suppose that Players 2, 3, and 5 are conforming with $\delta_2 = 5/9$, $\delta_3 = 2/3$, and $\delta_5 = 1/2$ (and $\delta_i = 0$ for all other players). Then wisdom levels can be calculated to be (4.9, 4.9, 4, 4, 2, 2, 4.9, 4, 2.225, 1.05625). Thus, increasing conformity can lead to a decrease in wisdom (as the first group's MSE becomes larger), the same wisdom (as the second group's MSE does not change), or an increase in wisdom (as the third group's MSE becomes smaller). We also find that the agents in the rest of the world are affected by the changes in conformity of the agents in the closed and strongly connected groups: the MSE of Players 7 and 9 increases, while Player 10's MSE decreases slightly. It still holds that Player 7 and 8's MSEs equal that of the first and second group, respectively.

We will now proceed by systematically analyzing the principles underlying the distribution of wisdom within the society.

5.2 Wisdom of Groups

Due to (13), a group \mathcal{C}_k will, given convergence, eventually end up reaching a consensus where all agents' opinions are equal to $c_k = v'_{|\mathcal{C}_k} x(0)_{|\mathcal{C}_k} =: \hat{\mu}_k$. Hence, we can directly derive group \mathcal{C}_k 's wisdom as the MSE of $\hat{\mu}_k$.

Lemma 1. *The MSE of $\hat{\mu}_k$ is given by*

$$\text{MSE}_k := E((\hat{\mu}_k - \mu)^2) = \sum_{i \in \mathcal{C}_k} v_i^2 \sigma_i^2 = \sum_{i \in \mathcal{C}_k} \left(\frac{(1 - \delta_i) w_i}{\sum_{j \in \mathcal{C}_k} (1 - \delta_j) w_j} \right)^2 \sigma_i^2.$$

We may use Lemma 1 to identify the individual contributions to the MSE in a given group \mathcal{C}_k . First, from Lemma 1 it follows directly that

$$\text{MSE}_k = \sum_{i \in \mathcal{C}_k} v_i^2 \sigma_i^2 \leq \sum_{i \in \mathcal{C}_k} v_i \sigma_i^2 \leq \max_{i \in \mathcal{C}_k} \sigma_i^2, \quad (18)$$

since $v_i^2 \leq v_i$ due to $v_i \in (0, 1]$ for all agents i . Thus, group \mathcal{C}_k 's long-run opinion is on average at least as close to the true value μ as that of the agent with the least precise

signal. This worst case is given when both inequalities in (18) become equalities, which is the case for $v_i \in \{0, 1\}$ for all $i \in \mathcal{C}_k$ (first inequality) and $v_i = 0$ for all i with $\sigma_i^2 < \max_{j \in \mathcal{C}_k} \sigma_j^2$ (second inequality). Therefore, information updating within group \mathcal{C}_k is worst when importance is given to only one agent whose signal is most imprecise. This case would be approached if all other agents were close to full conformity, i.e. δ_i close to 1. We now consider the comparative static effect of one agent's conformity on the wisdom of her group.

Proposition 4. *The wisdom of a closed and strongly connected group \mathcal{C}_k is increasing in the conformity level of a group member i if and only if i 's product of signal variance and power is larger than the group's MSE, i.e.*

$$\frac{\partial \text{MSE}_k}{\partial \delta_i} \leq 0 \Leftrightarrow v_i \sigma_i^2 \geq \text{MSE}_k.$$

To give an interpretation for Proposition 4, let us rewrite $v_i \sigma_i^2 = \frac{v_i}{1/\sigma_i^2}$ and $\text{MSE}_k = \sum_{j \in \mathcal{C}_k} v_j \frac{v_j}{1/\sigma_j^2}$. This shows that it is not a person's expertise alone which is decisive for the question of how this person can increase the group's wisdom, rather, it is the ratio of power over signal precision, $\frac{v_i}{1/\sigma_i^2}$: if agents with a high ratio as compared to the group's average are more conforming, then this will reduce their power within the group, decrease the group's MSE, and thereby increase its wisdom. Vice versa, agents who are not powerful enough in relation to their signal precision will increase the group's wisdom if they are less conforming, because this will increase their power, decrease the group's MSE, and foster its wisdom.³²

The above discussion implies that in the best possible case, the ratio of power over signal precision is constant within a group: $v_i \sigma_i^2 = v_j \sigma_j^2$ for all $i, j \in \mathcal{C}_k$. This is formalized in the following corollary of Proposition 4.

Corollary 2. *For the wisdom of group \mathcal{C}_k as measured by MSE_k , we have*

$$\text{MSE}_k \geq \frac{1}{\sum_{j \in \mathcal{C}_k} \frac{1}{\sigma_j^2}} =: \text{MSE}_k^*, \quad (19)$$

with equality in (19) if and only if $v_i \sigma_i^2 = v_j \sigma_j^2$ for all $i, j \in \mathcal{C}_k$. The latter condition is equivalent to

$$\delta_i = 1 - a \frac{1}{\sigma_i^2 w_i \sum_{j \in \mathcal{C}_k} \frac{1}{\sigma_j^2}} \text{ for all } i \in \mathcal{C}_k \quad (20)$$

for some constant $a \in (0, 2 \sum_{j \in \mathcal{C}_k} \frac{1}{\sigma_j^2} \min_{j \in \mathcal{C}_k} w_j \sigma_j^2)$.

³²An analogous discussion can be already found in DeMarzo et al. (2003) for the case where agents are honest.

Corollary 2 delivers the analogue to (18). While (18) describes the worst case with respect to wisdom, Corollary 2 considers the best scenario: all agents within the same closed and strongly connected group share the same ratio of power over signal precision, and this case can always be constructed if the agents' conformity is distributed suitably. In particular, choosing $a \in (0, \sum_{j \in \mathcal{C}_k} \frac{1}{\sigma_j^2} \min_{j \in \mathcal{C}_k} w_j \sigma_j^2]$ in (20) ensures $\delta_i \geq 0$ for all $i \in \mathcal{C}_k$ and therefore by Proposition 2 guarantees convergence of the opinions in \mathcal{C}_k to the best possible consensus $\hat{\mu}_k$. Notice also that the optimal MSE is smaller than individual signal variance σ_i^2 for all agents i in group \mathcal{C}_k , as is easily seen from (19). Therefore, under optimal conformity all agents within \mathcal{C}_k benefit from communication.

Reconsidering the example discussed in Subsection 5.1, we find the network centralities (the left-hand unit eigenvectors of G) to be $w_1 = 0.8$, $w_2 = 0.2$, $w_3 = 0.6$, $w_4 = 0.4$, $w_5 = 0.5$, and $w_6 = 0.4$. Therefore, in (20) the constant a can be chosen in $(0, 2/3)$ (group 1) and $(0, 3/2)$ (groups 2 and 3). Choosing $a = 1/3$ (group 1) and $a = 3/4$ (groups 2 and 3) delivers $\delta_1 = 5/6$, $\delta_3 = 5/12$, and $\delta_5 = 1/2$ (and $\delta_i = 0$ for all other agents). Thus, choosing the agents' degrees of conformity according to these values ensures the optimal wisdom within the respective groups, given by $(2.4, 2.4, 3.7\bar{3}, 3.7\bar{3}, 2, 2, 2.4, 3.7\bar{3}, 1.5\bar{3}, 0.88\bar{3})$. The same level could also be reached for other conformity levels, for instance, choosing $a = 1/4$ (first group), $a = 3/7$ (second group), and $a = 3/8$ (third group) in (20), we find that the conformity levels $\delta_{1:6} = (7/8, 1/4, 2/3, 3/7, 3/4, 1/2)$ also lead to the optimal wisdom. Notice that, as in Golub and Jackson (2010), wisdom thus is independent of the speed of convergence, as we have two examples with the same optimal wisdom but different speeds of convergence (the last-mentioned conformity levels lead to slightly slower convergence than the earlier mentioned ones).

5.3 Wisdom within the Rest of the World

Let us recall that agents in the rest of the world do not necessarily share a consensus opinion in the long run, so that we will typically have individual wisdom levels. Due to (14), we have the following formula for the long-run opinions within the rest of the world: $x(\infty)_{|\mathcal{R}} = \Gamma \hat{\mu}$, with $\hat{\mu} := (\hat{\mu}_1, \dots, \hat{\mu}_K)'$. Therefore, the wisdom levels in the rest of the world depend on the conformity levels of the agents in the closed and strongly connected groups as these affect the consensus opinions $\hat{\mu}_k$ of these groups. On the other hand, as neither the initial signals nor the conformity levels of the agents in the rest of the world play any role for their long-run opinions, these agents' wisdom is independent of their conformity levels as well as of their initial signals. In other words, if the rest of the world is non-empty, information processing in the society is necessarily inefficient as the information contained in these agents' initial signals is inevitably lost. Assuming convergence, let $\gamma_{i,k}$ denote the long-term weight of the group \mathcal{C}_k on the opinion of agent

$i \in \mathcal{R}$, i.e. $x_i(\infty) = \sum_{k=1}^K \gamma_{i,k} \hat{\mu}_k$ (cf. (14)). This immediately translates into the wisdom of an agent $i \in \mathcal{R}$ as follows:

$$E((x_i(\infty) - \mu)^2) = \sum_{k=1}^K \gamma_{i,k}^2 \text{MSE}_k \leq \max_{k=1, \dots, K} \text{MSE}_k. \quad (21)$$

The wisdom of an agent in the rest of the world depends on the wisdom within the closed and strongly connected groups. More precisely, an agent i 's wisdom only depends on the wisdom of groups \mathcal{C}_k to which there is a directed path in the network G because this corresponds to $\gamma_{i,k} > 0$. The worst case for an agent in the rest of the world is to be influenced only by agents of one closed and strongly connected group with maximal MSE. With regard to the example discussed in subsection 5.1 this is the case for Players 7 and 8 who have directed paths only into group 1 and group 2, respectively, such that they share their MSEs of 4 (cf. Figure 5). Player 9, however, who has directed paths into both groups with MSE of 4 reaches an MSE of 2 since the long-term weights $\gamma_{9,1} = 0.5$ and $\gamma_{9,2} = 0.5$ are squared in (21). Finally, Player 10 has directed paths into these groups via Player 9 and, moreover, has a directed path into group 3. Player 10 therefore is able to combine MSEs of 4, 4, and 2.25 into an MSE as low as 1.0625. It is intuitive that for maximal wisdom of an agent in the rest of the world, all groups' signals have to be accessed with some kind of balanced group weights. The following proposition confirms this intuition.

Proposition 5. *For agents $i \in \mathcal{R}$, we have:*

$$E((x_i(\infty) - \mu)^2) \geq \frac{1}{\sum_{k=1}^K \frac{1}{\text{MSE}_k}}, \quad (22)$$

with equality if and only if $\gamma_{i,k} = \frac{1}{\text{MSE}_k \sum_{l=1}^K \frac{1}{\text{MSE}_l}}$ for all $k = 1, \dots, K$.

Therefore, the highest wisdom is achieved if an agent in the rest of the world averages the different groups' opinions in such a way that the product of weight put on a group and its MSE is constant for all groups: the better a group's estimate, the more weight it should get. Nevertheless, as all the optimal weights are positive, this optimum can only be achieved if from agent i there is a directed path into all the closed and strongly connected groups. Notice also that the optimal weights depend on the groups' MSEs such that an agent in the rest of the world who is initially characterized by optimal weights would no longer average the groups' opinions optimally if conformity levels within the groups were to change.

It is remarkable that an agent in the rest of the world who is connected to multiple groups can reach a significantly lower MSE than the best informed agents from those

groups. Thus, the fact that agents in the rest of the world are absolutely powerless does not imply that they are not wise.

6 Concluding Remarks

So far, the literature on opinion dynamics has focused on truthful opinion representation either with a Bayesian approach (Banerjee, 1992; Bikhchandani et al., 1992; Smith and Sorensen, 2000; Gale and Kariv, 2003; Acemoglu et al., 2011) or assuming naïve updating according to a learning matrix (DeGroot, 1974; DeMarzo et al., 2003; Golub and Jackson, 2010; Acemoglu et al., 2010). Despite some disputable assumptions in both approaches, as Acemoglu and Ozdaglar (2011) point out, these models serve well to study conditions under which societies will eventually reach a state of agreement, i.e. consensus. Moreover, in both contexts the aggregation of initial opinions may, but need not, be “asymptotically efficient,” in the sense that social learning leads to a high accuracy of information in the long run. One basic force fostering efficient information aggregation even among naïve agents is a statistical effect of growing sample size (which is also called “the wisdom of crowds”) such as in Condorcet’s Jury Theorem. On the other hand, prominent agents or opinion leaders might reduce the accuracy of information aggregation by superseding valuable opinions of others.

To our best knowledge, this paper is the first contribution to incorporate misrepresentation of opinions among naïve agents. We assume that individuals depart from their true opinion by conforming or counter-conforming with their peer group which is a well documented phenomenon (Deutsch and Gerard, 1955; Jones, 1984; Zafar, 2011). While we follow the literature based on DeGroot (1974) in modeling *informational social influence* as naïve updating of opinions through the network, we, thus, also model *normative social influence* by including conforming/counter-conforming behavior. In order to study the effects of conformity on long-run opinions and information aggregation, we characterize sufficient conditions for convergence and characterize the long-run opinions in this dynamic framework. When all agents are conforming or honest, then opinions converge (Proposition 2).

Assuming convergence, we then characterize the long-run (consensus) opinion in each closed and strongly connected group under conformity (Theorem 1). Thereby, we are in a position to study the impact of the individual levels of conformity on opinion leadership and on wisdom of the society. Opinion leaders are those whose initial opinion has a high impact on consensus. We find that this influence is increasing in network centrality (as in the DeGroot model), but moreover decreasing in the individual level of conformity (Corollary 1). Thus, taking the network as given, we conclude that low conformity fosters opinion leadership while high conformity undermines opinion leadership. This re-

sult is fully in line with empirical evidence that opinion leaders are characterized by a higher inclination to “publicly individuate” themselves (Chan and Misra, 1990). Therefore, counter-conformity might be interpreted as a persuasion device since not only the connected agents’ opinions of next period are swayed towards own opinion but a higher impact on the consensus opinion is achieved.

The effect of heterogeneous levels of conformity on wisdom of the society is ambiguous. Here, wisdom is defined as the mean squared error (MSE) of the consensus opinion where agents’ initial opinions are noisy but unbiased signals about some true state of the world with heterogeneous signal precision. Increasing conformity of a given individual need not undermine the wisdom of the society, but can also enhance it or leave it unchanged. We find that increasing conformity of agents with high power and low signal precision increases the group’s wisdom (Proposition 4). In particular, optimal wisdom within a given closed and strongly connected group is achieved if distribution of conformity levels is such that ratio of power over signal precision is balanced across agents (Corollary 2). This result resembles the fact that reducing prominence of individuals – in particular prominence of uninformed agents – increases the accuracy of information aggregation. While in the previous literature reduction of prominence is achieved by increasing population size (see e.g. Golub and Jackson, 2010), in our model this can be achieved by conformity and therefore also holds for small groups. Finally, when considering agents in the rest of the world, we find that their levels of conformity have no influence on wisdom. Although powerless, individuals in the rest of the world can be quite wise since they may aggregate information from different groups.

The model presented here contains some simplifying assumptions which may be relaxed in future research. First, we assumed that the social network is exogenous and stays fixed over time. In the literature we can find models where the network structure may vary over time such that only agents with “close opinions” are listened to (Hegselmann and Krause, 2002), self-confidence varies (DeMarzo et al., 2003), and general changes are possible (Lorenz, 2005). It would be interesting to see how changes in the learning structure, either exogenously or endogenously, affect our results. Second, we assumed that interaction neighborhood equals observation neighborhood in the sense that agents conform or counter-conform with those agents they listen to. If this assumption is relaxed, the group structure may no longer be preserved and interesting applications to lobbying (addressing a certain group) become possible. We leave these ideas and possible other extensions to future research.

A Mathematical Appendix

A.1 Expressed Opinions

Proof of Proposition 1

First, notice that $s(t)$ by construction satisfies $s(t) = (I - \Delta)x(t) + \Delta Y s(t)$ with $Y := (I - D)^{-1}(G - D)$ and that for all $i \in A^\tau(t)$, $s_i^\tau(t)$ is the i -th component of $(I - \Delta)x(t) + \Delta Y s^{\tau-1}(t)$. For all $i \in A^\tau(t)$, we therefore find $s_i^\tau(t) - s_i(t)$ as the i -th component of $\Delta Y (s^{\tau-1}(t) - s(t))$. As Y is obviously a row-stochastic matrix, we immediately have $|s_i^\tau(t) - s_i(t)| \leq \delta^* \|s^{\tau-1}(t) - s(t)\|_\infty$ for all $i \in A^\tau(t)$, with $\delta^* := \max_{i \in \mathcal{N}} |\delta_i| < 1$, while we have $|s_i^\tau(t) - s_i(t)| = |s_i^{\tau-1}(t) - s_i(t)| \leq \|s^{\tau-1}(t) - s(t)\|_\infty$ for all $i \notin A^\tau(t)$. Together, we therefore have that $\|s^\tau(t) - s(t)\|_\infty \leq \|s^{\tau-1}(t) - s(t)\|_\infty$ for all τ , showing that the distance between $s^\tau(t)$ and $s(t)$ measured using the $\|\cdot\|_\infty$ -norm is a non-increasing and therefore converging sequence.

Now, let $U_i(t) := \{\tau \in \mathbb{N} : i \in A^\tau(t)\}$, for each agent i . Using the assumption that every agent i belongs almost surely to infinitely many $A^\tau(t)$, we define $\tau_1 := \min\{\tau \in \mathbb{N} : (\forall i \in \mathcal{N})(U_i(t) \cap \{1, \dots, \tau\} \neq \emptyset)\}$ as the first time-step where every agent has at least once been satisfied with her stated opinion.³³ Given the above, it is easy to see that $\|s^{\tau_1}(t) - s(t)\|_\infty \leq \delta^* \|s^0(t) - s(t)\|_\infty$. Proceeding in the same way by recursively defining $\tau_{k+1} := \min\{\tau > \tau_k : (\forall i \in \mathcal{N})(U_i(t) \cap \{\tau_k + 1, \dots, \tau\} \neq \emptyset)\}$ as the first time-step after τ_k such that all agents have at least been once satisfied with their stated opinion, we then have $\|s^{\tau_k}(t) - s(t)\|_\infty \leq (\delta^*)^k \|s^0(t) - s(t)\|_\infty$, yielding that $\|s^{\tau_k}(t) - s(t)\|_\infty$ and therefore also $\|s^\tau(t) - s(t)\|_\infty$ converges to 0. \square

A.2 Convergence

To prove Proposition 2 the following lemma is helpful.

Lemma A.1 (I-M). $I - M = (I - (G - D)\Delta(I - D)^{-1})^{-1}(I - G)$.

Proof of Lemma A.1 (I-M)

First, we can rewrite M , given by (6), to obtain

$$M = G - (G - D)(I - \Delta(I - D)^{-1}(G - D))^{-1}\Delta(I - (I - D)^{-1}(G - D)).$$

³³The assumption that all $U_i(t)$ are almost surely infinite guarantees that τ_1, τ_2, \dots are almost surely well-defined.

This can be verified by the following calculation.

$$\begin{aligned}
M &= D + (G - D)(I - \Delta(I - D)^{-1}(G - D))^{-1}(I - \Delta) \\
&= D + (G - D)[I - \Delta(I - D)^{-1}(G - D)]^{-1}[I - \Delta(I - D)^{-1}(G - D) \\
&\quad + \Delta(I - D)^{-1}(G - D) - \Delta] \\
&= D + (G - D)(I + [I - \Delta(I - D)^{-1}(G - D)]^{-1}[\Delta(I - D)^{-1}(G - D) - \Delta]) \\
&= G - (G - D)[I - \Delta(I - D)^{-1}(G - D)]^{-1}\Delta[I - (I - D)^{-1}(G - D)].
\end{aligned}$$

Thus,

$$\begin{aligned}
I - M &= I - G + (G - D)[I - \Delta(I - D)^{-1}(G - D)]^{-1}\Delta(I - D)^{-1}(I - G) \\
&= \left(I + (G - D)[I - \Delta(I - D)^{-1}(G - D)]^{-1}\Delta(I - D)^{-1} \right) (I - G). \tag{A.1}
\end{aligned}$$

Now, note that for any $n \times m$ -matrix A and any $m \times n$ -matrix B , with I_k the k -dimensional identity matrix ($k \in \{n, m\}$), we have that $I_n - AB$ is invertible if and only if $I_m - BA$ is invertible, and then $(I_n - AB)^{-1} = I_n + A(I_m - BA)^{-1}B$, since $(I_n + A(I_m - BA)^{-1}B)(I_n - AB) = I_n - AB + A(I_m - BA)^{-1}B - A(I_m - BA)^{-1}BAB = I_n - AB + A(I_m - BA)^{-1}(I_m - BA)B = I_n$. Taking $A = G - D$ and $B = \Delta(I - D)^{-1}$ in (A.1) then gives $I - M = (I - (G - D)\Delta(I - D)^{-1})^{-1}(I - G)$. \square

Proof of Proposition 2

Denote $Y := (I - D)^{-1}(G - D)$ which is row stochastic. Thus, as $|\delta_i| < 1$ for all $i \in \mathcal{N}$, we have that $I - \Delta Y$ is invertible and $(I - \Delta Y)^{-1} = \sum_{k=0}^{\infty} (\Delta Y)^k$. Moreover, if $\delta_i \geq 0$ for all $i \in \mathcal{N}$, the sum $\sum_{k=0}^{\infty} (\Delta Y)^k$ is a sum of non-negative matrices, implying that $(I - \Delta Y)^{-1}$ has only non-negative entries. Hence $M = D + (G - D)[I - \Delta Y]^{-1}(I - \Delta)$ is non-negative since it is the product of non-negative matrices (since $0 < g_{ii} < 1$) added to D , which is a diagonal matrix with strictly positive entries ($0 < g_{ii}$). Finally, since $M\mathbb{1} = \mathbb{1}$ by Lemma A.1, we get that M is row stochastic. Since the diagonal of D is strictly positive, we get that the diagonal of M is strictly positive, $m_{ii} > 0$, implying aperiodicity of M . Thus M^t converges. \square

Proposition A.1 (Blocks). *Let G be given as in (10), i.e. organized into blocks according*

to the partition $\Pi(\mathcal{N}, G) = \{\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_K, \mathcal{R}\}$. Then for every $t = 1, 2, \dots$ we have

$$M^t = \begin{pmatrix} M_{11}^t & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & M_{KK}^t & 0 \\ (M^t)_{\mathcal{R}1} & \cdots & \cdots & (M^t)_{\mathcal{R}K} & M_{\mathcal{R}\mathcal{R}}^t \end{pmatrix}$$

with

$$M_{kk}^t = [I - (I - (G_{kk} - D_{kk})\Delta_{kk}(I - D_{kk})^{-1})^{-1}(I - G_{kk})]^t$$

for all $k = 1, \dots, K, \mathcal{R}$, and

$$(M^t)_{\mathcal{R}k} = \sum_{l=0}^{t-1} M_{\mathcal{R}\mathcal{R}}^l M_{\mathcal{R}k} M_{kk}^{t-1-l},$$

where $M_{\mathcal{R}k} = (I - (G_{\mathcal{R}\mathcal{R}} - D_{\mathcal{R}\mathcal{R}})\Delta_{\mathcal{R}\mathcal{R}}(I - D_{\mathcal{R}\mathcal{R}})^{-1})^{-1}G_{\mathcal{R}k}[(I - \Delta_{kk}(I - D_{kk})^{-1}(G_{kk} - D_{kk}))^{-1}(I - \Delta_{kk})]$ for all $k = 1, \dots, K$.

Proof of Proposition A.1

Let $Z := [I - \Delta(I - D)^{-1}(G - D)]^{-1}(I - \Delta)$ to simplify $s = Zx$ and $M = D + (G - D)Z$. We now proceed in three steps: we first characterize Z , then M , and finally M^t . Let G be given as in (10). Then simple but tedious block matrix algebra together with Lemma A.1 yields:

1.

$$Z = \begin{pmatrix} Z_{11} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & Z_{KK} & 0 \\ Z_{\mathcal{R}1} & \cdots & \cdots & Z_{\mathcal{R}K} & Z_{\mathcal{R}\mathcal{R}} \end{pmatrix}$$

with

$$Z_{kk} = (I - \Delta_{kk}(I - D_{kk})^{-1}(G_{kk} - D_{kk}))^{-1}(I - \Delta_{kk}),$$

$$Z_{\mathcal{R}k} = Z_{\mathcal{R}\mathcal{R}}(I - \Delta_{\mathcal{R}\mathcal{R}})^{-1}\Delta_{\mathcal{R}\mathcal{R}}(I - D_{\mathcal{R}\mathcal{R}})^{-1}G_{\mathcal{R}k}Z_{kk}$$

for all $k = 1, \dots, K$, and

$$Z_{\mathcal{R}\mathcal{R}} = (I - \Delta_{\mathcal{R}\mathcal{R}}(I - D_{\mathcal{R}\mathcal{R}})^{-1}(G_{\mathcal{R}\mathcal{R}} - D_{\mathcal{R}\mathcal{R}}))^{-1}(I - \Delta_{\mathcal{R}\mathcal{R}}).$$

2. For $M = D + (G - D)Z = I - (I - (G - D)\Delta(I - D)^{-1})^{-1}(I - G)$, we get

$$M = \begin{pmatrix} M_{11} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & M_{KK} & 0 \\ M_{\mathcal{R}1} & \cdots & \cdots & M_{\mathcal{R}K} & M_{\mathcal{R}\mathcal{R}} \end{pmatrix}$$

with

$$\begin{aligned} M_{kk} &= D_{kk} + (G_{kk} - D_{kk})(I - \Delta_{kk}(I - D_{kk})^{-1}(G_{kk} - D_{kk}))^{-1}(I - \Delta_{kk}) \\ &= I - (I - (G_{kk} - D_{kk})\Delta_{kk}(I - D_{kk})^{-1})^{-1}(I - G_{kk}), \end{aligned}$$

$$\begin{aligned} M_{\mathcal{R}k} &= G_{\mathcal{R}k}Z_{kk} + (G_{\mathcal{R}\mathcal{R}} - D_{\mathcal{R}\mathcal{R}})Z_{\mathcal{R}k} \\ &= (I - (G_{\mathcal{R}\mathcal{R}} - D_{\mathcal{R}\mathcal{R}})\Delta_{\mathcal{R}\mathcal{R}}(I - D_{\mathcal{R}\mathcal{R}})^{-1})^{-1}G_{\mathcal{R}k}Z_{kk} \end{aligned}$$

for all $k = 1, \dots, K$, and

$$\begin{aligned} M_{\mathcal{R}\mathcal{R}} &= D_{\mathcal{R}\mathcal{R}} + (G_{\mathcal{R}\mathcal{R}} - D_{\mathcal{R}\mathcal{R}})(I - \Delta_{\mathcal{R}\mathcal{R}}(I - D_{\mathcal{R}\mathcal{R}})^{-1}(G_{\mathcal{R}\mathcal{R}} - D_{\mathcal{R}\mathcal{R}}))^{-1}(I - \Delta_{\mathcal{R}\mathcal{R}}) \\ &= I - (I - (G_{\mathcal{R}\mathcal{R}} - D_{\mathcal{R}\mathcal{R}})\Delta_{\mathcal{R}\mathcal{R}}(I - D_{\mathcal{R}\mathcal{R}})^{-1})^{-1}(I - G_{\mathcal{R}\mathcal{R}}). \end{aligned}$$

3. Finally, we claim that for every $t \in \mathbb{N} \setminus \{0\}$,

$$M^t = \begin{pmatrix} M_{11}^t & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & M_{KK}^t & 0 \\ (M^t)_{\mathcal{R}1} & \cdots & \cdots & (M^t)_{\mathcal{R}K} & M_{\mathcal{R}\mathcal{R}}^t \end{pmatrix}$$

with $(M^t)_{\mathcal{R}k} = \sum_{l=0}^{t-1} M_{\mathcal{R}\mathcal{R}}^l M_{\mathcal{R}k} M_{kk}^{t-1-l}$ for all $k = 1, \dots, K$.

The assertion for the diagonal elements $M_{11}^t, \dots, M_{KK}^t$ and $M_{\mathcal{R}\mathcal{R}}^t$ is trivial. We prove the formula for $M_{\mathcal{R}k}^t$ by induction.

- For $t = 1$, the assertion is trivial.
- $t \mapsto t + 1$: first, we have $(M^{t+1})_{\mathcal{R}k} = (M^t M)_{\mathcal{R}k} = (M^t)_{\mathcal{R}k} M_{kk} + M_{\mathcal{R}\mathcal{R}}^t M_{\mathcal{R}k}$ by simple matrix multiplication. Inserting $(M^t)_{\mathcal{R}k} = \sum_{l=0}^{t-1} M_{\mathcal{R}\mathcal{R}}^l M_{\mathcal{R}k} M_{kk}^{t-1-l}$, we

find

$$\begin{aligned}(M^{t+1})_{\mathcal{R}k} &= \left(\sum_{l=0}^{t-1} M_{\mathcal{R}\mathcal{R}}^l M_{\mathcal{R}k} M_{kk}^{t-1-l} \right) M_{kk} + M_{\mathcal{R}\mathcal{R}}^t M_{\mathcal{R}k} \\ &= \sum_{l=0}^{t+1-1} M_{\mathcal{R}\mathcal{R}}^l M_{\mathcal{R}k} M_{kk}^{t+1-1-l},\end{aligned}$$

which concludes the proof. \square

Proof of Proposition 3

1. ‘Only if’: this is proven in the first part of the proof of Theorem 1.
2. ‘If’: Suppose each M_{kk}^t converges and $M_{\mathcal{R}\mathcal{R}}^t$ converges to 0. First, since M_{kk}^t converges, its only eigenvalue with $|\lambda| \geq 1$ is $\lambda = 1$ with algebraic and geometric multiplicity equal to 1 for every $k = 1, \dots, K$. On the other hand, $M_{\mathcal{R}\mathcal{R}}^t \rightarrow 0$ implies that the eigenvalues of $M_{\mathcal{R}\mathcal{R}}$ are all smaller than 1 in absolute value and, thus, $M_{\mathcal{R}\mathcal{R}} - \lambda I$ is invertible for all complex numbers λ with $|\lambda| \geq 1$.

Now, let the complex number $\tilde{\lambda}$ be either outside of the unit circle ($|\tilde{\lambda}| > 1$) or exactly on the unit circle ($|\tilde{\lambda}| = 1$), but different from 1. Taking into account the block structure of M , we easily see that any solution of $(M - \tilde{\lambda}I)x = 0$ must satisfy $x_{|\mathcal{C}_1} = 0, \dots, x_{|\mathcal{C}_K} = 0$, and therefore also $x_{|\mathcal{C}_{\mathcal{R}}} = 0$, so that we can conclude that $\lambda = 1$ is the only possible eigenvalue of M with $|\lambda| \geq 1$.

In order to show convergence of M^t , we therefore have to show that algebraic and geometric multiplicity of $\lambda = 1$ coincide. With regard to algebraic multiplicity, the block structure of M implies $\det(M - \lambda I) = \prod_{k=1}^K \det(M_{kk} - \lambda I) \det(M_{\mathcal{R}\mathcal{R}} - \lambda I)$, such that the algebraic multiplicity of $\lambda = 1$ is the sum of the algebraic multiplicities of M_{11}, \dots, M_{KK} and $M_{\mathcal{R}\mathcal{R}}$, which are given by 1 and 0, respectively, since M_{kk} is by definition irreducible for all $k = 1, \dots, K$. Consequently, the algebraic multiplicity equals K . With regard to geometric multiplicity, the block structure of M implies that for any real numbers c_1, \dots, c_K , the vector x of the form $x_{|\mathcal{C}_k} = c_k \mathbb{1}_{|\mathcal{C}_k}$ ($k = 1, \dots, K$) and $x_{|\mathcal{C}_{\mathcal{R}}} = (I - M_{\mathcal{R}\mathcal{R}})^{-1} \sum_{k=1}^n c_k M_{\mathcal{R}k} \mathbb{1}_{|\mathcal{C}_k}$ is an eigenvector to M for $\lambda = 1$, implying that the geometric multiplicity is at least K , thereby concluding the proof. \square

A.3 Long run

Lemma A.2. *The following statements are equivalent:*

1. True opinions $x(t)$ converge for $t \rightarrow \infty$.
2. Stated opinions $s(t)$ converge for $t \rightarrow \infty$.
3. Perceived opinions $q(t)$ converge for $t \rightarrow \infty$.

Moreover, if the true, stated, and perceived opinions converge, then the limits coincide:

$$\lim_{t \rightarrow \infty} x(t) = \lim_{t \rightarrow \infty} s(t) = \lim_{t \rightarrow \infty} q(t).$$

Proof of Lemma A.2

From Proposition 1, we get that $s(t) = (I - \Delta(I - D)^{-1}(G - D))^{-1} (I - \Delta)x(t)$. Thus convergence of $x(t)$ implies convergence of $s(t)$. By definition we have that $q(t) = (I - D)^{-1}(G - D)s(t)$, and hence convergence of $s(t)$ implies convergence of $q(t)$. To see that convergence of $q(t)$ implies convergence of $x(t)$, we use that $x(t + 1) = Dx(t) + (G - D)s(t) = Dx(t) + (I - D)q(t)$. For all $t \geq 0$, this implies $x(t) = D^t x(0) + \sum_{l=0}^{t-1} D^{t-1-l} (I - D)q(l)$, the first part of which converges to 0 because all elements of the diagonal matrix D belong to $[0, 1)$. The limit of $x(t)$ therefore equals

$$\begin{aligned} \lim_{t \rightarrow \infty} \sum_{l=0}^{t-1} D^{t-1-l} (I - D)q(l) &= \lim_{t \rightarrow \infty} \sum_{l=0}^{t-1} D^{t-1-l} (I - D) (q(l) - q(\infty)) \\ &\quad + \lim_{t \rightarrow \infty} \sum_{l=0}^{t-1} D^{t-1-l} (I - D)q(\infty). \end{aligned}$$

First, note that the second limit equals $q(\infty)$, because $\sum_{l=0}^{\infty} D^l = (I - D)^{-1}$. For the first limit, note that for any $\varepsilon > 0$, we can find an index l_ε such that we have $\|q(l) - q(\infty)\| < \varepsilon$ for all $l > l_\varepsilon$. Splitting the sum into small l ($l \leq l_\varepsilon$) and large l ($l > l_\varepsilon$), we then easily see that the first term converges to 0. Therefore, $x(t)$ converges to $q(\infty)$. Since $s(t) = (I - \Delta)x(t) + \Delta q(t)$, $s(t)$ also shares the same limit. \square

To prove Theorem 1, the following Lemma is helpful.

Lemma A.3 (Convergence to Eigenvector). *Let A be an $n \times n$ -matrix with $A\mathbb{1} = \mathbb{1}$ and $\text{rk}(I - A) = n - 1$. If A^t converges to A^∞ for $t \rightarrow \infty$, then $A^\infty = \mathbb{1}w'$, with w' the unique normalized left eigenvector of A associated with the eigenvalue 1.*

Proof of Lemma A.3

Obviously, $AA^\infty = A^\infty = A^\infty A$. This implies that

- the columns of A^∞ must be multiples of $\mathbb{1}$,
- the rows of A^∞ must be multiples of w' ,

from which we find $A^\infty = r \mathbb{1} w'$ for some real number r which is found to be equal to 1 as $\mathbb{1} = A^\infty \mathbb{1} = r \mathbb{1} w' \mathbb{1} = r \mathbb{1}$. \square

Proof of Theorem 1

We first derive the formula for M_{kk}^∞ . Then we will turn to $M_{\mathcal{R}\mathcal{R}}^\infty$ and $M_{\mathcal{R}k}^\infty$.

Assume for the moment that $\text{rk}(I - G) = n - 1$. Then, as $v'(M - I) = 0$, we have due to Lemma A.1

$$0 = v'(I - M) = v' \left(I - (G - D)\Delta(I - D)^{-1} \right)^{-1} (I - G),$$

implying

$$v' \left(I - (G - D)\Delta(I - D)^{-1} \right)^{-1} = r w'$$

for some real number r . Using $w'G = w'$, we then find

$$v' = r w' \left(I - (G - D)\Delta(I - D)^{-1} \right) = r w' \left(I - (I - D)\Delta(I - D)^{-1} \right) = r w'(I - \Delta).$$

The normalization of v then entails $r = \frac{1}{w'(I - \Delta)\mathbb{1}}$, which shows that $v = \frac{(I - \Delta)w}{\mathbb{1}'(I - \Delta)w}$. Now, relaxing the assumption $\text{rk}(I - G) = n - 1$, the formula for M_{kk}^∞ follows.

To determine the formulas for $M_{\mathcal{R}\mathcal{R}}^\infty$ and $M_{\mathcal{R}k}^\infty$, we first establish that $GM^\infty = M^\infty$. We have $Gx = x \Leftrightarrow (I - G)x = 0 \Leftrightarrow [I - (G - D)\Delta(I - D)^{-1}]^{-1}(I - G)x = 0$, since by Lemma A.1 $[I - (G - D)\Delta(I - D)^{-1}]$ is invertible. Thus by Lemma A.1, $Gx = x$ if and only if $Mx = x$. Furthermore, $MM^\infty x = M^\infty x$ and therefore $GM^\infty x = M^\infty x$ for all n -dimensional vectors x , delivering $GM^\infty = M^\infty$. This implies

- $M_{\mathcal{R}\mathcal{R}}^\infty = G_{\mathcal{R}\mathcal{R}}M_{\mathcal{R}\mathcal{R}}^\infty$ and therefore $(I - G_{\mathcal{R}\mathcal{R}})M_{\mathcal{R}\mathcal{R}}^\infty = 0$, entailing $M_{\mathcal{R}\mathcal{R}}^\infty = 0$ because $I - G_{\mathcal{R}\mathcal{R}}$ is invertible,
- $M_{\mathcal{R}k}^\infty = G_{\mathcal{R}k}M_{kk}^\infty + G_{\mathcal{R}\mathcal{R}}M_{\mathcal{R}k}^\infty$, and therefore $M_{\mathcal{R}k}^\infty = (I - G_{\mathcal{R}\mathcal{R}})^{-1}G_{\mathcal{R}k}M_{kk}^\infty$. \square

A.4 Wisdom

Proof of Lemma 1

First, $\hat{\mu}_k$ is easily seen to be unbiased for μ because $\sum_{i \in \mathcal{C}_k} v_i = 1$. Therefore, its MSE equals its variance which is given by $\sum_{i \in \mathcal{C}_k} v_i^2 \sigma_i^2$ as the $x_i(0)$ are uncorrelated. \square

Proof of Proposition 4

$$\frac{\partial \text{MSE}_k}{\partial \delta_i} = \frac{\partial \sum_{j \in \mathcal{C}_k} v_j^2 \sigma_j^2}{\partial \delta_i} = \sum_{j \in \mathcal{C}_k} 2\sigma_j^2 v_j \frac{\partial v_j}{\partial \delta_i} \stackrel{(16)}{=} \frac{2w_i}{\sum_{j \in \mathcal{C}_k} w_j(1 - \delta_j)} \sum_{j \in \mathcal{C}_k} \sigma_j^2 v_j (v_j - 1_{j=i}).$$

The assertion follows easily noting that $\text{MSE}_k = \sum_{j \in \mathcal{C}_k} v_j v_j \sigma_j^2$. \square

Proof of Proposition 5 First, notice that $E((x_i(\infty) - \mu)^2) = \sum_{k=1}^K \gamma_{i,k}^2 \text{MSE}_k$, with $\sum_{k=1}^K \gamma_{i,k} = 1$ for all $i \in \mathcal{R}$. By the Cauchy-Schwarz inequality, we have

$$1 = \sum_{k=1}^K \gamma_{i,k} = \sum_{k=1}^K \left(\gamma_{i,k} \sqrt{\text{MSE}_k} \right) \frac{1}{\sqrt{\text{MSE}_k}} \leq \sqrt{\sum_{k=1}^K \gamma_{i,k}^2 \text{MSE}_k} \sqrt{\sum_{k=1}^K \frac{1}{\text{MSE}_k}},$$

with equality if and only if there exists some (necessarily positive) constant a such that $\gamma_{i,k} \sqrt{\text{MSE}_k} = a \frac{1}{\sqrt{\text{MSE}_k}}$ for all k . We therefore have $\sum_{k=1}^K \gamma_{i,k}^2 \text{MSE}_k \geq \frac{1}{\sum_{k=1}^K \frac{1}{\text{MSE}_k}}$, with equality if and only if $\gamma_{i,k} = \frac{1}{\text{MSE}_k \sum_{l=1}^K \frac{1}{\text{MSE}_l}}$ for all k . \square

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