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Robust Multidimensional Welfare Comparisons: One Vector of Weights, One Vote

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Summary

Many aspects of social welfare are intrinsically multidimensional. Composite indices attempting to reduce this complexity to a unique measure abound in many areas of economics and public policy. Comparisons based on such measures depend, sometimes critically, on how the di erent dimensions of performance are weighted. Thus, a policy maker maker may wish to take into account imprecision over composite index weights in a systematic manner. In this paper, such weight imprecision is parameterized via the ε-contamination framework of Bayesian statistics. Subsequently, combining results from polyhedral geometry, social choice, and theoretical computer science, an analytical procedure is presented that yields a provably robust ranking of the relevant alternatives in the presence of weight imprecision. The main idea is to consider a vector of weights as a voter and a continuum of weights as an electorate. The procedure is illustrated on recent versions of the Rule of Law and Human Development indices.

Keywords: Multidimensional Welfare, Composite Index, ε-Contamination, Polyhedral Geometry, Social Choice, Approximation Algorithms

JEL Classification: C02, C61, D04, D71, I31

I am grateful to Jim Lawrence for helpful and clarifying remarks and to Michaela Saisana for introducing me to this interesting area of research. The views expressed herein are purely those of the author and may not in any circumstances be regarded as stating an official position of the European Commission.

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Robust multidimensional welfare comparisons: one vector of weights, one vote*

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Abstract

Many aspects of social welfare are intrinsically multidimensional. Composite indices attempting to reduce this complexity to a unique measure abound in many areas of economics and public policy. Comparisons based on such measures depend, sometimes critically, on how the different dimensions of performance are weighted. Thus, a policy maker maker may wish to take into account imprecision over composite index weights in a systematic manner. In this paper, such weight imprecision is parameterized via the ϵ-contamination framework of Bayesian statistics. Subsequently, combining results from polyhedral geometry, social choice, and theoretical computer science, an analytical procedure is presented that yields a provably robust ranking of the relevant alternatives in the presence of weight imprecision. The main idea is to consider a vector of weights as a voter and a continuum of weights as an electorate. The procedure is illustrated on recent versions of the Rule of Law and Human Development indices.

Keywords: multidimensional welfare, composite index, ϵ-contamination, polyhedral geometry, social choice, approximation algorithms

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1 Introduction

Many aspects of social wellbeing are inherently multidimensional. Development, poverty, inequality, the rule of law, education: these are all concepts that depend on a number of different criteria that cannot be captured by simple quantitative measures. Yet, there is still a need to compare and eventually order possible alternatives on the basis of such multidimensional information. Composite indices attempt to accomplish this task by integrating the various dimensions into a single, one-dimensional measure. This is achieved by assigning weights to the

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different dimensions and aggregating over them. This aggregation is usually either additive or geometric.

It should be intuitively clear that the chosen weights can have a major effect on composite scores and therefore on the final ranking of the various alternatives. This means that weights need to be assigned in a systematic, transparent, and judicious fashion. Many different methods for doing so have been proposed by academics and practitioners, including principal component and factor analyses, data envelopment, public opinion polls, budget allocation, analytic hierarchy processes, and expert consultation, among others. The interested reader is referred to [9, 21] for a comprehensive survey.

Despite the wealth of available techniques to determine composite index weights, their determination often remains controversial. Indeed, there is frequently no one "right" way to set them and we are often justified, if not compelled to, consider the effect of many different weights at once. Such an analysis would serve two goals: (a) to examine how robust a given ranking of alternatives is to changes in weights, and (b) to determine a compromise ranking that is in some sense "optimal" in the presence of weight imprecision.

Earlier work in assessing the robustness of composite indices with respect to the choice of weights primarily focused on Monte Carlo simulation (Saisana et al. [24], OECD and JRC [21]). These computational approaches assessed the importance of weights in the context of a broader uncertainty and sensitivity analysis of given indices. Practically oriented, they did not propose a systematic theoretical framework to model weight imprecision and its effects. Of greater relevance to the work I will present, Foster et al. [11, 12] adopted a parametric structure for weight imprecision based on the ϵ-contamination model of Bayesian analysis (Hodges and Lehmann [13], Berger [7]).[1] In their setting, Foster et al [11] defined a pairwise comparison between alternatives to be robust with respect to a given level of weight imprecision if *all* of the weight vectors corresponding to it (i.e., this level of weight imprecision) produce composite scores that maintain the same relative ranking. Numerical examples applying their model to study the robustness of various indices were presented in [12]. Permanyer [22] generalized the approach of Foster et al. [11] by considering (i) alternative ways of parameterizing weight imprecision beyond ϵ-contamination, and (ii) the proportion of weights favoring one alternative over another given a level of weight imprecision. Permanyer's contribution is mostly conceptual, as he does not discuss how one can actually calculate the proportions in question. Moreover, beyond formal completeness, it is not clear what is gained by the discussion of alternative parameterizations of weight imprecision beyond ϵ-contamination[2], which itself is intuitive, theoretically grounded, and computationally tractable. Finally, both Foster et al. [11, 12] and Permanyer [22] are focused on providing rigorous methods for assessing the robustness of a given ranking to changes in weights, and are not directly concerned with proposing an optimal "compromise" ranking

[1] The ϵ-contamination model has also been studied in the economics and decision theoretic literature ([20, 15], among many others) on Knightian uncertainty, which however places more emphasis on the normative foundations and behavioral implications of such belief imprecision.

[2] No theoretical arguments or empirical illustrations are provided in the paper.

given weight imprecision.

My paper makes three contributions to the literature. First, I adopt the ϵ-contamination setting of Foster et al. [11][3], and use results from polyhedral geometry (Lawrence [17]) to efficiently compute the results of pairwise comparisons between alternatives. That is, given two alternatives a_1 and a_2 and a set of weights adhering to the parametric structure of ϵ-contamination, I adapt the insights of Lawrence [17] to present a closed-form formula for the proportion of those weights ranking a_1 above a_2 and vice-versa. Moreover, I am able to prove that these proportions are monotonic in ϵ, the magnitude of weight imprecision. I implement Lawrence's formula computationally in the context of the applied exercises of Section 5.

The second contribution of the paper is to use the theory of social choice to propose a "good" compromise ranking of the alternatives given weight imprecision. Viewing each vector of weights as a "voter" who expresses his/her preferences over alternatives via the composite score, the previous polyhedral analysis provides the results for all pairwise contests between alternatives over an "electorate" of weights defined by the ϵ-contamination structure. With this interpretation in mind, I propose the Kemeny aggregation procedure (Kemeny [14]) as a way of computing a compromise ranking, given the preferences of this electorate of weight vectors. Well-established in the social choice literature, Kemeny aggregation produces a ranking (referred to as "Kemeny-optimal") that minimizes the sum of pairwise rank disagreements with respect to stated voter preferences. In an important paper, Young and Levenglick [27] validated its intuitive attractiveness by showing that it rests on strong axiomatic foundations. Subsequently, Young [26] showed that Kemeny aggregation provides the maximum likelihood estimate (in the statistical sense of the term) of the true societal ranking of the alternatives. The intuitive, normative, and statistical appeal of Kemeny aggregation make it a very (if not the most) desirable aggregation procedure (Moulin [18]).

The paper's third contribution centers on the computational implementation of the above ideas. This is because, despite its many virtues, Kemeny aggregation suffers from one very serious drawback: the computation of a Kemeny-optimal ranking is NP-hard [6], even when the number of alternatives is just four [10]. In practical terms this means that we cannot hope to devise a fast algorithm to implement Kemeny aggregation. Thus, to deal with this difficulty, when the Kemeny-optimal ranking cannot be readily identified by first principles, I implement the best-known approximation algorithm available from the theoretical computer science literature to compute a provably-good approximation of it. This algorithm, due to Van Zuylen and Williamson [28], efficiently produces a compromise ranking whose sum of pairwise disagreements is guaranteed to be no greater than 4/3 times the minimum.[4] To my knowledge, this is the first paper in the economics and social science literature to apply rigorous approximation algorithms in the determination of Kemeny-optimal rankings. I proceed to illustrate the proposed method-

[3]This choice is not crucial, and the methods of the paper could extend to alternative parametric structures, albeit at a potentially higher computational cost.

[4]Using first principles and a "local Kemenization" procedure discussed in Dwork et al. [10], I am often able to improve this performance guarantee even further (see Sections 4.3 and 5.2).

ology to two well-known composite indices: the World Justice Project's Rule of Law Index [1], and the United Nations Human Development Index [25]. In both cases, my procedure correctly identifies Kemeny-optimal rankings under varying levels of weight imprecision.

Paper outline. The structure of the paper is as follows. Section 2 introduces the formal model and ϵ-contamination framework. Section 3 focuses on pairwise comparisons between alternatives and shows how to adapt the results of Lawrence [17] to my setting. Section 4 draws connections with social choice theory and Kemeny aggregation in particular, and presents a procedure for computing a provably-robust ranking given weight imprecision. Section 5 implements the proposed procedure to the Rule of Law and Human Development Indices, while Section 6 provides concluding remarks. All mathematical proofs, tables, and figures are collected in the Appendix.

2 Model Description

Consider a set of alternatives $\mathcal{A}$ indexed by $a = 1, 2, ..., A$ and a set of indicators $\mathcal{I}$ indexed by $i = 1, 2, ..., I$. Let $x_{ai} \in \Re$ denote alternative a's value of indicator i, and $\boldsymbol{x_a} \in \Re^I$ its "achievement" vector collecting all such information (all vectors are taken to be column vectors). The composite score corresponding to alternative a is computed through a weighted average of the components of its achievement vector, $\boldsymbol{x_a}$. For clarity I focus on standard linear aggregation, though the analysis easily extends to the generalized weighted means discussed in Permanyer [22].[5] The employed vector of weights is given by a vector $\boldsymbol{w}$ belonging in $\Delta^{I-1} = \{w \in \Re^I : \quad \boldsymbol{w} \geq \boldsymbol{0}, \quad \sum_{i=1}^{I} w_i = 1\}$, the $(I-1)$-dimensional simplex. Here, w_i represents the weight given to indicator i.

Clearly, the choice of weights $\boldsymbol{w}$ is very important in determining composite scores on the basis of which alternatives are ranked. Thus it is important to have a sense of how robust a ranking is with respect to changes in $\boldsymbol{w}$. Suppose we are given an initial vector of weights $\bar{\boldsymbol{w}} \in \Delta^{I-1}$. Now suppose that we are willing to consider weights deviating from $\bar{\boldsymbol{w}}$ that belong in the set $W^\epsilon(\bar{\boldsymbol{w}})$, where[6]

$$W^\epsilon(\bar{\boldsymbol{w}}) \equiv W^\epsilon = (1-\epsilon)\bar{\boldsymbol{w}} + \epsilon\Delta^{I-1} = \left\{ \boldsymbol{w} \in \Re^I : \quad \boldsymbol{w} \geq (1-\epsilon)\bar{\boldsymbol{w}}, \ \sum_{i=1}^{I} w_i = 1 \right\}. \tag{1}$$

Here, the parameter $\epsilon \in [0, 1]$ measures the imprecision associated with the initial vector of weights $\bar{\boldsymbol{w}}$. If $\epsilon = 0$, then we are completely confident in our choice of $\bar{\boldsymbol{w}}$, while if $\epsilon = 1$ we assign no special status to $\bar{\boldsymbol{w}}$ and consider all possible weight vectors equally valid. Originally proposed by Hodges and Lehmann [13] in Bayesian analysis, this way of parameterizing probabilistic imprecision is referred to as ϵ-contamination. From a statistical point of view, the parameter

[5] Indeed, see Section 5.2 for an application of our model to the geometric aggregation framework of the HDI.

[6] To avoid cumbersome notation, from now on I suppress dependence of $W^\epsilon(\bar{\boldsymbol{w}})$ on $\bar{\boldsymbol{w}}$.

ϵ may be interpreted as the amount of error attached to a prior $\bar{\boldsymbol{w}}$.[7] The ϵ-contamination parametric structure has also been studied in the economics and decision theoretic literature on Knightian uncertainty ([20, 15], among many others), which however places more emphasis on the normative foundations and behavioral implications of such belief imprecision.

3 Pairwise comparisons over a continuum of weights

Let us consider two alternatives $a_1, a_2 \in \mathcal{A}$ and their I-dimensional achievement vectors $\boldsymbol{x_{a_1}}$ and $\boldsymbol{x_{a_2}}$. Suppose, further, that we are given an initial vector of weights $\bar{\boldsymbol{w}} \in \Delta^{I-1}$ and a value of $\epsilon \in [0,1]$, capturing the degree of imprecision associated with $\bar{\boldsymbol{w}}$. How are we to decide which of the two alternatives fares "better" given the set of weights W^ϵ implied by Eq. (1)?

If $\epsilon = 0$ the answer to the above question is simple: just see which alternative's composite score is higher under weights $\bar{\boldsymbol{w}}$, the unique element of set W^0. That is, we need only compare $\bar{\boldsymbol{w}}'\boldsymbol{x_{a_1}}$ and $\bar{\boldsymbol{w}}'\boldsymbol{x_{a_2}}$ (the prime sign denotes the transpose operator). When we are dealing with $\epsilon > 0$ and a non-singleton set W^ϵ, the situation is more complex. Nonetheless we may ask an analogous question, namely: What *proportion* of weights belonging to W^ϵ lead to a higher composite score for a_1 than a_2?

Some additional notation would be useful. Let $W^\epsilon_{a_1a_2}$ denote the intersection of W^ϵ with the I-dimensional halfspace $\{\boldsymbol{w} \in \Re^I : \ \boldsymbol{w}'\boldsymbol{x_{a_1}} \geq \boldsymbol{w}'\boldsymbol{x_{a_2}}\}$. Introducing the difference vector $\boldsymbol{d} = \boldsymbol{x_{a_1}} - \boldsymbol{x_{a_2}}$, the polytope $W^\epsilon_{a_1a_2}$ is equal to

$$W^\epsilon_{a_1a_2} = \left\{ \boldsymbol{w} \in \Re^I : \ \ \boldsymbol{w} \geq (1-\epsilon)\bar{\boldsymbol{w}}, \ \sum_{i=1}^{I} w_i = 1, \ \ \boldsymbol{d}'\boldsymbol{w} \geq 0 \right\}. \tag{2}$$

The proportion we are interested in, denoted by $V^\epsilon_{a_1a_2}$, is defined as the ratio of the volumes of polytopes $W^\epsilon_{a_1a_2}$ and W^ϵ (Vol denotes volume):

$$V^\epsilon_{a_1a_2} = \frac{\int_{W^\epsilon_{a_1a_2}} \mathrm{d}u}{\int_{W^\epsilon} \mathrm{d}u} = \frac{Vol\left(W^\epsilon_{a_1a_2}\right)}{Vol(W^\epsilon)} \in [0,1]. \tag{3}$$

When $\boldsymbol{d}'\bar{\boldsymbol{w}} = 0$ and $\epsilon = 0$, I set $V^\epsilon_{a_1a_2} = 1/2$. Now, basic geometric reasoning allows us to establish an unambiguous monotonicity property of $V^\epsilon_{a_1a_2}$ with respect to the level of imprecision ϵ. In addition to its theoretical appeal, this property, summarized in Theorem 1, may be of considerable practical use (see Section 5.1).

Theorem 1 *(i) Suppose $\boldsymbol{d}'\bar{\boldsymbol{w}} \neq 0$. Then, $V^\epsilon_{a_1a_2}$ is monotonic in ϵ. It is increasing (decreasing) in ϵ if*

$$\boldsymbol{d}'\bar{\boldsymbol{w}} < (>) 0.$$

(ii) Suppose $\boldsymbol{d}'\bar{\boldsymbol{w}} = 0$. Then, $V^{\epsilon_1}_{a_1a_2} = V^{\epsilon_2}_{a_1a_2}$ for all $\epsilon_1, \epsilon_2 \in (0,1]$. There will in general be a discontinuity at $\epsilon = 0$.

[7] The interested reader may consult Chapter 4 in the survey by Berger [7] for many additional references on ϵ-contamination.

Proof. See Appendix. ■

Thus, when $\boldsymbol{d}'\bar{\boldsymbol{w}} \neq 0$ and a_1 and a_2 do not yield identical composite scores under the initial vector of weights $\bar{\boldsymbol{w}}$, Theorem 1 establishes that the proportion of weights favoring one alternative over another varies monotonically in the imprecision ϵ attached to $\bar{\boldsymbol{w}}$. The direction of the relationship depends on the comparison of alternatives a_1 and a_2 under weights $\bar{\boldsymbol{w}}$. It is decreasing if a_1 initially dominates a_2 and increasing otherwise. Conversely, when the initial weights yield identical composite scores for a_1 and a_2, the situation is different. For any two levels of imprecision ϵ_1 and ϵ_2 above 0, we will have $V_{a_1a_2}^{\epsilon_1} = V_{a_1a_2}^{\epsilon_2}$, while at $\epsilon = 0$ we will have (by definition) $V_{a_1a_2}^{0} = 1/2$. Thus, V^ϵ is constant when $\epsilon \in (0, 1]$ and will generally have a discontinuity at 0.

Define the function

$$D(\epsilon) = -\frac{1-\epsilon}{\epsilon}\boldsymbol{d}'\bar{\boldsymbol{w}}, \tag{4}$$

and suppose that there exists at least one indicator $i^* \in \mathcal{I}$ such that $\boldsymbol{d}_{i^*} \geq D(\epsilon)$. If such an i^* does not exist, then we may immediately conclude that $V_{a_1a_2}^{\epsilon} = 0$ (see proof of Theorem 1). Without loss of generality (upon possible relabeling) suppose that $i^* = I$ and define $\boldsymbol{w}^* \in \Re^{I-1}$ and $\boldsymbol{d}^* \in \Re^{I-1}$ as the restriction of vectors $\boldsymbol{w}$ and $\boldsymbol{d}$ to variables $\mathcal{I}\backslash\{i^*\} = \{1, 2, .., I-1\} \equiv \mathcal{I}^*$. Consider the following polytope $W_{a_1a_2}^{\epsilon,*}$ (here $\boldsymbol{e}$ denotes a vector of all ones of dimension $I-1$)

$$W_{a_1a_2}^{\epsilon,*} = \left\{ \boldsymbol{w}^* \in \Re^{I-1} : \;\; \boldsymbol{w}^* \geq 0, \;\; \sum_{i=1}^{I-1} w_i^* \leq 1, \;\; (\boldsymbol{d}^* - d_I\boldsymbol{e})'\boldsymbol{w}^* + d_I \geq D(\epsilon) \right\}. \tag{5}$$

Polytope $W_{a_1a_2}^{\epsilon,*}$ is obtained upon performing a sequence of simple affine transformations to polytope $W_{a_1a_2}^{\epsilon}$, ultimately reducing its dimension by 1 (see the proof of Theorems 1 and 2). Using basic results from linear algebra (Lang [16]) we arrive at the following Theorem.

Theorem 2 *Consider polytope $W_{a_1a_2}^{\epsilon,*}$ given by (5). The quantity $V_{a_1a_2}^{\epsilon}$ defined in Eq. (3) satisfies*

$$V_{a_1a_2}^{\epsilon} = (I-1)!\, Vol\left(W_{a_1a_2}^{\epsilon,*}\right).$$

Proof. See Appendix. ■

In light of Theorem 2, the main challenge now lies in calculating the volume of polytope $W_{a_1a_2}^{\epsilon,*}$. We make the following assumption.

Assumption 1 *There does not exist $i \in \mathcal{I}$ such that $\boldsymbol{d}_i = D(\epsilon)$.*

Assumption 1 ensures that polytope $W_{a_1a_2}^{\epsilon,*}$ is simple, i.e., that all of its vertices are nondegenerate. Primal nondegeneracy is a desirable property in linear programming as it facilitates the application of the simplex method (see Chapter 2 in Bertsimas and Tsitsiklis [5]).

Proposition 1 *Suppose Assumption 1 holds. Then the polytope $W_{a_1a_2}^{\epsilon,*}$ is simple. Moreover it has $O\left(I^2\right)$ vertices and $O\left(I^3\right)$ edges that can be readily identified (Eqs. (V1)-(V4) and (E1)-(E8) in Appendix). Using this information we can construct a vector $\boldsymbol{c} \in \Re^{I-1}$ such that the function $f(\boldsymbol{w}^*) = \boldsymbol{c}'\boldsymbol{w}^*$ is non-constant on each edge of $W_{a_1a_2}^{\epsilon,*}$ (Eq. (19) in Appendix). As a* *result, the volume of polytope $W^{\epsilon;*}_{a_1a_2}$ can be computed efficiently using the formula in Theorem 1 of Lawrence [17] (Eq. (20) in Appendix).*

Proof. See Appendix. ∎

Thus, by Proposition 3 we have an efficient method of computing $V^{\epsilon}_{a_1a_2}$ for any two alternatives a_1 and a_2 and $\epsilon \in [0,1]$. We conclude by performing the following transformation on $V^{\epsilon}_{a_1a_2}$ for all pairs $a_1, a_2 \in \mathcal{A}$:

$$V^{\epsilon}_{a_1a_2} \leftarrow V^{\epsilon}_{a_1a_2} + \frac{1-\left(V^{\epsilon}_{a_1a_2}+V^{\epsilon}_{a_2a_1}\right)}{2}, \quad V^{\epsilon}_{a_2a_1} \leftarrow V^{\epsilon}_{a_2a_1} + \frac{1-\left(V^{\epsilon}_{a_1a_2}+V^{\epsilon}_{a_2a_1}\right)}{2}. \tag{6}$$

This transformation provides a fair tie-breaking rule for vectors of weights yielding identical composite scores. It ensures that the volume of the polytope

$$\left\{ \boldsymbol{w} \in \Re^I : \ \boldsymbol{w} \geq (1-\epsilon)\bar{\boldsymbol{w}}, \ \sum_{i=1}^{I} w_i = 1, \ \boldsymbol{w}'\boldsymbol{x_{a_1}} = \boldsymbol{w}'\boldsymbol{x_{a_2}} \right\}$$

is equally divided between alternatives a_1 and a_2 so that $V^{\epsilon}_{a_1a_2}$ can be interpreted as the proportion of weights strictly favoring a_1 over a_2. Moreover, it implies that

$$V^{\epsilon}_{a_1a_2} + V^{\epsilon}_{a_2a_1} = 1, \ \ \forall a_1, a_2 \in \mathcal{A}.$$

In practice, these adjustments may often turn out to be negligible.

Remark 1. Assumption 1 is not strictly necessary for the implications of Proposition 3 to hold. As Lawrence himself notes in his paper's conclusion [17], his method can be extended to non-simple polytopes using standard linear programming techniques (see Bueler et al. [8] for an application). I choose to impose Assumption 1 because it can be always easily satisfied by a slight perturbation of $\bar{\boldsymbol{w}}$ or ϵ, while simplifying computations significantly.

Example 1. One of the strengths of the proposed framework is that it sheds light on subtle, but important differences among alternatives. This is illustrated by the following example:

$$\boldsymbol{x_{a_1}} = (1,2,3,4)', \ \ \boldsymbol{x_{a_2}} = (4,3,2,1.5)', \ \ \bar{\boldsymbol{w}} = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)'.$$

Looking at just the ordinal dimension of the indicator data, we see that both a_1 and a_2 dominate in exactly two dimensions. Moreover, the difference in the composite scores under zero imprecision is quite small: 2.5 for a_1 and 2.625 for a_2. This may lead us to think that the two alternatives are roughly equal, and remain all the more so when we take weight imprecision into account. However, this is not true. Indeed, we see that in reality a_2 fares significantly better than a_1 if we allow for weight imprecision (and our model makes quantitatively precise to what degree this is so), especially if we only wish to consider small levels of ϵ:

$$V^{0.1}_{a_1a_2} = 0.090, \ \ V^{0.25}_{a_1a_2} = 0.312, \ \ V^{0.5}_{a_1a_2} = 0.408, \ \ V^{1}_{a_1a_2} = 0.458.$$

Example 2. I provide an example of the discontinuity discussed in part (ii) of Theorem 1. Consider:

$$\boldsymbol{x_{a_1}} = (1, 2, 3.5, 4, 5)', \;\; \boldsymbol{x_{a_2}} = (3.5, 4, 5, 1, 2)', \;\; \bar{\boldsymbol{w}} = \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)'.$$

These data yield

$$V^0_{a_1a_2} = 0.50, \;\; V^\epsilon_{a_1a_2} = 0.478, \;\; \forall \epsilon \in (0, 1].$$

4 Weight imprecision and social choice theory

4.1 Weight vectors as voters

In Section 2 I discussed how an initial vector of weights $\bar{\boldsymbol{w}}$ and a value of ϵ imply, via the ϵ-contamination framework, the set of weights W^ϵ of Eq. (1). Subsequently, in Section 3 I demonstrated how, given a pair of alternatives $a_1, a_2 \in \mathcal{A}$ and a value $\epsilon \in [0, 1]$ we can use Lawrence's formula [17] to efficiently compute the proportion of weights within W^ϵ whose composite score for a_1 is at least as high as for a_2. After performing the transformation (6) for all pairs of alternatives in $\mathcal{A}$, this information can be summarized by a matrix $V^\epsilon = \left[V^\epsilon_{a_1a_2}\right]_{a_1,a_2\in\mathcal{A}}$ (whose diagonal entries are set to zero by definition, i.e., $V^\epsilon_{aa} = 0$ for all a).

Suppose now that we think of a vector of weights $\boldsymbol{w} \in W^\epsilon$ as a voter belonging to an electorate W^ϵ (the greater ϵ is, the larger the electorate). With this interpretation in mind, the quantity $V^\epsilon_{a_1a_2}$ defines the percentage of voters within the electorate W^ϵ preferring alternative a_1 to a_2. Thus, all information on the pairwise preferences of the electorate W^ϵ over the set of alternatives $\mathcal{A}$ is succinctly summarized by the matrix V^ϵ.

How can we use this information to compare and order alternatives? Given that the entries of matrix V^ϵ will generally fall strictly between 0 and 1, there will not be a ranking of $\mathcal{A}$ that is consistent with the preferences of all weights belonging to W^ϵ. Thus, the question arises: In view of this inconclusiveness, what would be a "good" compromise ranking that takes into account the results of pairwise comparisons across the electorate W^ϵ?

4.2 Kemeny aggregation

Given a set of individual ranked preferences that may conflict with each other, what procedure (i.e., rule) should society use to determine a consensus ranking? What properties should a compromise solution aspire to satisfy?

These fundamental questions have concerned philosophers and social scientists since the work of Condorcet and Borda in the 18th century. In a seminal paper, Arrow [4] famously proved that there does not exist an aggregation procedure[8] simultaneously satisfying a set of four plausible axioms: unrestricted domain, non-dictatorship, efficiency, and independence of alternative alternatives. Despite this negative result, a multitude of reasonable aggregation

[8]In the literature, the terms "voting rule" and "aggregation procedure" (and combinations thereof) are used interchangeably.

procedures have been proposed, and formal characterizations of such methods on the basis of desirable properties that they do or do not satisfy abound in the social theoretic literature (see Moulin [18]). For the purposes of this paper I focus on a particularly compelling property known as the *Condorcet criterion* (originally proposed in the 18th century by the Marquis de Condorcet). The Condorcet winner of an election is an alternative that, when compared with every other, is preferred by a majority of voters. An aggregation procedure satisfies the Condorcet criterion if it ranks the Condorcet winner first, whenever one exists. In turn, a ranking is referred to as Condorcet if it is consistent with the Condorcet criterion.

One well-known problem with Condorcet winners is that it is easy to construct examples of the so-called *Condorcet paradox* (originally noted by the Marquis himself), in which voters' ranked preferences preclude their existence [18]. In such instances, the Condorcet criterion is clearly of no help in choosing between different rankings. To deal with this issue, Kemeny [14] introduced an aggregation procedure satisfying a generalization of the Condorcet criterion, referred to in the literature as Kemeny optimality. Given a set of individual rankings, Kemeny aggregation produces a ranking (referred to as "Kemeny-optimal") that minimizes the sum of its pairwise disagreements with respect to voter preferences. As a corollary, when a Condorcet ranking exists, Kemeny's rule is guaranteed to choose it.[9] In an important article Young and Levenglick [27] confirmed the intuitive appeal of Kemeny aggregation by proving that it rests on solid axiomatic foundations. Moreover, Young [26] showed that, from a statistical standpoint, Kemeny aggregation can be viewed as a providing the maximum likelihood estimate of the "true" societal ranking of the alternatives.

The intuitive, normative, and statistical appeal of Kemeny aggregation make it a very desirable voting rule. Indeed, Moulin [18] goes so far as to state that it is the "correct method" for ranking alternatives. Unfortunately, however, Kemeny aggregation suffers from one very serious drawback: the computation of a Kemeny-optimal ranking is NP-hard [6], even when the number of alternatives is just four [10]. In practical terms this means that we cannot hope to devise a fast algorithm to implement Kemeny aggregation. Nevertheless, spurred by the applicability of rank aggregation methods for internet search, various fast heuristics have been proposed in the computer science literature (e.g. Dwork et al. [10]).

In recent years, a burgeoning theoretical computer science literature has emerged that proposes provably-good approximation algorithms for Kemeny aggregation. An intelligent synopsis of these contributions is beyond the scope of this work and the interested reader is encouraged to refer to Section 1 in Van Zuylen and Williamson [28] for more information. For the purposes of this paper, I draw particular attention to the work of Van Zuylen and Williamson [28]. Their algorithm (DerandFASLP-Pivot in Figure 1 of [28]) employs a polynomial-time recursive procedure to produce a ranking whose sum of pairwise disagreements is within 4/3 of the minimum, the best approximation guarantee currently available. Its running time is primarily constrained by the solution of the linear programming relaxation of Kemeny aggregation. The output rank-

[9] Indeed, Kemeny-optimal rankings satisfy a stronger version of the Condorcet criterion known as the extended Condorcet criterion [10].

ing of this algorithm can be potentially improved by applying to it a procedure known as "local Kemenization" (Dwork et al. [10]). Local Kemenization successively examines all adjacent pairs of alternatives in a ranking and flips them if they result in an improved Kemeny performance. It may be efficiently implemented in polynomial time and it is guaranteed to produce a unique ranking that satisfies an extended version of the Condorcet criterion [10].

4.3 Application to composite index rankings

I now proceed to apply the above concepts to composite index rankings under weight imprecision. Recall the set of alternatives $\mathcal{A}$. A ranking R is a bijective map from $\mathcal{A}$ to $\{1, 2, ..., A\}$, where $R(a)$ is interpreted as the rank of alternative a. Let $\mathcal{R}_{\mathcal{A}}$ denote the set of all rankings of $\mathcal{A}$. The Kendall-τ distance between two rankings R_1 and R_2, denoted by $\tau(R_1, R_2)$, is defined as the number of pairs (a_i, a_j) such that $R_1(a_i) > R_1(a_j)$ and $R_2(a_i) < R_2(a_j)$. Hence, $\tau(R_1, R_2)$ counts the number of (pairwise) relative rank disagreements between R_1 and R_2. Given a set of rankings $\mathcal{S}$, a ranking K is Kemeny-optimal if it minimizes the function $\sum_{S \in \mathcal{S}} \tau(\cdot, S)$ over the set of rankings $\mathcal{R}_{\mathcal{A}}$. In the usual formulation of the problem, the set $\mathcal{S}$ is finite so the above sum is well-defined. However, in our context each weight vector in W^ϵ corresponds to a different ranking, implying that $\mathcal{S}$ is uncountable. Thus, to make sure the Kemeny-optimal ranking is well-defined we normalize by $\int_{\mathcal{S}} du$ and write:

$$K = \underset{R \in \mathcal{R}_{\mathcal{A}}}{\arg\min} \left\{ \int_{\mathcal{S}} \frac{\tau(R, S)}{\int_{\mathcal{S}} du} dS \right\}.$$

To apply the preceding formula to our context, we need to identify the appropriate set of rankings $\mathcal{S}$, which would entail associating a ranking for every vector of weights belonging to W^ϵ. This task is already demanding when the set of voters is finite, let alone when it is uncountable. Fortunately, however, I am able to sidestep this concern. This is because, to calculate a Kemeny-optimal ranking we only need the results of all pairwise comparisons between elements in $\mathcal{A}$, given by matrix V^ϵ, which can be efficiently computed using the methods in Section 3. Thus, letting $(a_1, a_2) \in \mathcal{A} \times \mathcal{A}$ denote ordered pairs of alternatives in $\mathcal{A}$, and K^ϵ the Kemeny optimal ranking given the set of weights W^ϵ, we may write

$$K^\epsilon = \underset{R \in \mathcal{R}_{\mathcal{A}}}{\arg\min} \sum_{(a_1, a_2) \in \mathcal{A} \times \mathcal{A}} \mathbf{1}\{R(a_1) < R(a_2)\} V^\epsilon_{a_1 a_2}. \quad (7)$$

Integrating the above discussion with that of Sections 2 and 3, I propose the following procedure to address minimization problem (7) and provide a robust compromise ranking of the alternatives given weight imprecision.

Procedure 1 (Input: $\boldsymbol{x_a}$ for all $a \in \mathcal{A}$, $\epsilon, \bar{\boldsymbol{w}}$.)

(a) Use the method outlined in Proposition 3 of Section 3 to compute the matrix V^ϵ. Perform transformation (6) on the elements of V^ϵ.

(b) Given V^ϵ, attempt to explicitly find Kemeny-optimal ranking K^ϵ by first principles.

(c) If (b) is not possible, apply Van Zuylen-Williamson [28] algorithm to matrix V^ϵ to compute a 4/3-Kemeny optimal ranking $K^{ZW,\epsilon}$.

(d) Perform local Kemenization [10] on $K^{ZW,\epsilon}$ to, if possible, decrease its objective function value. Denote the final ranking by $\widehat{K}^{ZW,\epsilon}$.

Procedure 1's performance guarantee. When Step (b) of Procedure 1 cannot be accomplished, it is important to know how close to Kemeny-optimal the output ranking $\widehat{K}^{ZW,\epsilon}$ will be. To wit, given a ranking R and $\epsilon \in [0,1]$ define the function

$$\kappa\left(R,\epsilon\right) = \sum_{(a_1,a_2)\in\mathcal{A}\times\mathcal{A}} \mathbf{1}\{R(a_1) < R(a_2)\} V^\epsilon_{a_1 a_2}$$

and consider the output ranking produced by the Van Zuylen-Williamson algorithm *before* local Kemenization, denoted by $K^{ZW,\epsilon}$. Its 4/3 performance guarantee implies that

$$\kappa\left(K^{ZW,\epsilon},\epsilon\right) \le \frac{4}{3}\kappa\left(K^\epsilon,\epsilon\right) = \frac{4}{3}\min_{R\in\mathcal{R}_\mathcal{A}} \kappa\left(R,\epsilon\right). \tag{8}$$

Now, consider the ranking produced by the Van Zuylen-Williamson algorithm *after* local Kemenization, denoted by $\widehat{K}^{ZW,\epsilon}$, and define the constant $\alpha^\epsilon \ge 1$ such that

$$\kappa\left(K^{ZW,\epsilon},\epsilon\right) = \alpha^\epsilon \cdot \kappa\left(\widehat{K}^{ZW,\epsilon},\epsilon\right). \tag{9}$$

Conversely, by first principles we may immediately establish the following lower bound:

$$\min_{R\in\mathcal{R}_\mathcal{A}} \kappa\left(R,\epsilon\right) \ge \frac{1}{2}\sum_{(a_1,a_2)\in\mathcal{A}\times\mathcal{A}} \min\left\{V^\epsilon_{a_1a_2}, V^\epsilon_{a_2a_1}\right\} \equiv l^\epsilon. \tag{10}$$

With bound (10) in mind, define the constant $\beta^\epsilon \ge 1$ such that

$$\kappa\left(\widehat{K}^{ZW,\epsilon},\epsilon\right) = \beta^\epsilon \cdot l^\epsilon. \tag{11}$$

Putting Eqs. (8)-(9)-(10)-(11) together, we may deduce the following guarantee on the performance of $\widehat{K}^{ZW,\epsilon}$:

$$\kappa\left(\widehat{K}^{ZW,\epsilon},\epsilon\right) \le \min\left\{\beta^\epsilon, \frac{4}{3\alpha^\epsilon}\right\}\cdot\kappa\left(K^\epsilon,\epsilon\right) = \min\left\{\beta^\epsilon, \frac{4}{3\alpha^\epsilon}\right\}\cdot \min_{R\in\mathcal{R}_\mathcal{A}}\kappa\left(R,\epsilon\right). \tag{12}$$

Thus, we see that, depending on the values of α^ϵ and β^ϵ, the 4/3 performance guarantee of Van Zuylen and Williamson can potentially be tightened.

Note. I end this section by noting that the application of Kemeny aggregation to composite indices has been previously pursued by Munda and Nardo [19]. However, their work differs from mine in substantive ways. First, Munda and Nardo simply consider each indicator as a voter, and do not introduce a systematic parametric structure for weight imprecision such as ϵ-contamination. Hence, their "electorate" is simply the set of I weight vectors assigning full weight to the different dimensions. As a result, calculating the proportion of voters favoring one alternative over another reduces to simply counting the indicators resulting in a higher value for it and cardinal information on the magnitude of this pairwise dominance is lost. Second, even within this restricted setting, Munda and Nardo present a largely qualitative picture that, beyond mentioning the existence of heuristics, does not propose a method for computing a Kemeny-optimal ranking nor a provably-good approximation for it.

5 Applications

In this section, I apply Procedure 1 to two popular composite indices: (a) the World Justice Project's 2012 Rule of Law Index [1] and (b) the United Nations 2013 Human Development Index (HDI) [25]. In both cases, I am able to explicitly identify Kemeny-optimal rankings under different levels of weight imprecision. My analysis shows that the rankings of the Rule of Law Index are completely robust to departures from equal-weight aggregation, whereas the situation with respect to the HDI is a little more complex. Both Lawrence's method for obtaining the matrix V^ϵ and van Zuylen and Williamson's approximation algorithm are implemented in Matlab.[10]

5.1 2012 Rule of Law Index

The World Justice Project (WJP) is a multi-national, multi-disciplinary US-based organization. It produces an important multi-dimensional measure of the rule of law, the Rule of Law Index, that encompasses issues from government power and corruption to fundamental rights and civil justice. The WJP measures nine main dimensions of the rule of law: (1) limited government powers, (2) absence of corruption, (3) order and security, (4) fundamental rights, (5) open government, (6) regulatory enforcement (7) civil justice, (8) criminal justice and (9) informal justice. The 2012 version of the index covers 97 countries, which are ranked along dimensions (1)-(8) above.[11] An aggregation of these 8 dimensions into one single composite measure was not attempted by the authors of the 2012 report, presumably because there was no consensus on whether and how such an aggregation should be performed.

For the purposes of this paper, I focus on the 21 European Union countries for which the Rule of Law index has data. Table 1 summarizes the (normalized) data for this set of countries (countries are ordered alphabetically). For an initial vector of weights, I consider equal weights

[10] Programs available upon request.

[11] Data on dimension (9) was gathered but not used by the authors of the 2012 Index, as it was deemed too preliminary.

across all indicators, i.e., $\bar{\boldsymbol{w}} = (1/8, 1/8, ..., 1/8)'$. The second-to-last column of Table 1 presents the ranking of the 21 countries on the basis of their composite scores for $\bar{\boldsymbol{w}}$, that is when $\epsilon = 0$. Using the notation of Section 4, I denote it by K^0. Conversely, the last column of Table 1 shows that this same ranking will be Kemeny-optimal for *any* value of $\epsilon \in (0, 1]$. This strong conclusion is arrived at solely by focusing on $\epsilon = 1$ and inspecting the matrix V^1 in Table 2. Doing a simple check on the latter table, we see that there does not exist a pair of countries (a_1, a_2) such that $K^0(a_1) < K^0(a_2)$ and $V^1_{a_1 a_2} < .5$. By Theorem 1 this implies that no such pairs exist for any V^ϵ where $\epsilon < 1$ either. Hence, the original ranking K^0 based on equal weights $\bar{\boldsymbol{w}}$ will satisfy

$$\kappa\left(K^0, \epsilon\right) = l^\epsilon, \ \ \forall \epsilon \in [0, 1],$$

where l^ϵ are the lower bounds defined in Eq. (10). As a result, we can unequivocally conclude that K^0 will be the unique Kemeny-optimal ranking for all possible levels of imprecision $\epsilon \in [0, 1]$.

5.2 2013 United Nations Human Development Index

The United Nations Human Development Index (HDI) is a prominent composite index of development. The HDI focuses on three main dimensions of development: (1) life expectancy, measured at birth (2) education, measured by mean years of schooling and expected years of schooling (3) GNI per capita measured in US dollars by purchasing power parity. Each dimension constitutes its own subindex and the data are normalized to lie between 0 and 1. In its most recent versions, the HDI is the geometric mean of the three dimension scores, where each dimension is assigned equal weight. In light of its importance in international policy circles, testing the robustness of the HDI rankings with respect to changes in weights has been pursued in a number of alternative ways (e.g., Anderson et al. [3], Foster et al. [12], Pinar et al. [23]).

I proceed to apply Procedure 1 to the most recent version of the HDI published in 2013. That the HDI is not a linear but a multiplicative composite does not complicate the use of my methodology. Indeed, since achievement vectors are non-negative and the natural logarithm is a strictly increasing function, the following relation holds

$$\prod_{i=1}^{3} \left(\boldsymbol{x_{a_1,i}}\right)^{w_i} \geq \prod_{i=1}^{3} \left(\boldsymbol{x_{a_2,i}}\right)^{w_i} \Leftrightarrow \log\left(\prod_{i=1}^{3} \left(\boldsymbol{x_{a_1,i}}\right)^{w_i}\right) \geq \log\left(\prod_{i=1}^{3} \left(\boldsymbol{x_{a_2,i}}\right)^{w_i}\right) \Leftrightarrow \sum_{i=1}^{3} w_i \log\left(\boldsymbol{x_{a_2,i}}\right) \geq \sum_{i=1}^{3} w_i \log\left(\boldsymbol{x_{a_2,i}}\right),$$

which implies that I can safely apply Procedure 1 to a linear composite index utilizing the natural logarithms of the HDI data.

Table 3 summarizes 2012 data for the three dimensions of the HDI for the countries having the 20 highest HDI scores under equal weights $\bar{\boldsymbol{w}} = (1/3, 1/3, 1/3)'$ (this ranking appears as K^0 in the Table's fourth column). The last four columns of Table 3 present the rankings obtained by applying Procedure 1 (to the natural logarithm of the data) for $\epsilon \in \{1/4, 1/2, 3/4, 1\}$.

How close are these rankings to the Kemeny-optimal ones? To answer this question I refer to quantities α^ϵ and β^ϵ defined in Eqs. (9)-(11). Table 4, in combination with inequality (12), immediately establishes that Procedure 1 yields a Kemeny-optimal ranking for

$\epsilon \in \{1/4, 1/2, 3/4\}$. When $\epsilon = 1$, the performance guarantee given by bound (12) is slightly above 1 (i.e., $\min\{1.279, 1.001\} = 1.001$) and I cannot immediately establish the Kemeny optimality of $\widehat{K}^{ZW,1}$. Instead, it is necessary to examine the ranking $\widehat{K}^{ZW,1}$ in more detail. To account for the value of $\beta^1 > 1$, I search for pairs of countries a_1, a_2 simultaneously satisfying $V^1_{a_1a_2} < .5$ and $\widehat{K}^{ZW,1}(a_1) < \widehat{K}^{ZW,1}(a_2)$. With regard to these pairwise comparisons $\widehat{K}^{ZW,1}$ goes against the wishes of a majority of weights. We find two such pairs. They are

(1) Singapore-France ($a_1 = 17, a_2 = 6$). Here we have $V^1_{a_1a_2} = 0.498$ but $K^{ZW,1}(a_1) = 17 < \widehat{K}^{ZW,1}(a_2) = 20$.

(2) Switzerland-Japan ($a_1 = 19, a_2 = 12$). Here we have $V^1_{a_1a_2} = 0.484$ but $K^{ZW,1}(a_1) = 8 < \widehat{K}^{ZW,1}(a_2) = 11$.

Such issues turn out to be inevitable. This is because we find ourselves in an instance of the Condorcet paradox discussed in Section 4.2. Indeed, the electorate associated with the set of weights W^1 produces what are known as Condorcet *cycles* [18], implying that there exists no ordering of the countries that is able to respect majority rule for all pairwise comparisons. Consulting the matrix V^1 computed in step (a) of Procedure 1 we identify two groups of countries that form such Condorcet cycles:

(1) Singapore-Austria-Belgium-France ($a_1 = 17, a_2 = 2, a_3 = 3, a_4 = 6$). Here we have $V^1_{a_1a_2} = 0.507$, $V^1_{a_2a_3} = 0.513$, $V^1_{a_3a_4} = 0.663$, and $V^1_{a_4a_1} = 0.502$.

(2) Switzerland-Iceland-Canada-Japan ($a_1 = 19, a_2 = 10, a_3 = 4, a_4 = 12$). Here we have $V^1_{a_1a_2} = 0.511$, $V^1_{a_2a_3} = 0.563$, $V^1_{a_3a_4} = 0.549$, and $V^1_{a_4a_1} = 0.516$.

Focusing on the above two cycles, we may deduce by inspection that $\widehat{K}^{ZW,1}$ resolves them in a way that minimizes the total amount of pairwise disagreements. Thus, we conclude that, similarly to our results for $\epsilon \in \{.25, .50, .75\}$, $\widehat{K}^{ZW,1}$ will be Kemeny-optimal. Given that $\beta^1 = 1.001 \approx 1$, this does not come as much of a surprise.

6 Conclusion

Judgments based on composite indices depend, sometimes critically, on how different dimensions of performance are weighted. As there is frequently no single "right" way to assign such weights, it is important to take this imprecision into account in a systematic way. In this paper I have presented a procedure for determining a provably-robust ranking of the relevant alternatives, given a well-established parametric structure for weight imprecision. My procedure is based on a combination of results from polyhedral geometry, social choice, and theoretical computer science, and pays special attention to issues of practicality and computational tractability. Its applicability was illustrated through numerical examples based on recent versions of the Rule of Law and Human Development indices.

Interesting future avenues for research would include adding further structure to the basic ϵ-contamination framework (e.g., to reflect normative concerns and/or operational constraints), as well as pursuing more complex empirical applications (e.g., the poverty measurement framework of [2]). Broader connections with decision-theoretic models of Knightian uncertainty could also be explored.

Appendix

A1: Proofs

Theorem 1. Let us first concentrate on the denominator of (3). We perform the following two operations on the elements of W^ϵ: (a) we translate them by $-(1-\epsilon)\bar{\boldsymbol{w}}$, and then (b) multiply them by $1/\epsilon$. The resulting polytope is Δ^{I-1}, the standard $(I-1)$-simplex. The volume of the standard simplex Δ^{I-1} (which has a side length of $\sqrt{2}$) is given by

$$Vol\left(\Delta^{I-1}\right) = \frac{\sqrt{2}^{I-1}\sqrt{I}}{(I-1)!\sqrt{2^{I-1}}} = \frac{\sqrt{I}}{(I-1)!}.$$

Basic linear algebra (see Lang [16]) implies that

$$Vol(W^\epsilon) = \sqrt{\epsilon^{2I}}Vol\left(\Delta^{I-1}\right) = \epsilon^I\frac{\sqrt{I}}{(I-1)!}. \tag{13}$$

Now let us focus on the numerator. Recall the difference vector $\boldsymbol{d} = \boldsymbol{x_{a_1}} - \boldsymbol{x_{a_2}}$ and the function $D(\epsilon) = -\frac{1-\epsilon}{\epsilon}\boldsymbol{d'}\bar{\boldsymbol{w}}$, defined in Eq. (4). Performing the same affine transformation as before, namely $\boldsymbol{w} \leftarrow \frac{\boldsymbol{w}-(1-\epsilon)\bar{\boldsymbol{w}}}{\epsilon}$, the polytope $W^\epsilon_{a_1a_2}$ is transformed into

$$\widehat{W}^\epsilon_{a_1a_2} = \left\{\widehat{\boldsymbol{w}} \in \Re^I : \ \ \widehat{\boldsymbol{w}} \geq \boldsymbol{0}, \ \sum_{i=1}^{I}\widehat{w}_i = 1, \ \ \boldsymbol{d'}\widehat{\boldsymbol{w}} \geq D(\epsilon)\right\}, \tag{14}$$

which in turn implies

$$Vol\left(W^\epsilon_{a_1a_2}\right) = \epsilon^I Vol\left(\widehat{W}^\epsilon_{a_1a_2}\right). \tag{15}$$

Putting Eqs. (13)-(15) together, we obtain

$$V^\epsilon_{a_1a_2} = \frac{Vol\left(W^\epsilon_{a_1a_2}\right)}{Vol\left(W^\epsilon\right)} = \frac{(I-1)!}{\sqrt{I}}Vol\left(\widehat{W}^\epsilon_{a_1a_2}\right). \tag{16}$$

Eqs (4)-(14) imply that the volume of $\widehat{W}^\epsilon_{a_1a_2}$ is increasing in ϵ if $\boldsymbol{d'}\bar{\boldsymbol{w}} < 0$, decreasing if $\boldsymbol{d'}\bar{\boldsymbol{w}} = 0$, and constant if $\boldsymbol{d'}\bar{\boldsymbol{w}} = 0$. The result now follows from Eq. (16). When $\boldsymbol{d'}\bar{\boldsymbol{w}} = 0$ and $\epsilon = 0$ the quantity $D(\epsilon)$ is not well-defined, leading to the potential stated discontinuity. ■

Theorem 2. Consider the polytope $W^{\epsilon,*}_{a_1a_2}$ of Eq. (5), obtained by eliminating variable I from polytope $\widehat{W}^\epsilon$:

$$W^{\epsilon,*}_{a_1a_2} = \left\{\boldsymbol{w^*} \in \Re^{I-1} : \ \ \boldsymbol{w^*} \geq 0, \ \sum_{i=1}^{I-1} w^*_i \leq 1, \ \ (\boldsymbol{d^*} - d_I\boldsymbol{e})'\boldsymbol{w^*} + d_I \geq D(\epsilon)\right\}.$$

The affine transformation f which maps polytope $W^{\epsilon,*}_{a_1a_2}$ to $\widehat{W}^\epsilon_{a_1a_2}$ is given by $f : \Re^{I-1} \to \Re^I$, satisfying $f(\boldsymbol{w^*}) = T \cdot \boldsymbol{w^*} + [0, 0, ..., 0, 1]'$, where T is an $I \times (I-1)$ matrix equal to:

$$T = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \\ 0 & 0 & \cdots & & 1 \\ -1 & -1 & -1 & \cdots & -1 \end{bmatrix} \Rightarrow T' \cdot T = \begin{bmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & \cdots & 1 \\ \vdots & & \ddots & & \\ 1 & 1 & \cdots & 2 & 1 \\ 1 & 1 & 1 & \cdots & 2 \end{bmatrix}$$

Thus we have $\det [T' \cdot T] = I$. Once again, basic linear algebra [16] implies that

$$Vol\left(\widehat{W}^{\epsilon}_{a_1a_2}\right) = \sqrt{\det [T' \cdot T]} \cdot Vol\left(W^{\epsilon,*}_{a_1a_2}\right) = \sqrt{I} \cdot Vol\left(W^{\epsilon,*}_{a_1a_2}\right). \tag{17}$$

Eqs. (16)-(17) together imply

$$V^{\epsilon}_{a_1a_2} = (I-1)!\, Vol\left(W^{\epsilon,*}_{a_1a_2}\right).$$

Proposition 3. We first identify the vertices of polytope $W^{\epsilon,*}_{a_1a_2}$. In doing so, we divide the set of indicators $\mathcal{I}^* = \{1, 2, ..., I-1\}$ into $\mathcal{I}^*_1$ and $\mathcal{I}^*_2$, such that

$$\mathcal{I}^*_1 = \{i \in \mathcal{I}^* : \ \boldsymbol{d^*_i} > D(\epsilon)\}, \ \ \mathcal{I}^*_2 = \{i \in \mathcal{I}^* : \ \boldsymbol{d^*_i} < D(\epsilon)\}.$$

Assumption 1 ensures that $\{\mathcal{I}^*_1, \mathcal{I}^*_2\}$ is a partition of $\mathcal{I}^*$. It will be useful to express polytope $W^{\epsilon,*}_{a_1a_2}$ in the following way:

$$W^{\epsilon,*}_{a_1a_2} = \{w^* \in \Re^{I-1} : \ \boldsymbol{y'_k w^*} \le \boldsymbol{b}\}, \tag{18}$$

where the $(I-1)$-dimensional vectors $\boldsymbol{y_k}$, $k = 1, 2, ..., I+1$, and $\boldsymbol{b}$ satisfy (a) $\boldsymbol{y_k} = -\boldsymbol{e_k}$[12] and $\boldsymbol{b}_k = 0$ for $k = 1, ..., I-1$, (b) $\boldsymbol{y_I} = [1, 1, 1, ..., 1]'$ and $\boldsymbol{b}_I = 1$, and (c) $\boldsymbol{y_{I+1}} = -\boldsymbol{d^*} + d_I \boldsymbol{e}$ and $\boldsymbol{b}_{I+1} = -D(\epsilon) + d_I$.

With representation (18) in mind, a vector $\boldsymbol{v}$ is a vertex of $W^{\epsilon,*}_{a_1a_2}$ if it satisfies $I-1$ linearly independent inequality constraints with equality [5]. The structure of vectors $\boldsymbol{y_k}$ for $k = 1, 2, ..., I+1$ and $\boldsymbol{b}$ imply that a vertex of $W^{\epsilon,*}_{a_1a_2}$ can have at most 2 nonzero entries. Furthermore, Assumption 1 ensures primal nondegeneracy so that every vertex $\boldsymbol{v}$ will corresponds to a unique basis matrix B_v, i.e., a unique set of linearly independent constraints satisfied with equality.

We may distinguish between four kinds of vertices and their corresponding bases:

(V1) $\boldsymbol{v_0} = \boldsymbol{0}$. $B_0 = \{\boldsymbol{y'_k} : k = 1, 2, .., I-1\}$

(V2) $\boldsymbol{v_i} = \boldsymbol{e_i}$ for all $i \in \mathcal{I}^*_1$. Here $B_i = \{\boldsymbol{y'_k} : k = 1, ..., i-1, i+1, .., I-1, I\}$, for all $i \in \mathcal{I}^*_1$.

(V3) $\boldsymbol{v_j} = \pi_j \boldsymbol{e}_j$, where $\pi_j = \frac{D(\epsilon) - d_I}{\boldsymbol{d^*_j} - d_I}$, for all $j \in \mathcal{I}^*_2$. Here $B_j = \{\boldsymbol{y'_k} : k = 1, ..., j-1, j+1, .., I-1, I+1\}$, for all $j \in \mathcal{I}^*_2$.

(V4) $\boldsymbol{v_{ij}} = \pi_{ij} \boldsymbol{e}_i + (1 - \pi_{ij}) \boldsymbol{e}_j$, where $\pi_{ij} = \frac{D(\epsilon) - \boldsymbol{d^*_j}}{\boldsymbol{d^*_i} - \boldsymbol{d^*_j}}$, for all $i \in \mathcal{I}^*_1$ and $j \in \mathcal{I}^*_2$. Here $B_{ij} = \{\boldsymbol{y'_k} : k = 1, ..., i-1, i+1, .., j-1, j+1, ..., I, I+1\}$, for all $i \in \mathcal{I}^*_1$ and $j \in \mathcal{I}^*_2$.

[12] Here $\boldsymbol{e_k}$ denotes the corresponding standard basis vector in $\Re^{I-1}$.

Two vertices are connected by an edge if they share $I-2$ common linearly independent active constraints [5]. An examination of the preceding expressions for the vertices $W^{\epsilon,*}$ of and their bases, implies that we may identify the following eight kinds of undirected edges:

(E1) $(\boldsymbol{v_0}, \boldsymbol{v_i})$, for all $i \in \mathcal{I}_1^*$.

(E2) $(\boldsymbol{v_0}, \boldsymbol{v_j})$, for all $j \in \mathcal{I}_2^*$.

(E3) $(\boldsymbol{v_i}, \boldsymbol{v_k})$ for all pairs (i,k) where $i,k \in \mathcal{I}_1^*$.

(E4) $(\boldsymbol{v_j}, \boldsymbol{v_k})$ for all pairs (j,k) where $j,k \in \mathcal{I}_2^*$.

(E5) $(\boldsymbol{v_i}, \boldsymbol{v_{ij}})$ for all pairs (i,j) where $i \in \mathcal{I}_1^*$ and $j \in \mathcal{I}_2^*$.

(E6) $(\boldsymbol{v_j}, \boldsymbol{v_{ij}})$ for all pairs (i,j) where $i \in \mathcal{I}_1^*$ and $j \in \mathcal{I}_2^*$.

(E7) $(\boldsymbol{v_{ij}}, \boldsymbol{v_{ik}})$ for all triplets (i,j,k) where $i \in \mathcal{I}_1^*$ and $j,k \in \mathcal{I}_2^*$.

(E8) $(\boldsymbol{v_{ij}}, \boldsymbol{v_{kj}})$ for all triplets (i,j,k) where $i,k \in \mathcal{I}_1^*$ and $j \in \mathcal{I}_2^*$.

Recall that we wish to exhibit a vector $\boldsymbol{c} \in \Re^{I-1}$ such that the function $f(\boldsymbol{w^*}) = \boldsymbol{c}'\boldsymbol{w^*}$ is non-constant on each edge of $W^{\epsilon,*}_{a_1a_2}$. To wit, recall the vertices of $W^{\epsilon,*}_{a_1a_2}$ enumerated above and the defined values of π_j for $j \in \mathcal{I}_2^*$ and π_{ij} for all $i \in \mathcal{I}_1^*$ and $j \in \mathcal{I}_2^*$. Define the following four quantities

1. $\delta_1 = \min\limits_{j,k \in \mathcal{I}_2^*} \{|\pi_j - \pi_k| : \pi_j \neq \pi_k\}$. If undefined, set $\delta_1 = 1$.
2. $\delta_2 = \min\limits_{i \in \mathcal{I}_1^*,\ j,k \in \mathcal{I}_2^*} \{|\pi_{ij} - \pi_{ik}| : \pi_{ij} \neq \pi_{ik}\}$. If undefined, set $\delta_2 = 1$.
3. $\delta_3 = \min\limits_{i,k \in \mathcal{I}_1^*,\ j \in \mathcal{I}_2^*} \{|\pi_{ij} - \pi_{kj}| : \pi_{ij} \neq \pi_{kj}\}$. If undefined, set $\delta_3 = 1$.
4. $\delta_4 = \min\limits_{i \in \mathcal{I}_1^*,\ j \in \mathcal{I}_2^*} \{|\pi_{ij} - \pi_j| : \pi_{ij} \neq \pi_j\}$. If undefined, set $\delta_4 = 1$.

Consequently let $\delta = \min\{\delta_1, \delta_2, \delta_3, \delta_4\}$ and define $C = \frac{2}{\delta}$. Finally recalling sets $\mathcal{I}_1^* = \{i_1, i_2, ..., i_{I_1^*}\}$ and $\mathcal{I}_2^* = \{j_1, j_2, ..., j_{I_2^*}\}$ we define the vector $\boldsymbol{c}$ satisfying

$$c_{i_k} = \begin{cases} C + \frac{k}{I_1^*} & i_k \in \mathcal{I}_1^* \\ \frac{k}{I_2^* I_1^*} & i_k \in \mathcal{I}_2^*. \end{cases} \tag{19}$$

With this choice of $\boldsymbol{c}$ we can check all eight kinds of edges E1-E8 and verify that $\boldsymbol{c}'\boldsymbol{v} \neq \boldsymbol{c}'\boldsymbol{u}$ for all pairs of adjacent vertices $(\boldsymbol{v}, \boldsymbol{u})$. Thus the function $f(\boldsymbol{w^*}) = \boldsymbol{c}'\boldsymbol{w^*}$ is non-constant on each edge of $W^{\epsilon,*}_{a_1a_2}$. Hence, in conjunction with Assumption 1, we may apply Theorem 1 in Lawrence [17] to conclude

$$Vol\left(W^{\epsilon,*}_{a_1a_2}\right) = \sum_{\substack{\text{vertices } \boldsymbol{v} \\ \text{of } W^{\epsilon,*}_{a_1a_2}}} \frac{(\boldsymbol{c}'\boldsymbol{v})^{I-1}}{(I-1)!\,|\det(B_{\boldsymbol{v}})| \prod_{i=1}^{I-1} \left[\left(B'_{\boldsymbol{v}}\right)^{-1} \boldsymbol{c}\right]_i}. \tag{20}$$

■

A2: Tables and Figures

	Dimension								Rankings	
Country	1	2	3	4	5	6	7	8	K^0	K^ϵ ($\forall\epsilon \in (0,1]$)
1. Austria	.823	.773	.885	.820	.802	.845	.735	.748	5	5
2. Belgium	.781	.782	.837	.813	.668	.698	.670	.716	10	10
3. Bulgaria	.515	.457	.739	.681	.531	.500	.557	.387	21	21
4. Croatia	.611	.547	.768	.672	.529	.484	.503	.527	20	20
5. Czech Republic	.711	.618	.810	.785	.491	.592	.638	.696	13	13
6. Denmark	.928	.953	.913	.909	.824	.846	.778	.872	2	2
7. Estonia	.795	.773	.823	.787	.713	.728	.698	.748	9	9
8. Finland	.891	.931	.917	.900	.838	.821	.780	.867	3	3
9. France	.797	.802	.841	.786	.751	.762	.677	.688	8	8
10. Germany	.821	.820	.863	.804	.729	.732	.791	.760	6	6
11. Greece	.641	.563	.732	.719	.508	.540	.605	.503	19	19
12. Hungary	.629	.722	.830	.716	.518	.596	.542	.639	16	16
13. Italy	.671	.624	.765	.723	.487	.556	.550	.673	17	17
14. Netherlands	.858	.931	.859	.837	.903	.827	.795	.801	4	4
15. Poland	.784	.718	.809	.847	.594	.611	.620	.733	12	12
16. Portugal	.713	.679	.744	.750	.616	.572	.607	.625	14	14
17. Romania	.580	.500	.802	.730	.509	.538	.577	.598	18	18
18. Slovenia	.642	.621	.802	.775	.635	.586	.586	.592	15	15
19. Spain	.753	.801	.788	.857	.614	.674	.637	.692	11	11
20. Sweden	.916	.956	.889	.929	.935	.893	.770	.823	1	1
21. United Kingdom	.785	.798	.843	.783	.782	.790	.716	.755	7	7

Table 1: 2012 Rule of Law Index: Data and rankings for 21 EU countries.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	0	1	1	1	1	0	1	0	.999	.778	1	1	1	.002	1	1	1	1	.999	0	.993
2	0	0	1	1	1	0	.064	0	.067	0	1	1	1	0	.986	1	1	1	.946	0	.004
3	0	0	0	.042	0	0	0	0	0	0	0	0	.001	0	0	0	0	0	0	0	0
4	0	0	.958	0	0	0	0	0	0	0	.069	0	.003	0	0	0	.067	0	0	0	0
5	0	0	1	1	0	0	0	0	0	0	.999	.808	1	0	0	.624	1	.750	0	0	0
6	1	1	1	1	1	0	1	.970	1	1	1	1	1	.931	1	1	1	1	1	.242	1
7	0	.936	1	1	1	0	0	0	.293	0	1	.999	1	0	.986	1	1	1	.953	0	0
8	1	1	1	1	1	.030	1	0	1	1	1	1	1	.879	1	1	1	1	1	.081	1
9	.001	.932	1	1	1	0	.707	0	0	.023	1	1	1	0	.983	1	1	1	.963	0	.005
10	.222	1	1	1	1	0	1	0	.977	0	1	1	1	0	.999	1	1	1	.998	0	.738
11	0	0	.998	.931	.001	0	0	0	0	0	0	.021	.053	0	0	0	.456	0	0	0	0
12	0	0	.990	1	.192	0	.001	0	0	0	.979	0	.877	0	0	.251	.993	.394	0	0	0
13	0	0	.999	.997	0	0	0	0	0	0	.947	.123	0	0	0	.021	.927	.119	0	0	0
14	.998	1	1	1	1	.069	1	.121	1	1	1	1	1	0	1	1	1	1	1	.001	1
15	0	.014	1	1	1	0	.014	0	.017	.001	1	.952	1	0	0	1	1	.998	.180	0	.006
16	0	0	1	1	.376	0	0	0	0	0	1	.749	.979	0	0	0	.994	.719	0	0	0
17	0	0	1	.933	0	0	0	0	0	0	.544	.007	.073	0	0	.006	0	0	0	0	0
18	0	0	1	1	.250	0	0	0	0	0	1	.606	.881	0	.002	.281	1	0	0	0	0
19	.001	.054	1	1	1	0	.047	0	.037	.002	1	1	1	0	.820	1	1	1	0	0	.013
20	1	1	1	1	1	.758	1	.919	1	1	1	1	1	.999	1	1	1	1	1	0	1
21	.008	.996	1	1	1	0	.999	0	.995	.262	1	1	1	0	.994	1	1	1	.987	0	0

Table 2: Matrix V^1 for 2012 Rule of Law Index for 21 EU countries of Table 1

	Dimension			Rankings				
Country	1	2	3	K^0	$\widehat{K}^{ZW,\frac{1}{4}}$	$\widehat{K}^{ZW,\frac{1}{2}}$	$\widehat{K}^{ZW,\frac{3}{4}}$	$\widehat{K}^{ZW,1}$
1. Australia	.981	.978	.871	2	3	3	3	3
2. Austria	.859	.962	.871	18	18	18	19	18
3. Belgium	.890	.947	.858	17	17	17	18	19
4. Canada	.908	.964	.866	11	11	11	10	10
5. Denmark	.920	.930	.858	15	15	15	15	15
6. France	.871	.973	.843	20	20	20	20	20*
7. Germany	.944	.955	.867	5	5	5	5	5
8. Hong Kong	.831	.994	.904	13	13	12	12	12
9. Iceland	.912	.977	.838	14	14	14	14	14
10. Ireland	.964	.958	.835	8	8	8	8	9
11. Israel	.912	.976	.822	16	16	16	16	16
12. Japan	.888	1	.854	10	9	9	11	11°
13. South Korea	.942	.958	.833	12	12	13	13	13
14. Netherlands	.934	.960	.874	4	4	4	4	4
15. New Zealand	1	.959	.811	6	6	6	6	7
16. Norway	.990	.966	.913	1	1	1	1	1
17. Singapore	.804	.966	.925	19	19	19	17	17*
18. Sweden	.913	.971	.870	7	7	7	7	6
19. Switzerland	.873	.985	.886	9	10	10	9	8°
20. United States	.994	.926	.897	3	2	2	2	2

Table 3: 2013 HD Report data (recall that Procedure 1 will be applied to natural logarithms of these data) and rankings. For $\epsilon = 1$, the $*$ and $\circ$ signs identify the pairs of countries that eventually lead to Condorcet cycles.

ϵ	α^{ϵ}	β^{ϵ}
.25	1.021	1.000
.50	1.002	1.000
.75	1.011	1.000
1	1.028	1.001

Table 4: Performance guarantee of Procedure 1 for rankings shown in Table 3.

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