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By **Dominik Karos**, Department of Economics and Statistics, Saarland University

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By Dominik Karos, Department of Economics and Statistics
Saarland University

Summary

The class of games with one apex player is generalized to the class of games with a collection of apex sets. These simple games, together with a power index, canonically induce a hedonic coalition formation game. A monotonicity property of solutions is introduced and its meaning for the induced hedonic game is analyzed. Necessary and sufficient conditions for the existence of core stable partitions are stated and core stable partitions are characterized.

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JEL Classification: C71

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Address for correspondence:

Dominik Karos
Department of Economics and Statistics
Saarland University
D-66123 Saarbrücken
Germany
E-mail: doka@mx.uni-saarland.de

Coalition Formation in Generalized Apex Games

Dominik Karos*

Abstract The class of games with one apex player is generalized to the class of games with a collection of apex sets. These simple games, together with a power index, canonically induce a hedonic coalition formation game. A monotonicity property of solutions is introduced and its meaning for the induced hedonic game is analyzed. Necessary and sufficient conditions for the existence of core stable partitions are stated and core stable partitions are characterized.

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1 Introduction

This paper considers three concepts of cooperative game theory, namely coalition formation in simple games, apex games and strongly monotonic solutions. The aim is to find stable outcomes of a hedonic coalition formation game which is derived from a generalized apex game and a strongly monotonic solution.

If a group of players has to make a decision, there are subgroups (or coalitions) which are able to impose the will of their members. Usually, the grand coalition should be able to unanimously decide, whereas the empty coalition should not be

*Department of Economics and Statistics, Saarland University, D-66123 Saarbrücken, Germany. Contact: doka@mx.uni-saarland.de

able to do so. Coalitions which are able to impose their will are called winning. A situation in which the collection of winning coalitions is fully described is called a simple game. An important question which arises in those games is, how powerful various players are. There are different ways how to measure this power. Particularly, it is interesting not only to ask, what is the power of a player in the overall game, but also: How powerful is a player within a certain coalition? The power of a player might be used to distribute the worth of a coalition between its members (think of the number of ministries a party gets within a political coalition). Hence, a power measure provides each player with a tool to compare the different coalitions he might belong to. Preferences over coalitions thus derived induce a coalition formation game. The question is now: Which coalition shall form? Or: Which coalition is stable, in the sense that no member will leave it? We assume that the value a player receives in a coalition does not depend on the behaviour of players outside of the coalition, that is that there are no externalities. Such coalition formation games are called hedonic games. They have been introduced in Drèze and Greenberg (1980) and despite the absence of externalities, their analysis is quite complex. Particularly, it is not even clear under which circumstances a stable coalition exists. Although there are some general conditions which guarantee existence (see for instance Banerjee et al., 2001; Bogomolnaia and Jackson, 2002; Iehlé, 2007), an analysis of hedonic games which are derived from a simple game and a power measure has not led to sufficiently general results. A good basis of this topic can be found for instance in Dimitrov and Haake (2006, 2008).

A special subclass of simple games are apex games. They have already been studied in Morgenstern and von Neumann (1944). These games with one major player (originally called chief player in Morgenstern and von Neumann (1944), later called apex player) and a set of minor players have been investigated in many articles (see for instance Aumann and Myerson, 1988; Hart and Kurz, 1983, 1984; Montero, 2002).

An apex player can form a winning coalition with each of the minor players. But the set of all minor players together is winning as well. In that sense, the apex player is not able to block any decision of the coalition of minor players. A player which is able to block each coalition is called a veto player, hence an apex game can be interpreted as a 'weak' version of a veto game. This is the motivation for the class of generalized apex games which is introduced in this paper. Consider a game with a nonwinning coalition C such that a coalition is winning if and only if it contains C and at least one of the remaining players. Each member of C is a veto player, and again the members of C are considerably weakened if the game is changed such that the remaining players can unanimously form a winning coalition. Although this is already a slight generalization, one might still go further. Let there be several coalitions I_k in a simple game and a set J with an empty intersection with each I_k, such that a coalition is winning if and only if it contains some I_k and at least one $j \in J$. The apex version would be that J itself is a winning coalition. In such games the sets I_k are called apex sets, and J is the set of minor players.

Generalized apex games can occur in many situations. The outcome of political election might be considered (see for instance Example 3.2). But also in decision processes a generalized apex game can easily appear: Suppose that a decision can be made in a committee and let I_k be the coalitions in the committee which can impose their will. Suppose further that a decision needs approval by at least one person from a second committee (for instance a supervisory board). Such a voting game will end up in a generalized apex situation (see Example 3.4).

The third concept we consider is the one of monotonic solutions. If there are two different simple games u and v on the same set of players such that a player can block in v all coalitions which he can block in u and at least one more, then his power in v should be strictly greater than in u. This monotonicity property appears for example in Sagonti (1991), where it is shown to be satisfied by the Shapley-Shubik-Index (see Definition 5.1 as well as Shapley, 1953), and the Banzhaf value (see Definition 5.8, as well as Banzhaf, 1965; Coleman, 1971). We show that the normalized Banzhaf value is in general not monotonic in this sense, but that its behaviour on generalized apex games is quite similar to that of a monotonic solution.

After these three ideas have been introduced, we consider hedonic games which are derived from a generalized apex game together with a strongly monotonic solution. In Shenoy (1979) it is shown that the (simple) apex game is the only four player (proper monotonic) simple game which induces together with the Shapley value a hedonic game with an empty core. For games with more players the core of the respective hedonic game remains empty. In particular, the Shapley value of the apex player increases with the number of minor players. It will be shown that this comes from the strong monotonicity of the Shapley value and holds true also on generalized apex games. We will derive necessary and sufficient conditions for the nonemptiness of the core of hedonic games which are induced by a generalized apex game and a strongly monotonic solution.

Although the existence of a stable coalition in a hedonic game derived from a generalized apex game and a strongly monotonic solution highly depends on the structure of the family $\{I_k\}_k$, it will be shown that many insights can be derived if simple conditions hold true. Particularly, it will be shown that the players of J are an important part of any core stable coalition.

Section 2 develops the basics of simple games and hedonic games. Particularly, some important properties of hedonic games which are induced by simple games and solutions are stated. Section 3 introduces apex games and their generalization. Some basic properties of the family of apex sets as well as uniqueness of a representation are shown. Section 4 contains the main statements. Strongly monotonic solutions for simple games are introduced and hedonic games which are induced by generalized apex games together with a strongly monotonic solution are analyzed. Necessary and sufficient conditions for the existence of stable coalitions are stated and candidates for core stable partitions are given. Section 5 applies these results to the Shapley value, the Banzhaf value and the normalized Banzhaf value. Sharper conditions for the existence of stable coalitions are stated and for many cases stable coalitions are characterized. Section 6 presents insights for further generalizations and states some open questions.

2 Coalition Formation in Simple Games

Throughout the paper let N be a finite set of players. A *coalition* is a subset $S \subseteq N$ and the set $\mathfrak{P} = \mathfrak{P}(N)$ is the collection of all subsets of N. For $i \in N$, let $\mathfrak{P}_i(N)$ be the collection of all subsets of N which contain i. A *partition* is a set of nonempty coalitions $\pi = \{S_1, \ldots, S_m\}$ such that $S_k \cap S_l = \emptyset$ for all $k \neq l$ and $\bigcup_{k=1}^{m} S_k = N$. The collection of all partitions of N is denoted by $\Pi = \Pi(N)$. For a partition $\pi \in \Pi$ and a player $i \in N$, let $\pi(i)$ denote the unique coalition in π which contains i. For a coalition $S \subseteq N$, let $\Pi_S = \Pi_S(N)$ be the collection of all partitions containing S.

2.1 Simple Games

A *simple game* is a function $v : \mathfrak{P}(N) \to \{0,1\}$ with $v(\emptyset) = 0$. Particularly, the *zero game* which assigns 0 to each coalition $S \subseteq N$ is a simple game.
Simple games have already occurred at the very beginning of game theory in the book of Morgenstern and von Neumann (1944). The authors devoted a large chapter only to this class of games and investigated their structure. Since then simple games have been studied in many articles (see for instance Banzhaf, 1965; Shapley, 1966; Shenoy, 1979). Their importance comes from the fact that they represent situations in which the members of a society vote for or against an alternative. The analysis of simple games provides insights into the relation between a player's power in and the final outcome of such a game.
Let v be a simple game. A coalition $S \subseteq N$ with $v(S) = 1$ is called *winning*. If S is a winning coalition and $i \in S$ is such that $v(S \setminus \{i\}) = 0$, then i is called *pivotal* in S with respect to v. A winning coalition S in which each $i \in S$ is pivotal (in S with respect to v) is called *minimal winning*. If $i \in N$ is not contained in any minimal winning coalition then i is called a *null player* (with respect to v). Equivalently $v(S) - v(S \setminus \{i\}) = 0$ for all $S \subseteq N$. If $i \in N$ is contained in all minimal winning coalitions then i is called *veto player* with respect to v. Equivalently $v(S) - v(S \setminus \{i\}) = 1$ for all winning coalitions $S \subseteq N$. If i is a veto player such that $v(\{i\}) = 1$ then i is called a *dictator*. Two players $i, j \in N$ are called *symmetric* in v if $v(S \setminus \{i\}) = v(S \setminus \{j\})$ for all $S \in \mathfrak{P}_i \cap \mathfrak{P}_j$. Note that all veto players with respect v are symmetric in v, as well as all null players with respect to v are symmetric in v.
The following two possible properties of a simple game are intuitive. They would be satisfied, for instance, in a simple game derived from majority voting, i.e. a voting game in which a coalition is winning if and only if it has a (simple or qualified) majority of the votes.

Definition 2.1. A simple game v is called

1. *proper*, if $v(S) + v(N \setminus S) \leq 1$ for all $S \subseteq N$.

2. *monotonic*, if $v(S) \leq v(T)$ for all S, T with $S \subseteq T$.

The set of all proper monotonic games on N is denoted by $\mathcal{V} = \mathcal{V}(N)$.

A veto player seems to be very powerful, as he can bring down each winning coalition, while a null player seems to be quite weak, as he does not have any influence in any winning coalition. The 'power' of a player i in a simple game $v \in \mathcal{V}$ is naturally related to the number of minimal winning coalitions to which i belongs.
If a winning coalition has formed, the next question is how powerful each player is in this (fixed) coalition. Therefore, it is reasonable to consider proper monotonic simple games on subcoalitions of N (see for instance Dimitrov and Haake, 2008). For $v \in \mathcal{V}$ and a set $S \subseteq N$ the subgame (or restricted game) v_S on N is defined by

$$v_S(T) = v(S \cap T)$$

for all $T \subseteq N$.
This idea is quite related to the *carrier* of a simple game defined in Shapley (1953): A carrier of a simple game v is defined as a set N of players such that $v(S) = v(N \cap S)$ for all coalitions S of players in a (maybe infinite) universe U. In our case, $S \subseteq N$ is a carrier of the game v_S. In this new game v_S, all players from $N \setminus S$ are basically ignored: They are null players with respect to v_S. Null players are considered later in this section. First it should be clarified, which properties of v remain valid in v_S. Fortunately, the structure of a subgame v_S is the same as that of v in the following sense.

Proposition 2.2. *Let v be a simple game and $S \subseteq N$.*

1. *If v is proper, then v_S is proper.*
2. *If v is monotonic, then v_S is monotonic.*

The proof of this proposition is left to the reader. So, for any $v \in \mathcal{V}$ and any $S \subseteq N$, the subgame v_S is also a proper monotonic simple game. If S is not a winning coalition, then v_S is just the zero game; if $S = N$, then v_S coincides with v. As $v_S \in \mathcal{V}$ for each $S \subseteq N$, veto and null players are well defined with respect to v_S. A *veto player in S* with respect to v is a veto player with respect to v_S, a *null player in S* with respect to v is a null player with respect to v_S. Hence, $i \in S$ is pivotal in S with respect to v if and only if i is a veto player with respect to v_S. Particularly, a veto player (null player) in N with respect to v is also a veto player (null player) in v_S for all winning coalitions $S \in \mathfrak{P}_i$.

Simple games form a subclass of the class of transferable utility games (TU games). An outcome of a TU game on N is an allocation of $v(N)$ to the players of N. An allocation rule which is applicable to arbitrary TU games is called a 'solution'. We next define the concept of solutions for proper monotonic simple games and some properties.

Definition 2.3. A solution is a mapping $\varphi : \mathcal{V}(N) \to \mathbb{R}^N$.[1] The projection $(\varphi(v))_i$ is abbreviated by $\varphi_i(v)$. A solution φ

1. is *nonnegative* if $\varphi_i(v) \geq 0$ for all $i \in N$ and all $v \in \mathcal{V}$;
2. is *efficient* if $\sum_{i\in N} \varphi_i(v) = v(N)$ for all $v \in \mathcal{V}$;
3. is *coalitionally efficient* if $\sum_{i\in S} \varphi_i(v_S) = v(S)$ for all $v \in \mathcal{V}$ and all $S \subseteq N$;
4. satisfies the *null player property* if $\varphi_i(v) = 0$ for all $v \in \mathcal{V}$ and all null players $i \in N$ with respect to v;
5. satisfies the *equal treatment property* if $\varphi_i(v) = \varphi_j(v)$ for every game $v \in \mathcal{V}$ and all symmetric players i and j in v;
6. lives only on winning coalitions if $\varphi_i(v_0) = 0$ for all $i \in N$ and the zero game v_0.

Consider a proper monotonic simple game v and a coalition S which is not winning. Then v_S is the zero game. Hence, if φ is a solution which lives only on winning coalitions then $\varphi_i(v_S) = 0$ for all $i \in N$. Note that this property is implied for instance by the null player property.
We now come back to null players. Let $v \in \mathcal{V}$ be a proper monotonic simple game and let i be a null player in N with respect to v. Then

$$v_{N\setminus\{i\}}(S) = v(S \cap N \setminus \{i\}) = v(S \setminus \{i\}) = v(S)$$

for all coalitions $S \subseteq N$. Hence, games v and $v_{N\setminus\{i\}}$ coincide. This is particularly true for restricted games v_S and null players $i \in S$ with respect to v_S.

[1] We do not need that a solution is *feasible*, i.e. that $\sum_{i\in N} (\varphi(v))_i \leq v(N)$. In particular, the Banzhaf value (see Definition 5.8) is not feasible.

Lemma 2.4. *Let $v \in \mathcal{V}$ be a proper monotonic simple game. Let $S \subseteq N$ and let $i \in S$ be a null player in S with respect to v. Then $v_S = v_{S \setminus \{i\}}$.*

Proof. For all $T \subseteq N$ it holds that

$$v_{S\setminus\{i\}}(T) = v((S \setminus \{i\}) \cap T) = v(S \cap (T \setminus \{i\})) = v_S(T \setminus \{i\}) = v_S(T).$$

Hence, $v_{S\setminus\{i\}}$ and v_S coincide. □

The following corollary is an immediate consequence and is stated for easy reference.

Corollary 2.5. *Let $v \in \mathcal{V}$ be a proper monotonic simple game. Let $S \subseteq N$ be a winning coalition and let $i \in S$ be a null player with respect to v_S. Then $\varphi_j(v_S) = \varphi_j\left(v_{S\setminus\{i\}}\right)$ for all solutions φ and all $j \in N$.*

Note that in Corollary 2.5 the solution φ does not need to satisfy the null player property. However, it can be used to throw light on the relation between the null player property, efficiency and coalitional efficiency.

Lemma 2.6. *A solution φ is coalitionally efficient if and only if it is efficient and satisfies the null player property.*

Proof. Let φ be a solution which satisfies the null player property and efficiency and let $S \subseteq N$ be a winning coalition. Then all players $i \in N \setminus S$ are null players in the restricted game v_S. In this case efficiency implies

$$\sum_{i\in S} \varphi_i(v_S) = \sum_{i\in N} \varphi_i(v_S) = 1 = v(S).$$

If S is not winning then $\sum_{i\in S} \varphi_i(v_S) = 0 = v(S)$ by the null player property. Hence, efficiency and the null player property together imply coalitional efficiency. On the other hand, if a solution φ satisfies coalitional efficiency, then φ is efficient as well. Let $i \in N$ be a null player with respect to v. Then $\varphi_j(v) = \varphi_j\left(v_{N\setminus\{i\}}\right)$ for all $j \in N$ by Corollary 2.5. In this case coalitional efficiency implies

$$\varphi_i(v) = \sum_{j\in N} \varphi_j(v) - \sum_{j\in N\setminus\{i\}} \varphi_j(v) = \sum_{j\in N} \varphi_j(v) - \sum_{j\in N\setminus\{i\}} \varphi_j\left(v_{N\setminus\{i\}}\right) = 1 - 1 = 0.$$

Hence, coalitional efficiency is equivalent to efficiency together with the null player property. □

In case of voting games it is often asked what efficiency means. Particularly, several authors have tried to find axiomatizations of various solutions without imposing an efficiency requirement (see for instance Dubey et al., 1981). A recent example of such an axiomatization of the Shapley-Shubik-Index (see Definition 5.1) is given in Einy and Haimanko (2011). For the results of our paper, the interpretation of efficiency is not of importance. Later, solutions will be considered which are not even efficient.

2.2 Hedonic Games

A *hedonic game* is a set N together with a profile of preferences $\succeq = (\succeq_i)_{i \in N}$, where $\succeq_i$ is defined on $\mathfrak{P}_i(N)$ for all $i \in N$.[2]
Hedonic games belong to a special class of coalition formation games and have been introduced by Drèze and Greenberg (1980). The main characteristic is, that for each player $i \in N$, the preference relation $\succeq_i$ depends only on the coalitions to which player i belongs, and not on the behaviour of the remaining players. The crucial question is, whether there is a partition of the player set N which is 'sufficiently satisfying' for all players. Sufficiently satisfying (or 'stable') in this context means, that there is no group of players which would leave their coalitions and form a new one together (see Definition 2.7). There are several answers to this question; the probably best known are the sufficient conditions as *ordinal balancedness* and *consecutiveness* in Bogomolnaia and Jackson (2002) and the *weak top coalition property* in Banerjee et al. (2001). Unfortunately, they are not necessary. A characterization of hedonic games for which a stable partition of the player set exists is given in Iehlé (2007). The author gives a weaker version of ordinal balancedness of a hedonic game which is both necessary and sufficient. Unfortunately, this *pivot balancedness* is neither constructive, in the sense that a core stable partition could easily be found, nor can it be verified efficiently.
The idea of stability of a partition in a hedonic game is stated formally in the following definition.

Definition 2.7. Let $(N, \succeq)$ be a hedonic game and $\pi \in \Pi$.

[2] A preference relation is a complete and transitive binary relation.

1. A *deviation* of π is a coalition $S \subseteq N$ such that $S \succ_i \pi(i)$ for all $i \in S$.

2. π is called *core stable*, if it has no deviations.

A partition π is said to be *blocked by* S if S if a deviation of π. A partition π is said to *blocked*, if it is blocked by some $S \subseteq N$. The *core* of a hedonic game $(N, \succeq)$ is the set of all core stable partitions and is denoted by $\mathcal{C}(N, \succeq)$.
The remainder of this section shows, how hedonic games can be derived from proper monotonic simple games together with a solution. It further states some basic properties the resulting hedonic games have, depending on properties of the underlying solution.
Given a proper monotonic simple game v and a solution φ there is a payoff $\varphi_i(v_S)$ for each $i \in N$ in the subgame v_S for each $S \subseteq N$. Hence, a simple game $v \in \mathcal{V}$ together with a solution φ canonically induces the following profile of preferences $(\succeq_i)_{i \in N}$. For $i \in N$ define $\succeq_i$ on $\mathfrak{P}_i$ by

$$S \succeq_i T \qquad \text{if and only if} \qquad \varphi_i(v_S) \geq \varphi_i(v_T). \tag{1}$$

Thus, a proper monotonic simple game v, together with a solution φ, induces a hedonic game. The core of this hedonic game is denoted by $\mathcal{C}(N, v, \varphi)$.
In general there is not much that can be said about the structure of a hedonic game. This is different in the case of hedonic games derived from simple games. A proper monotonic simple game $v \in \mathcal{V}$ together with a solution φ which lives only on winning coalitions leads to a simple structure of the induced hedonic game.

Lemma 2.8. *Let $v \in \mathcal{V}$ and let φ be a solution which lives only on winning coalitions. Then for each winning coalition S either $\Pi_S \subseteq \mathcal{C}(N, v, \varphi)$ or $\Pi_S \cap \mathcal{C}(N, v, \varphi) = \emptyset$.*

Proof. Let $S \subseteq N$ be a winning coalition. By properness $v(N \setminus S) = 0$ and by monotonicity $v(T) = 0$ for all $T \subseteq N \setminus S$. Hence, for all $T \subseteq N \setminus S$ the game v_T is the zero game. Consequently, if φ lives only on winning coalitions then $\varphi_i\left(v_{\pi(i)}\right) = 0$ for all $i \in N \setminus S$ and all $\pi \in \Pi_S$.
It has to be shown that a partition $\pi \in \Pi_S$ is blocked if and only if all partitions in Π_S are blocked. The 'if'-part is obvious. So, let $\pi, \sigma \in \Pi_S$ and let $D \subseteq N$ be

a deviation of π. Then $\varphi_i(v_D) > 0$ for all $i \in D \setminus S$ and $\varphi_i(v_D) > \varphi_i(v_S)$ for all $i \in D \cap S$. In this case D is also a deviation of σ. Hence, if a partition $\pi \in \Pi_S$ is blocked by a coalition D, then each partition $\sigma \in \Pi_S$ is blocked by D. □

Given this result the following definition makes sense.

Definition 2.9. Let $v \in \mathcal{V}$ and φ be a solution which lives only on winning coalitions. A coalition $S \subseteq N$ is called *core stable* if each partition $\pi \in \Pi_S$ is core stable. In this case we write $S \in \mathcal{C}(N, v, \varphi)$.

At the end of this section, it makes sense to exclude some trivialities which could appear in the presence of null players.

Lemma 2.10. *Let N be a set of players, let $v \in \mathcal{V}$ be not the zero game and let φ be a solution. Let $N' \subseteq N$ be the set of players of N which are not null players with respect to v, and for a partition $\pi \in \Pi(N)$ let $\pi' \in \Pi(N')$ such that $S \in \pi$ if and only if $S \cap N' \in \pi'$, and let φ' be the solution on $\mathcal{V}(N')$ defined by $\varphi'_i(v_S) = \varphi_i(v_S)$ for all $S \subseteq N'$ and all $i \in N'$. Then $\pi \in \mathcal{C}(N, v, \varphi)$ if and only if $\pi' \in \mathcal{C}(N', v_{N'}, \varphi')$.*

Proof. For a coalition $S \subseteq N$ let $S' = S \cap N'$. Recall that a null player with respect to v is also null player with respect to the restricted game v_S for all $S \subseteq N$. Hence, by Corollary 2.5, $\varphi_i(v_S) = \varphi_i(v_{S'}) = \varphi'_i(v_{S'})$ for all coalitions $S \subseteq N$ and all $i \in N$.
Let $\pi \in \mathcal{C}(N, v, \varphi)$ and assume that $\pi' \notin \mathcal{C}(N', v_{N'}, \varphi')$. Then there must be $T \subseteq N'$ which blocks π'. In particular,

$$\varphi_i(v_T) = \varphi'_i(v_T) > \varphi'_i\left(v_{\pi'(i)}\right) = \varphi_i\left(v_{\pi(i)}\right)$$

for all $i \in T$. Hence, T blocks π, a contradiction.
On the other hand, let $\pi \in \Pi(N)$ such that $\pi' \in \mathcal{C}(N', v_{N'}, \varphi')$. Suppose that $\pi \notin \mathcal{C}(N, v, \varphi)$. Then there must be $T \subseteq N$ such that $\varphi_i(v_T) > \varphi_i\left(v_{\pi(i)}\right)$ for all $i \in T$. Particularly, for all $i \in T'$ it holds that

$$\varphi'_i(v_{T'}) = \varphi_i(v_T) > \varphi_i\left(v_{\pi(i)}\right) = \varphi'_i\left(v_{\pi'(i)}\right).$$

Hence, T' is a deviation of π', again a contradiction. □

3 Generalized Apex Games

Apex games have been studied already in Morgenstern and von Neumann (1944). They consist of one apex player and a set of symmetric minor players in the following sense.

Definition 3.1. Let N be set of players and let a be a proper monotonic simple game on N such that there are $i \in N$ and $J \subseteq N \setminus \{i\}$ with

$$a(S) = \begin{cases} 1, & \text{if } J \subseteq S \text{ or } (i \in S \text{ and } S \cap J \neq \emptyset), \\ 0, & \text{else.} \end{cases}$$

Then $a = a_{iJ}$ is called an *apex game* with *apex player* i and *minor players* $j \in J$.

If the apex game is interpreted as a weighted voting game, an interesting question is which coalitions will form. One possibility is that the coalition of minor players should form. This idea is supported for instance in Hart and Kurz (1984) or Aumann and Myerson (1988). While Aumann and Myerson investigate the Myerson value (see Myerson, 1977) on graph games, Hart and Kurz use the Owen value (see Owen, 1977) and introduce a partition function form game. The stability concepts differ in both cases from core stability in our sense. Nevertheless, some ideas of their proofs will be generalized later. A very broad overview about the behaviour of various solution concepts on apex games can be found in Montero (2002).

Example 3.2.

a) The German parliament currently contains five parties. The biggest party is the CDU which could form a minimal winning coalition with three of the remaining four parties (SPD, FDP, Die Linke). Exactly these three parties together could also form a minimal winning coalition. The fifth party (Bündnis 90 / Die Grünen) is not contained in any minimal winning coalition. Hence, Die Grünen is a null player in the simple game, CDU is an apex player and the remaining parties are minor players in an apex game. The government contains CDU together with FDP.

b) In the election of the Moldavian parliament in August 2009 the communist party (PCRM) was not able to get an absolute majority on its own.

> However, a coalition with any of the four smaller parties would have led to the majority. The four smaller parties on the other hand could form a winning coalition on their own and make decisions for which a simple majority was sufficient - although their votes did not exceed the 60% limit which was necessary to vote the president.[3] In the end the four smaller parties built the new government and the president's election was supported by the communists.

The idea of Hart and Kurz (1984) allows for two stable outcomes in an apex games with not more than four players (which are not null players), namely all minor players together or the apex player together with exactly one minor player. Aumann and Myerson (1988) come to the conclusion that only the coalition of all minor players should form. Both examples are supporting the idea of Hart and Kurz (1984), the last one supports Aumann and Myerson (1988).
However, both examples are strongly influenced by political differences. In Germany the coalition of minor players would contain the liberals (FDP) and the socialists (Die Linke). In Moldavia the four minor players are the four Europe-oriented parties and the apex player are the communists. But in both cases, there seems to be a special role for minimal winning coalitions.
At this point, there are a few things to say about apex games. If $|J| = 1$ the game a_{iJ} is a dictator game with dictator $j \in J$. If $|J| = 2$ then it is a symmetric game between i, j_1 and j_2, where $J = \{j_1, j_2\}$. In particular, the representation of the game is not unique as each player could be interpreted as apex player.
The more interesting cases start with $|J| \geq 3$. These games have a unique representation: i must be unique, else the game would not be proper. Since a player outside of $\{i\} \cup J$ is per definition not contained in any minimal winning coalition, J is the unique set of players in $N \setminus \{i\}$ which are not null players.
The class of apex games can be generalized in the following way.

Definition 3.3. Let N be a set of players and a be a proper monotonic simple game on N such that there are a collection $\mathcal{I} = \{I_k\}_{k=1}^{m}$ of nonempty subsets of

[3] In order to prevent misunderstandings, it should be mentioned that most decisions in the parliament can be imposed by simple majority; one exception is the election of the president.

N and a set $J \subseteq N \setminus \bigcup_{k=1}^{m} I_k$ with

$$a(S) = \begin{cases} 1, & \text{if } J \subseteq S \text{ or } \left(S \cap J \neq \emptyset \text{ and there is } I_k \in \mathcal{I} \text{ with } I_k \subseteq S\right), \\ 0, & \text{else.} \end{cases}$$

Then $a = a_{\mathcal{I}J}$ is a *generalized apex game* with *apex sets* $I_1, \ldots I_m$ and *minor players* $j \in J$.

Before starting with further investigations on generalized apex games, it is useful to consider an example.

Example 3.4. In a stock corporation under German law there is a two-tier board system. There is an executive board which is in charge of the management of the company and a supervisory board which has to ensure that the interests of the stakeholders are not violated. Particularly, there are decisions which the supervisory board has to approve. Suppose that there is a proper and monotonic voting system in the executive board to make a decision, and that such a decision is accepted by the supervisory board if at least one of its members agrees. On the other hand, let the supervisory board be able to force the executive board to follow a decision if it is reached unanimously. (This assumption is usually fulfilled as the supervisory board is allowed to fire executives and replace them). In this case each winning coalition in the executive board would be an apex set in the sense of this paper, and the supervisory board would consist of the minor players.

The set $\mathcal{I}$ is not necessarily unique due to monotonicity of the game. If $I_1, I_2 \in \mathcal{I}$, then $I_1 \cup I_2$ can be element of $\mathcal{I}$ or not, without any influence on the winning coalitions of $a_{\mathcal{I}J}$. Therefore, $\mathcal{I}$ is called a *minimal representation* if for all $I \in \mathcal{I}$ and all $j \in J$ the coalition $I \cup \{j\}$ is minimal winning.
Independently of whether or not a generalized apex game is given in minimal representation, the structure of apex sets is not arbitrary.

Lemma 3.5. *Let $a_{\mathcal{I}J}$ be a generalized apex game on N with $|J| \geq 2$ and let $I_1, I_2 \in \mathcal{I}$. Then $I_1 \cap I_2 \neq \emptyset$.*

Proof. Assume that $I_1 \cap I_2 = \emptyset$ and let $j_1, j_2 \in J$ with $j_1 \neq j_2$. Then $I_1 \cup \{j_1\}$ and $I_2 \cup \{j_2\}$ are winning coalitions with empty intersection. This contradicts the properness of $a_{\mathcal{I}J}$. □

Hence, for a generalized apex game $a_{\mathcal{I}J}$ and a apex set $I \in \mathcal{I}$ it always holds that $N \setminus I$ cannot contain any apex set.
While it may happen that a player from an apex set is a null player in the restriction of $a_{\mathcal{I}J}$ to a winning set, for the minor players the following holds true.

Lemma 3.6. *Let $a_{\mathcal{I}J}$ be a generalized apex game on a set N. Let S be a winning coalition. Then $j \in J$ is a null player in the restriction of $a_{\mathcal{I}J}$ to S if and only if $j \notin S$.*

Proof. If $j \notin S$ then j is a null player in the restriction of $a_{\mathcal{I}J}$ to S. Let $j \in S$. If there is $I \in \mathcal{I}$ such that $I \subseteq S$, then j is pivotal in $I \cup \{j\}$ with respect to v_S. If such an I does not exist, then by definition of a generalized apex game $J \subseteq S$, otherwise S cannot be winning. In this case j is pivotal in J. In any case, there is a winning coalition in which j is pivotal with respect to v_S. Hence, j cannot be a null player in v_S. □

The remaining players, i.e. players which are neither minor players nor contained in any apex set, have not much influence on the game.

Lemma 3.7. *Let $a_{\mathcal{I}J}$ be a generalized apex game on N in minimal representation and let $h \in N \setminus (\bigcup_{k=1}^{m} I_k \cup J)$. Then h is a null player in N with respect to $a_{\mathcal{I}J}$.*

Proof. By construction of the apex game there is no minimal winning coalition which contains h. This is the definition of a null player. □

We know that for each apex game there is a representation $a_{\mathcal{I}J}$. However, as seen before, this representation need not to be unique. It is useful to consider only generalized apex games in minimal representation as in this case a lot of ambiguity vanishes. But even then, uniqueness is not guaranteed: Let N contain three player and let v be the proper monotonic simple game in which each coalition containing at least two players is winning. Then v is an apex game. We refer to v as the *three player simple majority voting game.* Particularly, each player of N could be interpreted as apex player. The next lemma helps to find those generalized apex games which have a unique representation.

Lemma 3.8. *Let N be a set of players and let $a_{\mathcal{I}J}$, $a_{\mathcal{I}'J'}$ be generalized apex games on N in minimal representation with $a_{\mathcal{I}J}(S) = a_{\mathcal{I}'J'}(S)$ for all $S \subseteq N$. If $|J| \geq 3$ then $J = J'$ and $\mathcal{I} = \mathcal{I}'$.*

Proof. Let $|J| \geq 3$. It is sufficient to show that $J = J'$. In this case, since the minimal winning coalitions of both games are identical, the collections $\mathcal{I}$ and $\mathcal{I}'$ must be the same.
So, assume that $J \neq J'$. J is minimal winning in $a_{\mathcal{I}'J'}$. Therefore, J' cannot be a proper subset of J. By assumption $J' \neq J$ and, since J is winning, $J = I' \cup \{j\}$ for some $I' \in \mathcal{I}'$ and $j \in J' \cap J$. Let $j' \in J' \setminus J$ (such a j' exists as $J' \nsubseteq J$). Then $I' \cup \{j'\}$ is minimal winning as well by definition of a generalized apex game. Particularly, j is not contained in $I' \cup \{j'\}$, which means that $J \nsubseteq I' \cup \{j'\}$. Now, $I' \subsetneq J$ by construction and $I' \cup \{j'\}$ is winning in $a_{\mathcal{I}J}$. Therefore, all $i' \in I' \subseteq J$ are minor players with respect to $a_{\mathcal{I}J}$. Consequently, the singleton $\{j'\}$ must lie in $\mathcal{I}$. We have $\{j'\} \in \mathcal{I}$, $I' \subsetneq J$, and $I' \cup \{j'\}$ being minimal winning in $a_{\mathcal{I}J}$. Consequently, I' cannot contain more than one element. Hence, the set $J = I' \cup \{j\}$ cannot contain more than two elements. But this is a contradiction to $|J| \geq 3$. □

The condition $|J| \geq 3$ is already sufficient for uniqueness. But as it is, we can describe the set of apex games which do not have a unique representation very well.

Theorem 3.9. *The minimal representation of a generalized apex game $a_{\mathcal{I}J}$ on N is unique if and only if the restriction of $a_{\mathcal{I}J}$ to $\bigcup_{I \in \mathcal{I}} I \cup J$ is not the three player simple majority voting game.*

Proof. By Lemma 2.4 we can assume without loss of generality that N does not contain any null players. Hence, by Lemma 3.7 $N = \bigcup_{I \in \mathcal{I}} I \cup J$. If $|J| \geq 3$ then Lemma 3.8 applies and there is nothing to show. If $|J| = 1$ then all players in $N \setminus J$ are null players and the representation is unique. Hence, let $|J| = 2$. If $|N| = 2$ then the representation is unique. If N contains at least 4 players then there are at least two players in $N \setminus J$ which are not null players. Hence, by Lemma 3.5, there cannot be a minimal apex set which contains only one player. Consequently, each minimal winning coalition in $a_{\mathcal{I}J}$ except J contains at least three players. If $J' \neq J$, then $|J'| \geq 3$ and thus, $a_{\mathcal{I}'J'}$ has a unique minimal

representation by Lemma 3.8. As $a_{\mathcal{I}J}$ and $a_{\mathcal{I}'J'}$ coincide, they must have the same minimal representation.
Hence, there is no generalized apex game with $|N| \neq 3$ and $|J| \neq 2$ with a representation which is not unique. But if $|N| = 3$ and $|J| = 2$ then $a_{\mathcal{I}J}$ is the three player simple majority voting game. □

4 Monotonic Solutions and Core Stable Partitions

This section is devoted to the impact of monotonicity of a solution φ on the existence of core stable partitions in the hedonic game derived from a simple game v and φ as in (1). The first subsection defines a 'strong monotonicity' property which is based on Sagonti (1991) and quite related to strong monotonicity in the sense of Young (1985). The second subsection develops some properties of strongly monotonic solutions on generalized apex games. The third section states the main results of the paper concerning the existence of core stable partitions in hedonic games which are induced by generalized apex games and strongly monotonic solutions.

4.1 Strongly Monotonic Solutions on Simple Games

The following definition is due to Sagonti (1991), where the monotonicity properties of several solutions are analyzed.

Definition 4.1. A solution φ is called *strongly monotonic* if

$$\varphi_i(v) > \varphi_i(u)$$

for all $i \in N$ and all $u, v \in \mathcal{V}$ with

$$\begin{aligned} & v(S) - v(S \setminus \{i\}) \geq u(S) - u(S \setminus \{i\}) \quad \text{for all } S \subseteq N, \\ \text{and} \quad & v(S) - v(S \setminus \{i\}) > u(S) - u(S \setminus \{i\}) \quad \text{for some } S \subseteq N. \end{aligned}$$

A similar definition of strong monotonicity can be found in Young (1985). There, a solution φ is called strongly monotonic, if $\varphi_i(v) \geq \varphi_i(u)$ for all players $i \in N$

and all $u, v \in \mathcal{V}$ which satisfy

$$v(S) - v(S \setminus \{i\}) \geq u(S) - u(S \setminus \{i\}) \quad \text{for all } S \subseteq N.$$

Neither the strict inequality $v(S) - v(S \setminus \{i\}) > u(S) - u(S \setminus \{i\})$ is required, nor the strict inequality $\varphi_i(v) > \varphi_i(u)$ is expected. Hence, there is a crucial difference between the definitions of Sagonti and Young: If a player i is pivotal in exactly the same coalitions with respect to a game u and a game v, then Definition 4.1 does not make any statement about the relation between $\varphi_i(u)$ and $\varphi_i(v)$, whereas Young's notion of monotonicity claims $\varphi_i(v) = \varphi_i(u)$. This claim of Young's definition, together with efficiency and equal treatment, is already sufficient to characterize the Shapley-Shubik-Index (see Definition 5.1) on proper monotonic simple games.[4]

In the following, if a solution is called strongly monotonic, we always refer to Definition 4.1. A strongly monotonic solution which lives only on winning coalitions has a useful property concerning players which are not null players.

Lemma 4.2. *Let $v \in \mathcal{V}$ and let φ be a strongly monotonic solution which lives only on winning coalitions. If $i \in N$ is not a null player with respect to v, then $\varphi_i(v) > 0$.*

Proof. Let $i \in N$ be not a null player with respect to v. Then there is a minimal winning coalition S such that $i \in S$. As φ lives only on winning coalitions, $\varphi_i(v_T) = 0$ for each $T \subseteq N \setminus S$. Strong monotonicity applied to v and the zero game v_T, together with the fact that i is pivotal in S with respect to v, implies that $\varphi_i(v) > 0$. □

Recall that a solution which satisfies the null player property lives only on winning coalitions. Hence, the next corollary follows immediately.

Corollary 4.3. *Let $v \in \mathcal{V}$ and let φ be a strongly monotonic solution which satisfies the null player property. Then $\varphi_i(v) > 0$ if and only if $i \in N$ is not a null player with respect to v.*

[4] Although Young's original statement applies on general TU games, the proof uses the decomposition of an arbitrary game into primitive games following Shapley (1953). This idea also applies on the class of proper monotonic simple games. A proof which avoids this decomposition can be found in Peleg and Sudhölter (2007).

A strongly monotonic solution φ which lives only on winning coalitions assigns for each $v \in \mathcal{V}$ a strictly positive value to each player which is not a null player. Consequently, if v is not the zero game, then there must be a coalition S, such that $\varphi_i(v_S) > 0$ for each $i \in S$. This means that a partition which does not contain any winning coalition can never be core stable, for it is blocked by S. Together with Lemma 2.8 and Definition 2.9 it becomes clear that in this case knowing all core stable coalitions means knowing all core stable partitions.

4.2 Strongly Monotonic Solutions on Apex Games

In the following, let $a_{\mathcal{I}J}$ be a generalized apex game and let φ be a strongly monotonic solution. For convenience, set $\varphi_i(S) = \varphi_i(a_{\mathcal{I}J,S})$. The first lemma of this subsection gives some consequences of strong monotonicity on generalized apex games. It is the basis for most of the later results.

Lemma 4.4. *Let N be a set of players, let $a_{\mathcal{I}J}$ be a generalized apex game on N, and let φ be a strongly monotonic solution. Let $J_1, J_2 \subsetneq J$ with $|J_1| > |J_2|$ and let $I, I_1, I_2 \subseteq N \setminus J$ with $I_2 \subsetneq I_1$ be such that $I \cup \{j\}, I_1 \cup \{j\}, I_2 \cup \{j\}$ are winning for each $j \in J$. Then*

1. *$\varphi_j(I_1 \cup \{j\}) \geq \varphi_j(I_2 \cup \{j\})$ for all $j \in J$, where the equality holds if and only if each $i \in I_1 \setminus I_2$ is a null player in I_1 with respect to $a_{\mathcal{I}J}$.*

2. *If φ satisfies in addition the equal treatment property then $\varphi_i(I \cup J_1) > \varphi_i(I \cup J_2)$ for each $i \in I$ which is not a null player in $I \cup J_1$ with respect to $a_{\mathcal{I}J}$.*

3. *If φ satisfies in addition the equal treatment property then $\varphi_j(I \cup J_1) < \varphi_j(I \cup J_2)$ for all $j \in J_2$.*

The proof of Lemma 4.4 can be found in the appendix. Note that in case of an efficient solution which satisfies equal treatment part 3 can be formulated even sharper. Not only the power of one minor player is decreasing with increasing number of minor players in the winning coalition but also the aggregated power of all minor players is decreasing.

Corollary 4.5. *Let N be a set of players, let $a_{\mathcal{I}J}$ be a generalized apex game on N and let φ be a strongly monotonic solution which satisfies coalitional efficiency* *and equal treatment. Let $J_1, J_2 \subsetneq J$ with $|J_1| > |J_2|$ and $I \subseteq N \setminus J$ be such that $I \cup \{j\}$ is winning for all $j \in J$. Then $\sum_{j \in J_1} \varphi_j (I \cup J_1) < \sum_{j \in J_2} \varphi_j (I \cup J_2)$.*

Proof. Part 2 of Lemma 4.4 says that $\varphi_i (I \cup J_1) > \varphi_i (I \cup J_2)$ for each $i \in I$ which is not a null player in $I \cup J_1$ with respect to $a_{\mathcal{I}J}$. Hence,

$$\sum_{j \in J_1} \varphi_j (I \cup J_1) = 1 - \sum_{i \in I} \varphi_i (I \cup J_1) < 1 - \sum_{i \in I} \varphi_i (I \cup J_2) = \sum_{j \in J_2} \varphi_j (I \cup J_2) .$$

The equalities follow from coalitional efficiency and the strict inequality follows from the fact that not every $i \in I$ is a null player in $I \cup J_1$ with respect to $a_{\mathcal{I}J}$. □

Knowing when φ_j is increasing or decreasing for $j \in J$ from parts 1 and 3 of Lemma 4.4, it is not difficult to find the coalition, which maximizes φ_j for the minor players.

Corollary 4.6. *Let $a_{\mathcal{I}J}$ be a generalized apex game in minimal representation with $\mathcal{I} = \{I_k\}_{k=1}^n$ and let φ be a strongly monotonic solution which satisfies equal treatment. Let $j \in J$ and $S^* = \bigcup_{k=1}^n I_k \cup \{j\}$. Then*

$$\varphi_j (S^*) \geq \varphi_j (S)$$

for all $S \subsetneq N$ with $J \nsubseteq S$, where the equality holds if and only if $S^* \subseteq S$ and each $i \in S \setminus S^*$ is null player in S^* with respect to $a_{\mathcal{I}J}$.

Proof. Let S', $j \in S'$ be such that $\varphi_j (S') \geq \varphi_j (S)$ for all $S \subsetneq N$ with $J \nsubseteq S$ (such S' exists as $\mathfrak{P}_j$ is finite. If $|S' \cap J| \geq 2$ then $\varphi_j (S') < \varphi_j \left((S' \cap J) \cup \{j\}'\right)$ by part 3 of Lemma 4.4. Hence, $S' \cap J = \{j\}$. If there is $I \subseteq \bigcup_{k=1}^n I_k \setminus S'$ with $i \in I$ which is not a null player in $S' \cup I$ with respect to $a_{\mathcal{I}J}$ then $\varphi_j (S' \cup I) > \varphi_j (S')$ by part 1 of Lemma 4.4. Hence, $\bigcup_{k=1}^n I_k \cup \{j\} \subseteq S'$. Particularly, $S^* \subseteq S'$ and all $i \in S' \setminus S^*$ must be null players with respect to $a_{\mathcal{I}J}$. Thus, by Corollary 2.5, $\varphi_j (S^*) = \varphi_j (S') \geq \varphi_j (S)$ for all for all $S \subsetneq N$ with $J \nsubseteq S$. □

Remark 4.7. The condition $J \nsubseteq S$ (or $J_1 \neq J$ in Lemma 4.4) is necessary as otherwise the main argument of the proofs gets lost. As long as S does not contain J, each winning subcoalition of S must contain an apex set. This highlights the role of strong monotonicity: If the number of minimal apex sets strictly increases, the number of coalitions in which $j \in J$ is pivotal strictly increases as well. On the other hand, if $J \subseteq S$ and $I_k \nsubseteq S$ for some minimal $I_k \in \mathcal{I}$ then j is pivotal in $J \cup I_k$ with respect to the restriction of $a_{\mathcal{I}J}$ to S (note that the only minimal winning coalition in $S \cap (J \cup I_k)$ is J). Consider now the restriction of $a_{\mathcal{I}J}$ to $S \cup I_k$. For each $j' \in J \setminus \{j\}$, the coalition $I_k \cup \{j'\} \subseteq (J \cup I_k) \setminus \{j\}$ is winning. Particularly, j would not be pivotal in $I_k \cup J$ with respect to this subgame. Hence, strong monotonicity would not apply.[5]

4.3 Core Stability in the Induced Hedonic Game

It has been shown in Lemma 3.7 that all players in a generalized apex game $a_{\mathcal{I}J}$ in minimal representation which belong neither to an apex set nor to the set of minor players are null players with respect to $a_{\mathcal{I}J}$. It has further been shown in Lemma 2.10 that null players can be excluded from all core stable partitions in a hedonic game derived from a simple game which is not the zero game. Therefore, in the remainder, it will be assumed that $a_{\mathcal{I}J}$ is a generalized apex game on the set $N = \bigcup_{I \in \mathcal{I}} I \cup J$.

The first Lemma is stated for completeness. It eases the later proofs as we need not to take care of trivialities.

Lemma 4.8. *Let $a_{\mathcal{I}J}$ be a generalized apex game with $|J| = 1$ and let φ be a strongly monotonic solution which lives only on winning coalitions. Then $\mathcal{C}(N, a_{\mathcal{I}J}, \varphi) \neq \emptyset$. In particular, $S \in \mathcal{C}(N, a_{\mathcal{I}J}, \varphi)$ if and only if S is winning.*

Proof. If $|J| = 1$ then $j \in J$ is a dictator and contained in all winning coalitions, and all other players are null players. As φ lives only on winning coalitions $\varphi_i(S) = 0$ for each coalition S which is not winning and each $i \in S$. As j is pivotal in all winning coalitions and φ is strongly monotonic, $\varphi_j(S) > 0$ for each winning coalition by Lemma 4.2. Hence, a coalition which is not winning can never block a winning coalition and is always blocked by J. Let S be a winning coalition. As all players in $N \setminus J$ are null players with respect $a_{\mathcal{I}J}$, they are null players in S with respect to $a_{\mathcal{I}J}$. Hence, by Corollary 2.5, $\varphi_j(S) = \varphi_j(J)$. Consequently, winning coalitions cannot block each other, so they all are core stable. □

[5] In Lemma 5.10 there is an analysis of the Banzhaf value on winning coalitions which contain J. In particular, it is shown that part 3 of Lemma 4.4 does not hold on coalitions which contain J.

Also for generalized apex games with $|J| \geq 2$, a first result can be proven. Lemma 4.4 made statements about the payoffs of different players in different coalitions of a generalized apex game under a strongly monotonic solution. As these payoffs are used to develop preferences in the induced hedonic game, the following theorem can be derived. The proof can be found in the appendix.

Theorem 4.9. *Let $a_{\mathcal{I}J}$ be a generalized apex game and let φ be a strongly monotonic solution which satisfies equal treatment and which lives only on winning coalitions. Then $|S \cap J| \geq \frac{1}{2}|J|$ for each core stable coalition $S \in \mathcal{C}(N, a_{\mathcal{I}J}, \varphi)$.*

Theorem 4.9 states that a core stable coalition must contain at least half of the minor players. Consequently, in a weighted voting game which has the structure of a generalized apex game (as the two tier board system of Example 3.4) a stable majority can be reached only with a majority of the minor players in J (the supervisory board in the example). This is surprising from a theoretical point of view: Players in J can replace each other (from the view of players from the apex sets); this makes them weak in the simple game without coalition formation. However, in the hedonic version, each core stable coalition must contain at least half of them.
The previous result can be strengthened even further if one assumes coalitional efficiency. Again, the proof can be found in the appendix.

Theorem 4.10. *Let $a_{\mathcal{I}J}$ be a generalized apex game with $\mathcal{I} = \{I_k\}_{k=1}^{n}$ and $|J| \geq 2$. Let φ be a strongly monotonic solution which satisfies equal treatment and coalitional efficiency. Let $I = \bigcup_{k=1}^{n} I_k$ and let $j \in J$.*

1. *If $\varphi_j(I \cup \{j\}) \leq \frac{1}{|J|}$, then $\mathcal{C}(N, a_{\mathcal{I}J}, \varphi) \neq \emptyset$. In particular, $S \in \mathcal{C}(N, a_{\mathcal{I}J}, \varphi)$ if and only if $J \subseteq S$ and all $i \in S \setminus J$ are null players with respect to the restriction of $a_{\mathcal{I}J}$ to S.*

2. *If $\frac{1}{|J|} < \varphi_j(I \cup \{j\}) \leq \frac{1}{2}$, then $\mathcal{C}(N, a_{\mathcal{I}J}, \varphi) = \emptyset$.*

3. *If $q = \varphi_j(I \cup \{j\}) > \frac{1}{2}$, then for each $S \in \mathcal{C}(N, a_{\mathcal{I}J}, \varphi)$*

$$\frac{1}{2} \leq \frac{|S \cap J|}{|J|} \leq q.$$

5 Sharper Results for Specific Solutions

5.1 The Shapley-Shubik-Index

Among many different solutions on simple games the probably best known is the Shapley-Shubik-Index (or Shapley value). In the first part of this section it will be the solution of interest.

Definition 5.1. Let $v \in \mathcal{V}$. The Shapley value of a player $i \in N$ in v is defined as

$$Sh_i(v) = \sum_{S \subseteq N,\, i \in S} \frac{(|N| - |S|)!\,(|S| - 1)!}{|N|!} \left(v(S) - v(S \setminus \{i\})\right).$$

As we are interested in the hedonic game derived from a simple game as in (1), the computation of the Shapley value on restricted games is important. But this computation is quite intuitive as it can be seen from the next proposition (the proof is left to the reader).

Proposition 5.2. *Let N be a set of players and $v \in \mathcal{V}$. Then*

$$Sh_i(v_S) = \sum_{T \subseteq S,\, i \in T} \frac{(|S| - |T|)!\,(|T| - 1)!}{|S|!} \left(v(T) - v(T \setminus \{i\})\right)$$

for all $S \subseteq N$ and $i \in S$.

For convenience the notation $Sh_i(v_S)$ will be abbreviated by $Sh_i(S)$, if it is clear which game v is meant.
It is well known that the Shapley value satisfies many of the previously discussed properties: It is nonnegative, coalitionally efficient, strongly monotonic, and satisfies equal treatment. Particularly, the condition of Lemma 4.8 are satisfied for a generalized apex game $a_{\mathcal{I}J}$ with $|J| = 1$. Hence, in this case each winning coalition is core stable in the induced hedonic game.
In a simple apex game a_{iJ} the players in J are called minor players, since their power in the game is quite small. Particularly, the Shapley value of any $j \in J$ is equal to $\frac{1}{2}$ in the minimal winning coalition $\{i, j\}$ and coalitions with higher cardinality will lead to even lower Shapley values for $j \in J$, as the Shapley value satisfies the conditions of Lemma 4.4 and Corollary 4.5.

In a generalized apex game $a_{\mathcal{I}J}$ with $\mathcal{I} = \{I_k\}_{k=1}^n$ the Shapley value φ_j is maximized in $S^* = \bigcup_{k=1}^n I_k \cup \{j\}$ for each $j \in J$. But there is an upper bound for $Sh_j(S)$ which does not depend on the size of S^*. The proof of the next lemma can be found in the appendix.

Lemma 5.3. *Let $a_{\mathcal{I}J}$ be a generalized apex game on N and let $|J| \geq 2$. Then $Sh_j(S) \leq \frac{1}{2}$ for all $j \in J$ and all $S \in \mathfrak{P}_j(N)$.*

A *strong* monotonic simple game v is a proper monotonic simple game, such that $v(S) + v(N \setminus S) = 1$.[6] Examples are voting games (with tie breaking) in which each coalition containing a simple majority of votes is winning. Also simple apex games are strong. The next corollary shows that if a generalized apex game is strong, the upper bound in Lemma 5.3 is even sharp. For easiness of reading the proof is stated in the appendix.

Corollary 5.4. *Let $a_{\mathcal{I}J}$ be a generalized apex game on N which is strong. Let $\mathcal{I} = \{I_k\}_{k=1}^m$ and $|J| \geq 2$. Then $Sh_j(I \cup \{j\}) = \frac{1}{2}$ for all $j \in J$ where $I = \bigcup_{k=1}^m I_k$.*

Examples for strong generalized apex games can easily be found. The first example is of course the simple apex game. Considering again the executive board in Example 3.4, one finds a strong apex game if this board can make decisions with simple majority (and tie breaking, if the number of members is even). If, on the other hand, the executive board has to make a decision unanimously (and contains more than one member), then the resulting game is not strong (see also Corollary 5.7).
Knowing the bounds as stated in Lemma 5.3, we can now apply Theorems 4.9 and 4.10 of the previous section to the Shapley value.

Theorem 5.5. *Let $a_{\mathcal{I}J}$ be an apex game on N with $\mathcal{I} = \{I_k\}_{k=1}^n$ and let $I = \bigcup_{k=1}^n I_k$.*

1. *If $|J| = 2$ then $\mathcal{C}(N, a_{\mathcal{I}J}, Sh) \neq \emptyset$. In particular, if $J \subseteq S$ and all $i \in S \setminus J$ are null players with respect to the restriction of $a_{\mathcal{I}J}$ to S, then $S \in \mathcal{C}(N, v, Sh)$.*

[6] Although there are different definitions of a strong monotonic simple game, we follow the definition in Peleg and Sudhölter (2007).

2. *If* $|J| \geq 3$ *then* $\mathcal{C}(N, a_{\mathcal{I}J}, Sh) \neq \emptyset$ *if and only if* $Sh_j(I \cup \{j\}) \leq \frac{1}{|J|}$ *for all* $j \in J$. *In particular* $S \in \mathcal{C}(N, a_{\mathcal{I}J}, Sh)$ *if and only if* $J \subseteq S$ *and all* $i \in S \setminus J$ *are null players with respect to the restriction of* $a_{\mathcal{I}J}$ *to* S.

Although the condition $Sh_j(I \cup \{j\}) \leq \frac{1}{|J|}$ for all $j \in J$ in the second part of Theorem 5.5 is easy to verify, it highly depends on the structure of $\mathcal{I}$. The next corollaries capture two special cases of a monotonic simple game.

Corollary 5.6. *Let the generalized apex game* $a_{\mathcal{I}J}$ *on* N *with* $\mathcal{I} = \{I_k\}_{k=1}^n$ *be strong. Then* $\mathcal{C}(N, a_{\mathcal{I}J}, Sh) \neq \emptyset$ *if and only if* $|J| \leq 2$.

Proof. In Corollary 5.4 it has been shown that in this case $Sh_j(I \cup \{j\}) = \frac{1}{2}$ for $I = \bigcup_{k=1}^n I_k$ and all $j \in J$. If $|J| = 1$ then Lemma 4.8 applies and $\mathcal{C}(N, a_{\mathcal{I}J}, Sh) \neq \emptyset$. If $|J| = 2$, then $\mathcal{C}(N, a_{\mathcal{I}J}, Sh) \neq \emptyset$ by part 1 of Theorem 5.5. If $|J| \geq 3$, then $\frac{1}{2} = Sh_j(I \cup \{j\}) > \frac{1}{3} \geq \frac{1}{|J|}$. Hence, by part 2 of Theorem 5.5, $\mathcal{C}(N, a_{\mathcal{I}J}, Sh) = \emptyset$. □

Corollary 5.7. *Let* $a_{\mathcal{I}J}$ *be a generalized apex game on* N *such that* $\mathcal{I}$ *contains only one apex set, namely* I. *Then* $\mathcal{C}(N, a_{\mathcal{I}J}, Sh) \neq \emptyset$ *if and only if* $|J| \leq |I| + 1$.

Proof. $Sh_j(I \cup \{j\}) = \frac{1}{|I|+1}$ for all $j \in J$ as $I \cup \{j\}$ is minimal winning. If $|J| \leq 2$ then $\mathcal{C}(N, a_{\mathcal{I}J}, Sh) \neq \emptyset$ by Lemma 4.8 and part 1 of Theorem 5.5. In this case $|J| \leq 2 \leq |I| + 1$ as I is nonempty. If $|J| \geq 3$, by part 2 of Theorem 5.5, $\mathcal{C}(N, a_{\mathcal{I}J}, Sh) \neq \emptyset$ if and only if $\frac{1}{|I|+1} \leq \frac{1}{|J|}$. This is the case if and only if $|J| \geq |I| + 1$. □

5.2 The Banzhaf Value

Another example of a strongly monotonic solution on proper monotonic simple games is the Banzhaf value. It has been introduced in Banzhaf (1965) and is basically the same as the index presented in Coleman (1971). Although the original index counted for each player the number of coalitions in which he is pivotal (see for instance Dubey and Shapley, 1979), here the version given in Owen (1978) is used.

Definition 5.8. Let $v \in \mathcal{V}$. The Banzhaf value of a player $i \in N$ in v is defined as

$$\eta_i(v) = \frac{1}{2^{|N|-1}} \sum_{S \subseteq N \setminus \{i\}} (v(S \cup \{i\}) - v(S)).$$

It can easily be verified that the Banzhaf value is nonnegative, strongly monotonic and satisfies the null player property as well as equal treatment on $\mathcal{V}$. Particularly, Theorem 4.9 applies, hence for each generalized apex game $a_{\mathcal{I}J}$ and each $S \in \mathcal{C}(N, a_{\mathcal{I}J}, \eta)$ it holds that $|S \cap J| \geq \frac{1}{2}|J|$. As the null player property implies that η lives only on winning coalitions, Lemma 4.8 applies. Hence, if $|J| = 1$ then each winning coalition is core stable in the induced hedonic game.
A weakness of the Banzhaf value is that it is not efficient. Hence, Theorem 4.10 cannot be applied. The normalized version of the Banzhaf value (see for instance Dubey and Shapley, 1979) is under consideration later in this section. First, we want to find a statement which is stronger than Theorem 4.10. To do so it is important to analyze the behaviour of the Banzhaf value on restricted games. For convenience, $\eta_i(v_S)$ will again be abbreviated by $\eta_i(S)$ for $i \in S$, if the game v is fixed. The proof of the following proposition is a simple computation and omitted.

Proposition 5.9. *Let $v \in \mathcal{V}$ and let $S \subseteq N$ be a winning coalition with respect to v. Then*

$$\eta_i(S) = \frac{1}{2^{|S|-1}} \sum_{T \subseteq S \setminus \{i\}} (v(T \cup \{i\}) - v(T))$$

for all $i \in S$.

An easy consequence is that $\eta_i(S) = \frac{1}{2^{|S|-1}}$ for each minimal winning coalition $S \subseteq N$ and each $i \in S$.
There are two types of winning sets which are not minimal, namely those containing J and those not containing J. For both of them Lemma 4.4 can be strengthen as follows. The proof of the lemma can be found in the appendix.

Lemma 5.10. *Let $a_{\mathcal{I}J}$ be a generalized apex game on N and let $S \subseteq N$ be a winning coalition.*

1. *If $J \nsubseteq S$ then $\eta_j(S) = \frac{1}{2^{|S \cap J|-1}} \cdot \eta_j((S \setminus J) \cup \{j\})$ for all $j \in J \cap S$.*

2. *If $J \subseteq S$ then $\eta_j(S) = \frac{1}{2^{|J|-1}}$ for all $j \in J$.*

Note that part 2 of Lemma 5.10 does not contradict the strong monotonicity of η (see also Remark 4.7).
From part 1 it follows that $\eta_j(S)$ decreases for $j \in J$ with the number of minor players in S as long as $J \nsubseteq S$. As η is strongly monotonic and satisfies equal treatment, Corollary 4.6 applies and $\eta_j(S)$ takes its maximum (over all coalitions S which do not contain J) in $S^* = \bigcup_{k=1}^{n} I_k \cup \{j\}$.
As in the case of the Shapley value there is an upper bound for $\eta_j(S)$ for all $j \in J$ in a generalized apex game $a_{\mathcal{I}J}$. The proof follows similar arguments as the proof of Lemma 5.3 and is omitted.

Lemma 5.11. *Let $a_{\mathcal{I}J}$ be generalized apex game on N with $|J| \geq 2$. Then $\eta_j(S) \leq \frac{1}{2}$ for all $j \in J$ and $S \in \mathfrak{P}_j(N)$. Particularly, the equality holds if and only if $a_{\mathcal{I}J}$ is strong and $S = \bigcup_{I \in \mathcal{I}} I \cup \{j\}$.*

A first corollary for generalized apex games with two minor players follows immediately.

Corollary 5.12. *Let $a_{\mathcal{I}J}$ be a generalized apex game with $\mathcal{I} = \{I_k\}_{k=1}^{n}$ and $|J| = 2$. Let $I = \bigcup_{I_k \in \mathcal{I}} I_k$. Then $\mathcal{C}(N, a_{\mathcal{I}J}, \eta) \neq \emptyset$, in particular $S \in \mathcal{C}(N, a_{\mathcal{I}J}, \eta)$ for all $S \subseteq N$ with $J \subseteq S$.*

Proof. By Lemma 5.11, $\eta_j(S) \leq \frac{1}{2}$ for all winning coalitions $S \subseteq N$ and all $j \in S \cap J$. By Lemma 5.10, $\eta_j(S) = \frac{1}{2}$ for all $S \subseteq N$ with $J \subseteq N$. As by definition of a generalized apex game each winning coalition S must contain some $j \in J$, it follows that winning coalitions $S \supseteq J$ cannot be blocked. $\square$

We now come to the main theorem concerning the Banzhaf value for generalized apex games with at least three minor players. The proof can be found in the appendix.

Theorem 5.13. *Let $a_{\mathcal{I}J}$ be a generalized apex game with $\mathcal{I} = \{I_k\}_{k=1}^{n}$ and $|J| \geq 3$. Let $I = \bigcup_{I_k \in \mathcal{I}} I_k$ and let $j \in J$.*

1. *If $\eta_j(I \cup \{j\}) \leq \frac{1}{2^{|J|-1}}$ then*
   - *(a) $S \in \mathcal{C}(N, v, \eta)$ for all $S \subseteq N$ with $J \subseteq S$.*
   - *(b) There is $S \in \mathcal{C}(N, v, \eta)$ with $J \nsubseteq S$, if and only if (i) $|J| = 3$, (ii) $|S \cap J| = 2$, and (iii) $a_{\mathcal{I}J}$ is strong. In this case $I \subseteq S$.*

2. *If* $\frac{1}{2^{|J|-1}} < \eta_j(I \cup \{j\}) \leq \frac{1}{2^{\frac{1}{2}|J|+1}}$, *then* $\mathcal{C}(N, v, \eta) = \emptyset$.

3. *If* $\frac{1}{2^{\frac{1}{2}|J|+1}} < \eta_j(I \cup \{j\})$, *then* $\frac{1}{2^{\frac{1}{2}|J|}} \leq \frac{2^{|S \cap J|}}{2^{|J|}} \leq \eta_j(I \cup \{j\})$ *for each core stable coalition* S.

The last case of Theorem 5.13 does not appear in case of the Shapley value. There, the important bound for the two cases has been $Sh_j(J) = \frac{1}{|J|}$. As the Banzhaf value is not coalitionally efficient there is a new case which has to be considered in Theorem 5.13. The following example shows, that this case could actually appear.

Example 5.14. Let $N = \{1, 2, 3, 4\}$ and let a_{1J} be the simple apex game on N with $J = \{2, 3, 4\}$ and apex player 1. Then the condition in part 3 of Theorem 5.13 is satisfied, which implies that the only candidate for a core stable partition is $S \subseteq N$ with $1 \in S$ and $|S \cap J| = 2$. We prove that such S is indeed core stable. We have $\eta_j(S) = \frac{1}{4}$ for all $j \in S \cap J$ and $\eta_1(S) = \frac{1}{2}$. Hence, S is not blocked by J of N, as $\eta_j(N) = \eta_j(J) = \frac{1}{2^{|J|-1}} = \frac{1}{4}$. Further, for each coalition T containing only one minor player, $\eta_1(T) = \frac{1}{2} = \eta_1(S)$. Hence, T cannot block S either. By equal treatment of η and symmetry between players in J, S cannot be blocked by a coalition containing two minor players. Hence, S is core stable.

A useful property of the Banzhaf value is that the value for any player $j \in J$ of any coalition S containing j can easily be related to $\eta_j((S \setminus J) \cup \{j\})$. This simplifies many calculations. However, since η not efficient, there is an additional case in theorem 5.13 compared to the Shapley value. This makes a characterization of core stable partitions (if any) much more difficult.
There is a natural way to make the Banzhaf value efficient (see for instance Owen, 1978; Dubey and Shapley, 1979).

Definition 5.15. Let $v \in \mathcal{V}$. The normalized Banzhaf value is defined as

$$\beta_i(v) = \frac{\eta_i(v)}{\sum_{j \in N} \eta_j(v)}.$$

The normalization ensures that the value is coalitionally efficient. Hence, the normalized Banzhaf value is nonnegative, coalitionally efficient and satisfies equal treatment.

For a player i and a proper monotonic simple game $v \in \mathcal{V}$, let $\mu_i(v)$ be the number of coalitions in which i is pivotal with respect to v. Then the normalized Banzhaf value is exactly

$$\beta_i(v) = \frac{\mu_i(v)}{\sum_{j \in N} \mu_j(v)}.$$

In the following we write $\beta_i(S)$ (respectively $\mu_i(S)$) for $\beta_i(v_S)$ (respectively $\mu_i(v_S)$) if it is clear which game v is meant.
For a dictator game with dictator j it is clear that $\beta_j(S) = 1$ for the dictator j in each winning coalition S. Hence, for the generalized apex game with $|J| = 1$ each winning coalition is core stable in the induced hedonic game as before.
In Sagonti (1991) the question is asked whether or not the normalized Banzhaf value is strongly monotonic. The following example shows that in general it is not - not even on simple games. This is the price paid for the normalization.

Example 5.16. Consider the player set $N = \{1, 2, 3, 4, 5\}$ and the apex game $a_{\mathcal{I}J}$ on N with $J = \{1, 2\}$ and apex sets

$$\mathcal{I} = \{\{3, 4\}, \{3, 5\}, \{4, 5\}\}.$$

Let $S = \{1, 3, 4\}$ and $T = \{1, 3, 4, 5\}$. If β were strongly monotonic, then $\beta_1(T) > \beta_1(S)$ by Corollary 4.6. But $\mu_1(S) = \mu_3(S) = \mu_4(S) = 1$, and on the other hand $\mu_1(T) = 3$ and $\mu_3(T) = \mu_4(T) = \mu_5(T) = 2$. Hence,

$$\beta_1(S) = \frac{1}{3} = \frac{3}{9} = \beta_1(T).$$

This implies that β is not strongly monotonic.

It can be shown that if $|N| \geq 6$, there is also a collection of apex sets such that even $\beta_1(S) > \beta_1(T)$ for $S \subsetneq T$.
Although β is not strongly monotonic, at least the following monotonicity properties on generalized apex games which are analogous to Lemma 4.4 can be verified.

Lemma 5.17. *Let N be a set of players and let $a_{\mathcal{I}J}$ be a generalized apex game with $\mathcal{I} = \{I_k\}_{k=1}^n$ on N. Let $J_1, J_2 \subsetneq J$ with $|J_1| > |J_2|$ and $I \subseteq \bigcup_{k=1}^m I_k$ be such that $I \cup \{j\}$ is winning for all $j \in J$. Then:*

1. $\beta_j(I \cup J_1) < \beta_j(I \cup J_2)$ *for all* $j \in J_2$.

2. $\beta_i(I \cup J_1) > \beta_i(I \cup J_2)$ *for all* $i \in I$.

As before, it is useful to look for an upper bound of $\beta_j(S)$ with $S \subseteq N$ in a generalized apex game $a_{\mathcal{I}J}$ with $|J| \geq 2$. For the proof, which can be found in the appendix, a result from Dubey and Shapley (1979) is used.

Lemma 5.18. *Let* $a_{\mathcal{I}J}$ *be a generalized apex game on a player set* N *with* $|J| \geq 2$. *Then* $\beta_j(S) \leq \frac{1}{2}$ *for all* $S \subseteq N$ *and all* $j \in S \cap J$.

This is sufficient to make statements about the normalized Banzhaf value which are analogous to Theorem 5.13. The only difference is that it is unclear where the value is maximal for elements from J.

Theorem 5.19. *Let* $a_{\mathcal{I}J}$ *be a generalized apex game with* $\mathcal{I} = \{I_k\}_{k=1}^n$. *Let* $j \in J$ *and let* $I^* \subseteq \bigcup_{k=1}^n I_k$ *maximize* $\beta_j(I \cup \{j\})$ *over all* $I \subseteq \bigcup_{k=1}^n I_k$.

1. *If* $|J| = 2$ *then* $\mathcal{C}(N, v, \beta) \neq \emptyset$. *In particular, if* $J \subseteq S$ *and all* $i \in S \setminus J$ *are null players with respect to the restriction of* $a_{\mathcal{I}J}$ *to* S, *then* $S \in \mathcal{C}(N, v, \beta)$.

2. *If* $|J| \geq 3$ *then* $\mathcal{C}(N, v, \beta) \neq \emptyset$ *if and only if* $\beta_j(I^* \cup \{j\}) \leq \frac{1}{|J|}$. *In this case* $S \in \mathcal{C}(N, v, \beta)$ *if and only if* $J \subseteq S$ *and all* $i \in S \setminus J$ *are null players with respect to the restriction of* $a_{\mathcal{I}J}$ *to* S.

6 Further Generalizations

Generalized apex sets can be interpreted as a composition of monotonic simple games in the following sense. Let N_1, N_2 be two player sets with an empty intersection. Let v_1 be a proper monotonic simple game on N_1 and let $\mathcal{I}$ be the collection of all minimal winning coalitions. Let v_2 be the monotonic simple game on N_2 with $v_2(S) = 1$ for all $S \subseteq N_2$ with $S \neq \emptyset$. The game v defined on $N_1 \cup N_2$ by

$$v(S) = \begin{cases} 1, \text{ if } (v_1(S \cap N_1) = 1 \text{ and } v_2(S \cap N_2) = 1) \text{ or } N_2 \subseteq S, \\ 0, \text{ else} \end{cases}$$

is a generalized apex game with apex sets $I \in \mathcal{I}$ and minor players $j \in N_2$. The question is for which other games $\tilde{v}_2$ instead of v_2 the results can be generalized. If all players are symmetric in $\tilde{v}_2$, a strongly monotonic solution applied on the composed game $\tilde{v}$ will behave similarly compared to a generalized apex game. But for general $\tilde{v}_2$ it is quite difficult to find any results.

A Proofs

Proof of Lemma 4.4. 1. Let $j \in J$, let v be the apex game restricted to $I_1 \cup \{j\}$ and let u be the apex game restricted to $I_2 \cup \{j\}$. If $T \subseteq N$ is such that $j \in T$ is pivotal in T with respect to u then T is a winning coalition in u. In this case T is also winning in v due to properness. Particularly, $T \cap (I_1 \cup \{j\}) \cap J = \{j\}$ and hence, by definition of a generalized apex game, j is pivotal in T with respect to v. Consider two cases:

   (i) If $I_1 \setminus I_2$ contains only null players with respect to v then u and v are identical (Lemma 2.4) and hence, $\varphi_i(u) = \varphi_i(v)$ for all $i \in N$.

   (ii) If there is $i \in I_1 \setminus I_2$ which is not a null player, then there is a minimal winning coalition T such that i and j are pivotal in T with respect to v. On the other hand, $u(T) = 0$ and hence, j is not pivotal with respect to u. Hence, $varphi_j(I_1 \cup \{j\}) = \varphi_j(v) > \varphi_j(u) = varphi_j(I_2 \cup \{j\})$ due to strong monotonicity of φ.

2. Since all $j \in J$ are symmetric with respect to $a_{\mathcal{I}J}$ and since φ satisfies the equal treatment property, it can be assumed without loss of generality that $J_2 \subsetneq J_1$. Let v be the restricted game $a_{\mathcal{I}J}$ on $I \cup J_1$ and u be the restricted game $a_{\mathcal{I}J}$ on $I \cup J_2$. Let $i \in I$ be not a null player with respect to v. First, we show that in this case i is not a null player with respect to u either. As i is not a null player with respect to v, there is a minimal winning set S containing i such that $S \cap J_1 = \{j\}$. Particularly, $S' = S \setminus \{j\} \cup \{j'\}$ is minimal winning in v for each $j' \in J_2 \subsetneq J_1$. This means, that i is pivotal in S' with respect to u. Hence, i is not a null player with respect to u.

   Let now i be pivotal in $S \subseteq N$ with respect to u. Then i is also pivotal in S with respect to v by definition of a generalized apex game. Hence, $v(S \cup \{i\}) - v(S) \geq u(S \cup \{i\}) - u(S)$ for all $S \subseteq N \setminus \{i\}$. Particularly, for $S = I \cup J_1 \setminus J_2$ the inequality is strict. Hence, by monotonicity of φ we have $\varphi_i (I \cup J_1) = \varphi_i (v) > \varphi_i (u) = \varphi_i (I \cup J_2)$.

3. If $j \in J_2 \setminus J_1$ there is nothing to show. Since all $j \in J$ are symmetric with respect to $a_{\mathcal{I}J}$ and since φ satisfies the equal treatment property, it can be assumed without loss of generality that $J_2 \subsetneq J_1$. Let v be the apex game restricted on $I \cup J_1$ and let u be the restricted apex game on $I \cup J_2$. Let $j \in J_2 \subseteq J_1$ and let $S \subseteq N$ such that j is pivotal in S with respect to v. Then S contains at least one apex set and S contains j, hence S is winning in u. Particularly, S cannot contain any $j' \in J_1 \setminus \{j\} \supsetneq J_2 \setminus \{j\}$ as otherwise j would not be pivotal in S with respect to v. Hence, j is also pivotal in S with respect to u. Consequently, $u (S \cup \{i\}) - u (S) \geq v (S \cup \{i\}) - v (S)$ for all $S \subseteq N \setminus \{i\}$.

   On the other hand, consider $T = I \cup \{j\} \cup (J_1 \setminus J_2)$. Then j is pivotal in T with respect to u but not with respect to v. Strong monotonicity implies $\varphi_j (I \cup J_2) = \varphi_j (u) > \varphi_j (v) = \varphi_j (I \cup J_1)$.

□

Proof of Theorem 4.9. If $|J| \leq 2$, there is not much to show. As $a_{\mathcal{I}J}$ is not the zero game, each core stable coalition must be winning and each winning coalition must contain at least one member of J.
Hence, suppose that $|J| \geq 3$. Let $S \in \mathcal{C} (N, a_{\mathcal{I}J}, \varphi)$ and assume that $|S \cap J| < \frac{1}{2} |J|$. As S must be winning, S contains an apex set $I \subseteq S \setminus J$. By properness of the game the coalitions $\pi (j)$ are not winning for any $j \in J \setminus S$. As φ lives only on winning coalitions, $\varphi_j (\pi (j)) = 0$ for all $\pi \in \Pi_S$ and all $j \in J \setminus S$. For each winning coalition T' containing $j \in J$, j is not a null player with respect to restriction of $a_{\mathcal{I}J}$ to T' by Lemma 3.6. Thus, by Lemma 4.2 $\varphi_j (T') > 0$ for all $j \in T' \cap J$. The coalition $T = (S \setminus J) \cup (J \setminus S)$ contains the apex set $I \subseteq S \setminus J$ and minor players $j \in J \setminus S \neq \emptyset$. Hence, T is a winning coalition in $a_{\mathcal{I}J}$ and

$$\varphi_j (T) > \varphi_j (\pi (j)) = 0$$

for all $j \in J \setminus S$.
Since φ satisfies the equal treatment property, part 2 of Lemma 4.4 applies. Hence, $\varphi_i (S)$ is strictly increasing in the size of $S \cap J$ for all $i \in S \setminus J$. As

$|S \cap J| < \frac{1}{2}|J| < |J \setminus S| = T \cap J$, it follows in particular

$$\varphi_i(T) = \varphi_i((S \setminus J) \cup (J \setminus S)) > \varphi_i((S \setminus J) \cup (J \cap S)) = \varphi_i(S)$$

for all $i \in S \setminus J$. Thus, S is blocked by T. $\square$

Proof of Theorem 4.10. If $|J| = 1$ then Lemma 4.8 applies. In particular, $S \subseteq N$ is winning if and only if $J \subseteq S$. In this case every $i \in S \setminus J$ is a null player in S with respect to $a_{\mathcal{I}J}$ Hence, only the first case has to be considered. But it has be shown that it holds true in Lemma 4.8.
Hence, let $|J| \geq 2$. Note that by Lemma 2.6 coalitional efficiency implies the null player property. So, φ lives only on winning coalitions and Theorem 4.9 applies.

1. Let $\varphi_j(I \cup \{j\}) \leq \frac{1}{|J|}$ and let $S \subseteq N$ with $J \nsubseteq S$. If $|S \cap J| < \frac{1}{2}|J|$, then by Theorem 4.9 S is not core stable. Hence, suppose $|S \cap J| \geq \frac{1}{2}|J|$. From Corollary 4.5 and equal treatment it follows, that

   $$|S \cap J| \cdot \varphi_j(S) = \sum_{j' \in S \cap J} \varphi_{j'}(S) < \varphi_j(S \setminus J \cup \{j\}).$$

   Hence,

   $$\varphi_j(S) < \frac{1}{|S \cap J|}\varphi_j(S \setminus J \cup \{j\}) \leq \frac{1}{\frac{1}{2}|J|} \cdot \frac{1}{|J|} \leq \frac{1}{|J|},$$

   as $|J| \geq 2$. So, S is blocked by J.

   If $J \subseteq S$ and S contains $i \in S \setminus J$ which is not a null player, then $\varphi_i(S) > 0$ as shown in Lemma 4.2. Consequently, $\varphi_j(S) < \frac{1}{|J|}$ for all $j \in J$ by equal treatment. Hence, S is blocked by J.

   If $J \subseteq S$ and all $i \in S$ are null players in the restriction of $a_{\mathcal{I}J}$ to S, then the restriction of $a_{\mathcal{I}J}$ to S coincides with the restriction of $a_{\mathcal{I}J}$ to J. Thus, S is core stable if and only if J is core stable. J cannot be blocked by any coalition T containing J as $\varphi_j(T) \leq \frac{1}{|J|} = \varphi_j(J)$ for all such T. Hence, it remains to be shown that J is not blocked by any coalition which does not contain J. Each winning coalition S must contain some $j \in J$. By Corollary 4.6 φ_j is maximized by $I \cup \{j\}$. Hence, it is sufficient to show that J is not blocked by $I \cup \{j\}$ for any $j \in J$. But this follows from $\varphi_j(I \cup \{j\}) \leq \frac{1}{|J|} = \varphi_j(J)$. Thus, $J \in \mathcal{C}(N, a_{\mathcal{I}J}, \varphi)$.

2. Let $\varphi_j(I \cup \{j\}) > \frac{1}{|J|}$. Then J is blocked by $I \cup \{j\}$ for each $j \in J$. If S consists only of J and only null players with respect to the restriction of $a_{\mathcal{I}J}$ to S, then $\varphi_j(J) = \frac{1}{|J|}$ and S is blocked. If S contains J and some players which are not null players in S with respect to $a_{\mathcal{I}J}$ then S is blocked by J as before. If S is winning with $|S \cap J| < \frac{1}{2}|J|$ then S cannot be core stable by Theorem 4.9. Let $|S \cap J| \geq \frac{1}{2}|J|$. As $\varphi_j(I \cup \{j\}) \leq \frac{1}{2}$ we have

$$\varphi_j(S) < \frac{1}{|S \cap J|}\varphi_j(S \setminus J \cup \{j\}) \leq \frac{1}{\frac{1}{2}|J|} \cdot \frac{1}{|2|} \leq \frac{1}{|J|}.$$

Hence, S is blocked by J. Therefore, there are no core stable coalitions.

3. Let $q = \varphi_j(I \cup \{j\}) > \frac{1}{2}$ and let $S \subseteq N$. As before, Theorem 4.9 implies the first inequality.

   Since $\varphi_j(J) = \frac{1}{|J|} \leq \frac{1}{2}$, J is blocked by $I \cup \{j\}$, and so is each coalition S containing J and only null players in S with respect to $a_{\mathcal{I}J}$. If $S \supseteq J$ contains any $i \in S \setminus J$ which is not a null player in S then S is blocked by J.

   Using Corollary 4.5 as before implies

$$\varphi_j(S) < \frac{1}{|S \cap J|}\varphi_j(S \setminus J \cup \{j\}) = \frac{q}{|S \cap J|}.$$

   If now $\frac{|S \cap J|}{|J|} > q$, then $\varphi_j(S) < \frac{1}{|J|}$. In this case S is blocked by J. Hence, $\frac{|S \cap J|}{|J|} \leq q$.

□

Proof of Lemma 5.3. Let $\mathcal{I} = \{I_k\}_{k=1^m}$ and let $S \subseteq N$. If $J \subseteq S$, the inequality is implied by equal treatment. So, let $J \nsubseteq S$. Due to Corollary 4.6, Sh_j is maximal on $I \cup \{j\}$, where $I = \bigcup_{k=1}^m I_k$. In this case

$$\begin{aligned} Sh_j(I \cup \{j\}) &= \sum_{T \subseteq I} \frac{(|I| - |T|)!\,|T|!}{(|I|+1)!}\left(v(T \cup \{j\}) - v(T)\right) \\ &= \frac{1}{|I|+1} \cdot \frac{1}{2}\left[\sum_{T \subseteq I} \binom{|I|}{|T|}^{-1} \left(v(T \cup \{j\}) - v(T)\right)\right. \\ &\quad \left. + \sum_{T \subseteq I} \binom{|I|}{|I \setminus T|}^{-1} \left(v((I \setminus T) \cup \{j\}) - v(I \setminus T)\right)\right]. \end{aligned}$$

By Lemma 3.5 the intersection of two apex sets must be nonempty. Let $T' \subseteq I$ such that $T' \cup \{j\}$ is winning for all $j \in J$. Then there is $I_l \subseteq T'$ and therefore $T' \cap I_k \neq \emptyset$ for all $k = 1, \ldots n$. Particularly, $I \backslash T'$ does not contain any apex set and hence, $I \backslash T' \cup \{j\}$ is not winning. Consequently, $v\left((I \setminus T') \cup \{j\}\right) + v\left(T' \cup \{j\}\right) \leq 1$.
Further, for each integer $l \leq |I|$ there are $\binom{|I|}{l}$ subsets of I of size l. Altogether

$$\begin{aligned} Sh_j\left(I \cup \{j\}\right) &\leq \frac{1}{|I|+1} \cdot \frac{1}{2} \sum_{T \subseteq I} \binom{|I|}{|T|}^{-1} \cdot 1 \qquad (2) \\ &= \frac{1}{|I|+1} \cdot \frac{1}{2} \sum_{k=0}^{|I|} \binom{|I|}{k} \binom{|I|}{k}^{-1} \\ &= \frac{1}{2}. \end{aligned}$$

This completes the proof. □

Proof of Corollary 5.4. Let $T \subseteq I$ such that $T \cup \{j\}$ is winning for all $j \in J$. Then $(I \setminus T) \cup \{j\}$ is not winning. On the other hand if $T \subseteq I$ is such that $T \cup \{j\}$ is not winning for any $j \in J$, then $N \setminus (T \cup \{j\})$ is winning in a strong game. Let $S = I \setminus T$. Then $N \setminus (T \cup \{j\}) = S \cup (J \setminus \{j\})$. As this set is winning and does not contain J, there must be an apex set $I_k \subseteq S$. Hence, $S \cup \{j'\}$. In this case $S \cup \{j\}$ is also winning since j and j' are symmetric. Hence, inequality (2) in the proof of Lemma 5.3 becomes an equality and $Sh_j\left(I \cup \{j\}\right) = \frac{1}{2}$ for all $j \in J$. □

Proof of Theorem 5.5.

1. If $|J| = 2$ then $Sh_j(J) = \frac{1}{2}$. As shown in Lemma 5.3 this is the upper bound for $Sh_j(S)$ for each $S \subseteq N$. Hence, there is no deviation of J. If S is such that $J \subseteq S$ and all players in $S \setminus J$ are null players in the restriction of $a_{\mathcal{I}J}$ to S then by Corollary 2.5 $Sh_i(S) = Sh_i(J)$ for all $i \in S$. Hence, S is core stable as well.

2. Let $|J| \geq 3$. If $Sh_j\left(I \cup \{j\}\right) > \frac{1}{|J|}$ then by part 2 of Theorem 4.10 the core is empty (note that part 3 of Theorem 4.10 never applies as $Sh_j\left(I \cup \{j\}\right) \leq \frac{1}{2}$ by Lemma 5.3). If, on the other hand, $Sh_j\left(I \cup \{j\}\right) \leq \frac{1}{|J|}$ then J is not blocked by $S^* = I \cup \{j\}$ for any j. Due to Corollary 4.6 S^* maximizes Sh_j over all coalitions which do not contain J. Hence, J is not blocked by any coalition which does not contain J. Due to symmetry and efficiency of the

Shapley value, J cannot be blocked by any coalition containing J. Hence, $J \in \mathcal{C}(N, a_{\mathcal{I}J}, Sh)$. If S contains J and is such that all players in $S \setminus J$ are null players with respect to the restriction of $a_{\mathcal{I}J}$ to S then by Corollary 2.5 $Sh_i(S) = Sh_i(J)$ for all $i \in N$. Particularly, such S cannot be blocked as J is not blocked.

$\square$

Proof of Lemma 5.10. 1. Let $S \subseteq N$ be winning such that $J \not\subseteq S$ and let $j \in S \cap J$. Then each subset of S for which $j \in J$ is pivotal must be a subset of $(S \setminus J) \cup \{j\}$. Hence,

$$\begin{aligned}
\eta_j(S) &= \frac{1}{2^{|S|-1}} \sum_{T \subseteq S \setminus \{j\}} v(T \cup \{j\}) - v(T) \\
&= \frac{2^{|S \setminus J|}}{2^{|S|-1}} \cdot \frac{1}{2^{|S \setminus J|}} \sum_{T \subseteq S \setminus J} v(T \cup \{j\}) - v(T) \\
&= \frac{1}{2^{|S \cap J|-1}} \cdot \eta_j((S \setminus J) \cup \{j\}).
\end{aligned}$$

2. As J is minimal winning, $\eta_j(J) = \frac{1}{2^{|J|-1}}$ for each $j \in J$. Let now S be a coalition which contains J. We consider two cases:

   (i) Let S not contain any apex set. Then each $i \in S \setminus J$ is a null player with respect to v_S. Consequently, $\eta_i(S) = \eta_i(J)$ for all $i \in N$ by Corollary 2.5. In particular $\eta_j(S) = \eta_j(J) = \frac{1}{2^{|J|-1}}$ for all $j \in J$.

   (ii) Let S contain at least one apex set, let $T' \subseteq S$ and let $j \in T' \cap J$. j can be pivotal in T' with respect to $a_{\mathcal{I}J}$ only if T' contains not more but one, or all minor players. Hence,

$$\begin{aligned}
\eta_j(S) = \frac{1}{2^{|S|-1}} \Bigg(&\sum_{T \subseteq S \setminus J} \left(a_{\mathcal{I}J}(T \cup \{j\}) - a_{\mathcal{I}J}(T)\right) \\
&+ \sum_{T \subseteq S \setminus J} \left(a_{\mathcal{I}J}(T \cup J) - a_{\mathcal{I}J}(T \cup J \setminus \{j\})\right) \Bigg).
\end{aligned}$$

   Let $T \subseteq S \setminus J$. If T contains an apex set, then j is pivotal $T \cup \{j\}$ but not in $T \cup J$. If T does not contain any apex set, then j is pivotal in

$T \cup J$ but not in $T \cup \{j\}$. Hence, for all $T \subseteq S \setminus J$

$$(a_{\mathcal{I}J}(T \cup \{j\}) - a_{\mathcal{I}J}(T)) + (a_{\mathcal{I}J}(T \cup J) - a_{\mathcal{I}J}(T \cup J \setminus \{j\})) = 1.$$

As there are exactly $2^{|S \setminus J|}$ different subsets $T \subseteq S \setminus T$,

$$\eta_j(S) = \frac{1}{2^{|S|-1}} \cdot 2^{|S \setminus J|} = \frac{1}{2^{|J|-1}}.$$

□

Proof of Theorem 5.13. 1. Let $\eta_j(I \cup \{j\}) \leq \frac{1}{2^{|J|-1}}$.

(a) $I \cup \{j\}$ maximizes $\eta_j(S)$ over all coalitions $S \subseteq N$ which do not contain J. As $I \cup \{j\}$ does not block J and as each winning coalition S contains some $j' \in J$ (which are symmetric to j in $a_{\mathcal{I}J}$), there is no coalition which blocks J. Hence, a coalition $S \supseteq J$ cannot be blocked by part 2 of Lemma 5.10. Therefore, each coalition containing J is core stable.

(b) Let $S \in \mathcal{C}(N, v, \eta)$ such that $J \nsubseteq S$. Then $\eta_j(S) \geq \eta_j(J) = \frac{1}{2^{|J|-1}}$ for all $j \in S \cap J$. Further, $\eta_j(S) = \frac{1}{2^{|S \cap J|-1}} \cdot \eta_j(S \setminus J \cup \{j\})$ for all $j \in S$ by part 1 of Lemma 5.10, and $|S \cap J| \geq \frac{1}{2}|J|$ by Theorem 4.9. Now, we show that the three conditions must be satisfied.

(i) If $|J| \geq 4$, then $\eta_j(I \cup \{j\}) \leq \frac{1}{2^{|J|-1}} \leq \frac{1}{2^{\frac{1}{2}|J|+1}}$. Hence,

$$\begin{aligned} \eta_j(S) &= \frac{1}{2^{|S \cap J|-1}} \cdot \eta_j(S \setminus J \cup \{j\}) \leq \frac{1}{2^{\frac{1}{2}|J|-1}} \cdot \frac{1}{2^{\frac{1}{2}|J|+1}} \\ &= \frac{1}{2^{|J|}} < \eta_j(J). \end{aligned}$$

In this case S would be blocked by J. Hence, $|J| = 3$.

(ii) As $|S \cap J| \geq \frac{1}{2}|J|$, $J \nsubseteq S$, and $|J| = 3$, it follows that $|S \cap J| = 2$.

(iii) As $|S \cap J| = 2$ it follows

$$\frac{1}{2} \cdot \eta_j(S \setminus J \cup \{j\}) = \eta_j(S) \geq \frac{1}{2^{|J|-1}} = \frac{1}{4}$$

Hence, $\eta_j(S \setminus J \cup \{j\}) \geq \frac{1}{2}$. By Lemma 5.11 this is the upper bound for $\eta_j(S)$ which can be reached only if $a_{\mathcal{I}J}$ is strong.

Lemma 5.11 said that S must contain all apex sets, that is $I \subseteq S$.

Let on the other hand conditions (i)-(iii) be satisfied. Then

$$\eta_j(S) = \frac{1}{4} = \eta_j(J)$$

for all $j \in S \cap J$ if and only if $I \subseteq S$. In this case S cannot be blocked.

2. Any coalition $S \supseteq J$ containing some $i \in S \setminus J$ which are not null players in S is blocked by J. If $S \supseteq J$ does not contain any such i, then S is blocked by $I \cup \{j\}$. Hence, let $J \nsubseteq S$. By Theorem 4.9, each coalition $S \in \mathcal{C}(N, a_{\mathcal{I}J}, \eta)$ must contain the majority of players from J. Consider a winning coalition S such that $|S \cap J| \geq \frac{1}{2}|J|$. By part 1 of Lemma 5.10

$$\eta_j(S) = \frac{1}{2^{|S\cap J|-1}} \cdot \eta_j(S \setminus J \cup \{j\}) \leq \frac{1}{2^{\frac{1}{2}|J|-1}} \cdot \eta_j(S \setminus J \cup \{j\}).$$

Thus, if $\eta_j(S \setminus J \cup \{j\}) \leq \frac{1}{2^{\frac{1}{2}|J|+1}}$ then

$$\eta_j(S) \leq \frac{1}{2^{\frac{1}{2}|J|-1}} \cdot \frac{1}{2^{\frac{1}{2}|J|+1}} = \frac{1}{2^{|J|}} < \frac{1}{2^{|J|-1}} = \eta_j(J).$$

In this case S is blocked by J.

3. From Theorem 4.9 it follows that each coalition $S \in \mathcal{C}(N, a_{\mathcal{I}J}, \eta)$ must contain the majority of players from J. Hence, $|S \cap J| \geq \frac{1}{2}|J|$ for each $S \in \mathcal{C}(N, v, \eta)$. This gives the first inequality. Since S is not dominated by J, it follows that $\eta_j(S) \geq \frac{1}{2^{|J|-1}}$. Hence, using part 1 of Lemma 5.10,

$$\frac{1}{2^{|S\cap J|-1}} \cdot \eta_j(S \setminus J \cup \{j\}) = \eta_j(S) \geq \frac{1}{2^{|J|-1}}.$$

It follows that

$$\frac{2^{|S\cap J|}}{2^{|J|}} \leq \eta_j(S \setminus J \cup \{j\}) \leq \eta_j(I \cup \{j\}).$$

□

Proof of Lemma 5.17. Let $S = I \cup J_1$ and let $\mu(S)$ be defined as $\sum_{i \in S\setminus J} \mu_i(S)$. $j \in J_1$ is pivotal in $T \subseteq S$ if and only if T is winning and $T \cap J_1 = \{j\}$. There are exactly $\mu_j(I \cup \{j\})$ such sets. For each such set T there are $2^{|J|-|J_1|}$ subsets $U \subseteq J \setminus J_1$ such that j is pivotal in $T \cup U$ with respect to the apex game restricted

to S. Hence, there are $\mu_j(I \cup \{j\}) \cdot 2^{|J|-|J_1|}$ different coalitions $T \subseteq S$, such that j is pivotal in T with respect to the restriction of $a_{\mathcal{I}J}$ to S.
On the other hand let $i \in I$. Then i is pivotal in $T \subseteq S$ if and only if T is winning i is pivotal in $(T \setminus J_1) \cup \{j\} \subseteq S$. Hence, there are $\mu_i(I \cup \{j\}) \cdot \left(2^{|J_1|} - 1\right)$ different coalitions $T \subseteq S$ such that i is pivotal in T with respect to $a_{\mathcal{I}J}$.
For each of theses sets there are $2^{|J|-|J_1|}$ subsets U of $J \setminus J_1$ such that i is pivotal in $T \cup U$. Altogether,

$$\begin{aligned}
\beta_j(I \cup J_1) &= \frac{\mu_j(I \cup \{j\}) \cdot 2^{|J|-|J_1|}}{|J_1| \cdot \mu_j(I \cup \{j\}) \cdot 2^{|J|-|J_1|} + \mu(I \cup \{j\}) \cdot (2^{|J_1|} - 1) \cdot 2^{|J|-|J_1|}} \\
&= \frac{\mu_j(I \cup \{j\})}{|J_1| \cdot \mu_j(I \cup \{j\}) + \mu(I \cup \{j\}) \cdot (2^{|J_1|} - 1)} \qquad (3) \\
\beta_i(I \cup J_1) &= \frac{\mu_i(I \cup \{j\}) \cdot \left(2^{|J_1|} - 1\right) \cdot 2^{|J|-|J_1|}}{|J_1| \cdot \mu_j(I \cup \{j\}) \cdot 2^{|J|-|J_1|} + \mu(I \cup \{j\}) \cdot (2^{|J_1|} - 1) \cdot 2^{|J|-|J_1|}} \\
&= \frac{\mu_i(I \cup \{j\}) \cdot \left(2^{|J_1|} - 1\right)}{|J_1| \cdot \mu_j(I \cup \{j\}) + \mu(I \cup \{j\}) \cdot (2^{|J_1|} - 1)} \qquad (4)
\end{aligned}$$

The rest of the proof is now straightforward.

1. If $j \in J_2 \setminus J_1$ there is nothing to show. For $j \in J_2 \cap J_1$ it follows from equation (3) that $\beta_j(I \cup J_1)$ is decreasing in $|J_1|$.

2. From equation (4) follows that $\beta_i(I \cup J_1)$ is increasing in $|J_1|$.

□

Proof of Lemma 5.18. We use the following result from Dubey and Shapley (1979):
Let v be a (not necessarily proper monotonic) simple game on a player set N. Let ω be the number of winning coalitions $S \subseteq N$ in v and let ν be the number of losing coalitions $S \subseteq N$ with respect to v. Then

$$\sum_{i \in N} \mu_i(v) \geq \lambda \cdot \lfloor |N| - \log_2(\lambda) \rfloor, \qquad (5)$$

where $\lambda = \min(\omega, \nu)$ and $\lfloor x \rfloor$ is the greatest integer k with $k \leq x$.
If $S \subseteq N$ is a winning coalition with $J \subseteq S$, then the claim follows from the equal treatment property, so let $J \nsubseteq S$. By part 1 of Lemma 5.17 it is sufficient to show, that this bound holds for each winning coalition $S \subseteq N$ with $|S \cap J| = 1$. Let

$j \in S \cap J$. Then j is a veto player in the restriction v of $a_{\mathcal{I}J}$ to S. By properness of the game, there are not more than $2^{|S|-2}$ subsets $T \subseteq S \setminus J = S \setminus \{j\}$ such that $T \cup \{j\}$ is winning. As there are $2^{|N|-|S|}$ different subsets of $N \setminus S$, we get

$$\mu_j(S) = \omega(v) \leq 2^{|S|-2} \cdot 2^{|N|-|S|} = 2^{|N|-2} \leq \nu(v).$$

With equation (5) we find

$$\begin{aligned}
\sum_{i \in S} \mu_i(S) &\geq \mu_j(S) \cdot \lfloor |N| - \log_2(\omega(v)) \rfloor \\
&\geq \mu_j(S) \cdot \lfloor |N| - \log_2\left(2^{|N|-2}\right) \rfloor \\
&= \mu_j(S) \cdot \lfloor |N| - |N| + 2 \rfloor \\
&= 2\mu_j(S).
\end{aligned}$$

Thus, $\beta_j(S) = \frac{\mu_j(S)}{\sum_{i \in S} \mu_i(S)} \leq \frac{1}{2}$. □

Proof of Theorem 5.19. As β is coalitional efficient and satisfies equal treatment, $\beta_j(J) = \frac{1}{|J|}$ for all $j \in J$. If $S \subseteq N$ is such that $J \subseteq S$ and all $i \in S \setminus J$ are null players with respect to the restriction of $a_{\mathcal{I}J}$ to S then by Corollary 2.5 $\beta_j(S) = \beta_j(J) = \frac{1}{|J|}$ for all $j \in J$. Therefore, S is blocked if and only if J is blocked. For $\beta_j(J) = \frac{1}{|J|}$ and $I^* \cup \{j\}$ maximizes $\beta_j(S)$ over all S which do not contain J, J is not blocked if and only if it is not blocked by $I^* \cup \{j\}$.

1. If $|J| = 2$, then $\beta_j(J) = \frac{1}{2}$ for all $j \in J$. As this is the upper bound for β_j by Lemma 5.18, J cannot be blocked.

2. We first show, that $S \subseteq N$ can be core stable, only if $J \subseteq S$ and all $i \in S \setminus J$. Let therefore $J \nsubseteq S$ and assume $S \in \mathcal{C}(N, a_{\mathcal{I}J}, \beta)$. From Theorem 4.9 it follows that a coalition $S \in \mathcal{C}(N, a_{\mathcal{I}J}, \beta)$ must contain the majority of players from J. Let $I = S \cap \bigcup_{k=1}^{n} I_k$ and $J_1 = S \cap J$. As $|J| \geq 3$, $|J_1| \geq 2$. Then from equation (3) follows that

$$\begin{aligned}
\beta_j(S) &= \frac{\mu_j(I \cup \{j\})}{|J_1| \cdot \mu_j(I \cup \{j\}) + \nu(I \cup \{j\}) \cdot (2^{|J_1|} - 1)} \\
&= \frac{1}{|J_1| + (2^{|J_1|} - 1) \cdot \frac{\nu(I \cup \{j\})}{\mu_j(I \cup \{j\})}}
\end{aligned}$$

If $|J_1| = 1$, this value is maximized by $I = I^*$. Hence, $\frac{\nu(I \cup \{j\})}{\mu_j(I \cup \{j\})}$ is minimized by I^* and $\beta_j(I \cup J_1)$ is maximized by I^* for all $J_1 \subsetneq J$. Particularly, since $|J_1| < 2^{|J_1|-1}$ for $|J_1| \geq 2$ it follows that

$$\beta_j(I \cup J_1) < \frac{1}{|J_1|} \cdot \frac{1}{1 + \frac{\nu(I \cup \{j\})}{\mu_j(I \cup \{j\})}} = \frac{1}{|J_1|} \cdot \beta_j(I \cup \{j\}).$$

Since $\beta_j(I^* \cup \{j\}) \leq \frac{1}{2}$ and $|J_1| \geq \frac{1}{2}|J|$, it follows that $\beta_j(S) < \frac{1}{|J|}$. This means that S is blocked by J.

Let now $J \subseteq S$. If S contains any $i \in S \setminus J$ which is not a null player, then $\beta_j(S) < \frac{1}{|J|}$. Thus S is blocked by J. Hence, S can be core stable only if $J \subseteq S$ and all$i \in S$ are null players in the restriction of $a_{\mathcal{I}J}$ to S. It has been shown at the beginning of the proof that such S is core stable if and only if J is core stable. As J is blocked by $I^* \cup \{j\}$ if and only if $\beta_j(I^* \cup \{j\}) > \frac{1}{|J|}$, the claim is proven.

□

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