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Modeling Ambiguity in Expert Elicitation Surveys: Theory and Application to Solar Technology R&D

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Summary

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Keywords: Ambiguity, Expert Elicitation, Convex Optimization, Solar Energy

JEL Classification: C61, D81

.We thank Gauthier de Maere d'Aertrycke for many illuminating discussions on convex optimization. The research leading to these results has received funding from the European Research Council under the European Community's Seventh Framework Programme (FP7/2007-2013)/ERC grant agreement n. 240895 project ICARUS Innovation for Climate Change Mitigation: a Study of energy R&D, its Uncertain Effectiveness and Spillovers.

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Modeling ambiguity in expert elicitation surveys: theory and application to solar technology R&D*

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Abstract

Optimal R&D investment is defined by deep uncertainty that can only partially be addressed through historical data. Thus, expert judgments expressed as subjective probability distributions are seen as an alternative way of assessing the potential of new technologies. In this paper we propose a simple decision-theoretic framework that takes into account ambiguity over expert opinion and helps decision makers visualize the full range of R&D outcomes given a particular level of ambiguity. Our model is intuitive, captures decision makers' ambiguity attitudes, and enables simple sensitivity analysis across levels of ambiguity. We apply our framework to original data from a recent expert elicitation survey on solar technology. The analysis suggests that ambiguity plays an important role in assessing the potential of a breakthrough in solar technology given different R&D investments.

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1 Introduction

Innovation is an uncertain process. The history of public R&D programs is paved with failures and dead ends and, eventually, successes. Failures can derive from a funding tap closed too early, or from the plain fact that the technology ultimately proves to be technically or economically infeasible. The bottom line is that making decisions over competing R&D programs under budget constraints is an important but considerably complex task. Above all, it is a task that needs to take into account uncertainty.

Addressing the uncertainty of R&D programs is complicated by the fact that probabilities of success are very hard to estimate. They tend to be functions of R&D investment itself, and this endogeneity adds formidable challenges to their econometric estimation. Nonetheless, there exists a vast literature, studying patent numbers and/or productivity levels, that provides empirical support to the idea of a positive and strong relationship between R&D funding and innovation both in general purpose innovation (Grossman and Helpman [15]) and more specifically in energy innovation (Newell et al. [21], Popp [24]).

Although the positive relationship between R&D and technological breakthroughs is well established, characterizing the probability of success given different levels of R&D expenditure is a question that can only partially be addressed by past data. Historical information on costs, patents, and R&D expenditure may be used to get an idea of the general trends, but research programs differ vastly and are, most of the time, not reproducible. Therefore, to account for the uncertainty of specific R&D programs it is often necessary to resort to expert judgments and subjective probabilities. Structured expert judgment, pioneered in the Rasmussen Report on risks of nuclear power plants (Rasmussen et al. [25]), derives probabilistic input for decision problems through experts' quantification of their subjective uncertainties (Morgan and Henrion [19], Cooke [10], O'Hagan et al. [23]). Experts' probability distributions are elicited via transparent protocols and treated as scientific data (Cooke [10]). The employed elicitation techniques involve recognizing and removing, as much as possible, known psychological biases in judgment (Tversky and Kahneman [27, 28]). They further incorporate consistency checks and structure the variables to be estimated in such a way that experts are called to respond to cognitively simple assessment questions.

Expert elicitation methods have been used in a variety of contexts (Cooke [10]). Numerous studies have attempted to elicit climate scientists' opinions to characterize the probability associated to reactions of the climatic system as well as to climatic impacts (Nordhaus [22], Morgan et al. [20], Vaughn and Spouge [29]). Recently, expert elicitation surveys have also been pursued in the assessment of energy technologies in relation to R&D budgets (the empirical focus of the present paper). Baker et al. [1, 2], Baker and Keisler [3], and Curtright et al. [11] use expert elicitation to investigate the uncertain effects of R&D investments on the prospect of success of carbon capture and storage, hybrid electric vehicles, cellulosic biofuels, and solar PV technologies.

Despite its compelling features, expert elicitation often generates widely-divergent opinions across experts, implying fundamentally different and competing views of how the innovation process works. To overcome this difficulty, researchers typically aggregate over expert estimates in some fashion and consider their average. Indeed, there is a rich decision-theoretic literature that studies the many different ways such aggregations may be performed. In a critical survey of the literature, Clemen and Winkler [8] broadly distinguish between (i) mathematical approaches and (ii) behavioral approaches. Mathematical aggregation methods are primarily concerned with constructing a single probability distribution on the basis of individual elicited distributions. This is usually pursued either through axiomatic treatments of mathematical formulas of aggregation, or, where possible, through Bayeasian statistical methods. A formidable amount of research has been pursued on such mathematical methods of aggregation and the reader is referred to Clemen and Winkler [9] for a comprehensive review. By contrast, behavioral approaches to expert aggregation involve the direct interaction between experts in order to reach consensus on a single "group" estimate. This interaction can be structured in a number of different ways according to the particular application at hand. More information on this literature can be found in Clemen and Winkler [8].

We take a fundamentally different approach to the ones outlined above. Our model is inspired by economic-theory advances in models of decision making under ambiguity and as such is not concerned with determining a single probability distribution reflecting expert opinion. Instead, it nests in a parametric fashion simple averaging and best/worst-case analysis and allows for an expression of decision-makers' beliefs regarding, and attitude towards, the underlying ambiguity in expert opinion. Our paper is, to the best of our knowledge, the first attempt to develop and apply a model of ambiguity aversion to the standard expert elicitation process.

Decision-theoretic models of ambiguity are designed to address situations in which a decision maker is unable to posit precise probabilistic structure to physical and economic models. This framework derives from the concept of uncertainty as introduced by Knight [17] to represent a situation in which a decision maker lacks adequate information to assign probabilities to events. Knight argued that this deeper kind of uncertainty is quite common in economic decision-making, and thus deserving of systematic study. Knightian uncertainty is contrasted to risk (measurable or probabilistic uncertainty) where probabilistic structure can be fully captured by a *single* Bayesian prior.[1]

In a seminal contribution, Gilboa and Schmeidler [14] developed the axiomatic foundations of *maxmin* expected utility (MEU), a substitute of classical expected utility for economic environments featuring unknown risk. They argued that when the underlying uncertainty of an economic system is not well understood, it is sensible, and axiomatically sound, to optimize over the worst-case outcome (i.e. the worst-case *prior*) that may conceivably come to pass. Ghirardato et al. [12] took a step further and axiomatized a generalization of the Gilboa and Schmeidler model. A compelling special example of their framework is their α-MEU model in which a decision maker's preferences are captured by a convex combination of the worst- and best-case expected payoffs over a set of uncertain priors. The assigned weights vary parametrically, so that the more weight is placed on an agent's minimum payoff, the more ambiguity averse he is considered to be (when there is zero weight on the best case we recover Gilboa and Schmeidler [14]).

The decision-theoretic model we propose in this work borrows from simplified versions of [14] and [12]. It begins by positing a benchmark second-order distribution over expert opinion that assigns equal weight to each expert. Subsequently, we consider intuitive and mathematically straightforward enlargements of the set of possible second-order distributions by parametrizing over an ambiguity parameter. This parameterization admits a practical interpretation in terms of the maximum possible weight that can be assigned to a single expert. Next, given a metric of R&D effectiveness, we calculate the best-and worst-case expected outcomes of a given level of R&D investment, subject to the feasible set of second-order distributions that is implied by assigned levels of ambiguity. Finally, we follow Ghirardato et al. [12] and consider a convex combination of best and worst-case expected outcomes as a reasonable way to model decision makers' preferences under ambiguity.

Our model's simple mathematical structure allows for sharp quantitative insights and we investigate its theoretical properties in considerable depth. Using results from convex optimization and duality theory, we are able to provide a closed-form expression for the best- and worst-case value functions and establish their differentiability in the ambiguity parameter. These results may be of independent mathematical interest and enable simple sensitivity analysis across different levels of ambiguity and ambiguity attitude.

Before discussing the empirical application of our model, we would like to emphasize that we

[1]The interested reader is referred to Gilboa and Marinacci [13] for a comprehensive survey of the literature of decision-making under ambiguity, a field of economic theory that is both vast and deep. In what follows, we focus purely on the contributions that are directly relevant to our purposes.

do not wish to oversell its theoretical virtues. Unlike the more fundamental literature on ambiguity aversion (such as [14, 12, 16]) we do not pursue an axiomatic characterization of our framework. Instead, we see the primary advantages of our approach as being those of transparency, intuitiveness, and practicality. Admitting simple mathematical structure and easy interpretation/calibration, it is a straightforward way of modelling ambiguity in expert elicitation, that managers and policy makers alike may find useful in visualizing the set of possible alternatives and making informed decisions given their particular beliefs about, and attitude towards, ambiguous expert opinion.

We base the empirical analysis of our paper on the ICARUS expert elicitation (for more information see www.icarus-project.org and Bosetti et al. [5]) in which sixteen leading European experts from the academic world, the private sector, and international institutions were interviewed to assess the potential of R&D investments in solar technologies, both Photovoltaic (PV) and Concentrated Solar Power (CSP). We use an integrated assessment model [6] to calculate the benefits of R&D investment (in the form of lower future solar-electricitly costs) and use these estimates to inform our assesment of the relevant R&D investment alternatives. Our subsequent analysis suggests that ambiguity plays an important role in assessing the potential of a breakthrough in solar technology. The policy implication we are able to cautiously draw is that more aggressive investment in solar technology R&D is likely to yield significant dividends, even (or perhaps especially) after taking ambiguity into account.

2 Expert Elicitation under Ambiguity

Consider a set $\mathcal{N}$ of N technological experts indexed by $n = 1, 2, ..., N$. R&D investment is denoted by a variable r and the technology's cost by c. An expert n's probability distribution of the future cost of technology given investment r is captured by a random variable $C_n(r)$ having a probability distribution function

$$\pi_n(c|r).^2 \tag{1}$$

Expert beliefs over the economic potential of R&D investment may, and usually do, vary significantly. The question thus naturally arises: How do we make sense of this divergence when studying optimal R&D investment? In the absence of data that could lend greater credibility to one expert

[2] Note that the decision variables of our model (R&D investment) directly affect the subjective probability distributions of the technology's costs. This means that our setting is not amenable to standard decision-theoretic frameworks going back to Savage [26]. In particular, we cannot use the increasingly influential smooth ambiguity model of Klibanoff et al. [16].

over another and form the basis of a Bayesian analysis, one straightforward way would be to simply linearly aggregate over all pdfs π_n as given by Eq. (1), so that we obtain an "aggregate" joint pdf $\bar{\pi}$, where

$$\bar{\pi}(c|r) = \sum_{n=1}^{N} \frac{1}{N} \pi_n(c|r). \tag{2}$$

This approach inherently assumes that each and every expert is equally likely to represent reality, and makes use of simple but powerful linear aggregation. While this is standard practice in the expert elicitation literature, a great deal of information may be lost in such an averaging-out process, especially when there are huge differences among experts.

Thus, we move beyond simple averaging and develop an intuitive decision-theoretic framework based on second-order probabilities, associated to experts. Our model enhances transparency and conveys a better and deeper understanding of the collected distributions, facilitating extensional reasoning and shedding light on the credibility of the simple average.[3] In addition, it allows for an expression of decision makers' attitude towards ambiguity.

In our framework each expert n's pdf $\pi_n(c|r)$ is weighted by the decision maker through a second-order probability p_n. The set of admissible second-order distributions $\mathbf{p}$ depends on the amount of ambiguity the decision maker is willing to take into account, a model input. Specifically, we consider the set of second-order distributions $\mathrm{P}(b)$ over a set of n experts, parametrized by $b \in [0, \frac{N-1}{N}]$ where

$$\mathrm{P}(b) = \left\{ \mathbf{p} \in \Re^N : \;\; \mathbf{p} \geq \mathbf{0}, \;\; \sum_{n=1}^{N} p_n = 1, \;\; \sum_{n=1}^{N} \left(p_n - \frac{1}{N} \right)^2 \leq b \right\}. \tag{3}$$

Here, the set $\mathrm{P}(b)$ measures the *ambiguity* of the experts' subjective beliefs. Setting $b = 0$ implies complete certainty and adoption of the equal-weight singleton, while $b = \frac{N-1}{N}$ complete ambiguity over the set of all possible second-order distributions.[4]

We now briefly explain the practical interpretation of an ambiguity level b in our model. Consider the benchmark equal-weight distribution $\boldsymbol{p} = \frac{1}{N}\, \boldsymbol{e_N}$, where $\boldsymbol{e_N}$ is a unit vector of dimension N. Now take any expert $\tilde{n}$ and begin increasing the second-order probability attached to her esti-

[3]Indeed, Larrick and Soll [18] argue that "The benefits of averaging may be appreciated at a deep level by careful extensional reasoning –by imagining the space of possible outcomes and their implications." Our hope is that our contribution accomplishes exactly this purpose by providing a useful guide around the space of possible outcomes and ultimately improving the understanding and appreciation of the average of experts' judgments.

[4]The latter statement holds in light of the fact that values of $b > \frac{N-1}{N}$ cannot enlarge the feasible set. This is because the maximizers of $\sum_{n=1}^{N} \left(p_n - \frac{1}{N}\right)^2$ over the set of probability vectors concentrate all probability mass on one expert, leading to an ambiguity level of $\left(1 - \frac{1}{N}\right)^2 + (N-1) \cdot \left(\frac{1}{N}\right)^2 = \frac{N-1}{N}$.

mate by an amount $\epsilon_b \geq 0$ so that $p_{\tilde{n}} = \frac{1}{N} + \epsilon_b$. The convexity of the feasible set $\mathrm{P}(b)$ implies that, in order to maximize the value of ϵ_b (over the feasible set $\mathrm{P}(b)$), we need to offset the increase of ϵ_b in $p_{\tilde{n}}$ by decreasing equally all p_n $n \neq \tilde{n}$ by an amount $\frac{\epsilon_b}{N-1}$. Subsequently, inclusion in $\mathrm{P}(b)$ implies

$$\epsilon_b^2 + (N-1)\left(\frac{\epsilon_b}{N-1}\right)^2 \leq b \quad \Rightarrow \quad \epsilon_b \leq \sqrt{\frac{N-1}{N}b}.$$

Thus, we are able to provide a tight upper bound on the maximum second-order probability $p^*(b;N)$ that can be placed on the estimate of any *single* expert, as a function of b:

$$p^*(b;N) = \max_{\boldsymbol{p}\in\mathrm{P}(b)} \max_{n\in\mathcal{N}} p_n = \frac{1}{N} + \sqrt{\frac{N-1}{N}b}. \tag{4}$$

To provide a sense of what this would mean, increasing b from zero to 0.1 would imply a maximum second-order probability $p^*(b;N)$ of 0.48, 0.4, and 0.35 for $N = 5, 10,$ and 20 experts, respectively.

Denote our cost domain by D and define the real-valued function

$$f\left(C_n(r), r\right) : D \times \Re_+ \mapsto \Re, \quad n \in \mathcal{N},$$

as representing a *metric* of the economic impact of investment r, under expert n. Individual metrics across experts are collected in the vector-valued function

$$f = \{f(C_1(r), r), f(C_2(r), r), ..., f(C_N(r), r)\}.$$

An example of a possible metric we could consider is:

$$f\left(C_n(r), r\right) = \mathbf{1}\{C_n(r) \leq z\}, \tag{5}$$

where $\mathbf{1}$ is an indicator function. In expectation it will be equal

$$\mathbf{E}[\mathbf{1}\{C_n(r) \leq z\}] = \mathbf{Pr}[C_n(r) \leq z] = \int_{c\in D: c\leq z} \pi_n(c|r)\mathrm{d}c,$$

which represents the probability of the cost the technology being less than or equal to a value z.

A different metric we could consider could be the net payoff of an investment r. Denoting the payoff associated to a technology cost c by the function $g(c)$, this would be given by

$$f(C_n(r), r) = g(C_n(r)) - r. \tag{6}$$

Now, given R&D investment r, metric f, and the set of second-order distributions $\mathrm{P}(b)$ introduced in (3), we can calculate the worst- and best-case expected outcomes of our investment decision given ambiguity b. This provides a measure of the spread, as measured by metric f, between the worst and best-cases, given a "willingness" to stray from the benchmark equal-weight distribution that is constrained by b. More formally, we calculate the value functions

$$V_{max}(r|f,b) \quad = \quad \max_{\mathbf{p}\in \mathrm{P}(b)} \sum_{n=1}^{N} p_n \mathbf{E}\left[f\left(C_n(r), r\right)\right] \tag{7}$$

$$V_{min}(r|f,b) \quad = \quad \min_{\mathbf{p}\in \mathrm{P}(b)} \sum_{n=1}^{N} p_n \mathbf{E}\left[f\left(C_n(r), r\right)\right]. \tag{8}$$

Plotting functions (7) and (8) over $b \in [0,(N-1)/N]$ gives decision makers a comprehensive picture of the effectiveness of R&D investment r (as measured by metric f), and could potentially provide novel insights that are not normally captured by simple averaging.

The value functions (7)-(8) fix a level of ambiguity b and subsequently focus on the absolute best and worst-cases. As such they capture extreme attitudes towards ambiguity. To express more nuanced decision-maker preferences we consider the following function

$$V(r|f,b,\alpha) = \alpha \cdot V_{min}(r|f,b) + (1-\alpha)\cdot V_{max}(r|f,b) \quad \alpha \in [0,1], \tag{9}$$

representing a convex combination of the worst- and best-cases. The parameter α above captures the decision maker's *ambiguity attitude*. It measures his degree of pessimism given ambiguity b: the greater (smaller) α is, the more (less) weight is placed on the worst-case scenario. Eq. (9) is reminiscent of the α-maxmin model of Ghirardato et al. [12] and could operate as an objective function when searching for optimal investment decisions.

3 Theoretical Analysis

In this section we focus on the decision maker's optimization problems (7) and (8) and analyze the behavior of functions V_{max} and V_{min} as we vary ambiguity levels b. Using results from convex optimization and duality theory we are able to derive relatively simple closed-form expressions for these functions and establish their differentiability in b. To the best of our knowledge these results are novel, and could be of independent mathematical interest.

To avoid cumbersome notation, throughout this section we suppress dependence on f and r by fixing a metric and an investment level as defined below:

$$x_n \quad \equiv \quad \mathbf{E}\left[f\left(C_n(r)\right)\right],$$

$$V_{max}(r|f,b) \quad \equiv \quad V_{max}(b) = \max_{\mathbf{p}\in \mathrm{P}(b)} \sum_{n=1}^{N} p_n x_n \tag{10}$$

$$V_{min}(r|f,b) \quad \equiv \quad V_{min}(b) = \min_{\mathbf{p}\in \mathrm{P}(b)} \sum_{n=1}^{N} p_n x_n. \tag{11}$$

Optimization problems (10) and (11) are conic programs with a simple structure and thus amenable to rich analysis. We begin by proving that their optimal cost functions are continuous, monotonic, and concave/convex in b.

Proposition 1 *The function $V_{max}(b)$ ($V_{min}(b)$) defined in Eq. (10) (Eq. 11) is increasing (decreasing) and concave (convex) in b. Both functions are continuous in b.*

Proof. See Appendix. ■

Before we state our next result we need to introduce some new notation. First, let $\mathcal{N}_k$ denote the set of experts sharing the k'th order statistic of $\{x_1, x_2, ..., x_N\}$. There are a total of k_N such sets where, depending on the problem instance, k_N can be any number in $\{1, 2, ..., N\}$, and we define $N_k = |\mathcal{N}_k|$. Furthermore, let $\mathcal{N}_k^+ = \bigcup_{i=k}^{k_N} \mathcal{N}_i$, $\mathcal{N}_k^- = \bigcup_{i=1}^{k} \mathcal{N}_i$ and $N_k^+ = \left|\mathcal{N}_k^+\right|$, $N_k^- = \left|\mathcal{N}_k^-\right|$. Our model structure enables us to easily show the following Lemma.

Lemma 1 *Consider the optimization problems (10) and (11). Define ambiguity levels $b^*_{max} \equiv \frac{1}{N_{k_N}} - \frac{1}{N}$ and $b^*_{min} \equiv \frac{1}{N_1} - \frac{1}{N}$. $V_{max}(b)$ is strictly increasing in $[0, b^*_{max}]$ and equal to $\max_{n\in\mathcal{N}} x_n$ in $[b^*_{max}, \frac{N-1}{N}]$. $V_{min}(b)$ is strictly decreasing in $[0, b^*_{min}]$ and equal to $\min_{n\in\mathcal{N}} x_n$ in $[b^*_{min}, \frac{N-1}{N}]$.*

Proof. See Appendix. ■

Lemma 1 suggests that b^*_{max} and b^*_{min} are important thresholds. They represent the level of ambiguity above which the set $\mathrm{P}(b)$ allows for the maximum/minimum expert estimate to be attained in the optimal solution of (10)/ (11). The closer b^*_{max} and b^*_{min} are to 0, the less relevant the issue of ambiguity is. Our next result establishes that for levels of ambiguity smaller than these extreme values, the optimal solutions of problems (10) and (11) will be unique and bind the quadratic ambiguity constraint associated with set $\mathrm{P}(b)$.

Proposition 2 *Suppose $b \in [0, b^*_{max}]$ and consider the optimization problem (10). There exists a unique optimal solution $\mathbf{p^{max}}(b)$ and it must satisfy the quadratic ambiguity constraint with equality. Equivalent results apply to the minimization problem (11).*

Proof. See Appendix. ■

The following Lemma formalizes an straightforward property of the optimization problems we are concerned with.

Lemma 2 *Consider the optimization problem (10) for $b_1 \in [0, b^*_{max}]$ and let $\mathbf{p^{max}}(b_1)$ denote its unique optimal solution. The following holds for any $n \in \mathcal{N}$*

$$p_n^{max}(b_1) = 0 \;\; \Rightarrow p_n^{max}(b) = 0, \;\; \textit{for all } b > b_1. \tag{12}$$

If a level of ambiguity $b > b^*_{max}$ admits multiple optima, the above relation is understood to hold for all optimal solutions $\boldsymbol{p^{max}}(b)$. Equivalent results apply to the minimization problem (11).

Proof. See Appendix.

Before stating our next result we define the following levels of ambiguity

$$b^{max}_{(k)} = \min\left\{b : p^{max}_n(b) = 0, \ \ \forall n \in \mathcal{N}^-_k\right\}, \ \ k \in \{1, 2, ..., k_N - 1\}, \tag{13}$$
$$b^{min}_{(k)} = \min\left\{b : p^{min}_n(b) = 0, \ \ \forall n \in \mathcal{N}^+_{k_N - k+1}\right\} \ \ k \in \{1, 2, ..., k_N - 1\}. \tag{14}$$

Lemma 2 implies that $b^{max}_{(k)}$ and $b^{min}_{(k)}$ can be interpreted in the following way. In the case of problem (10), $b^{max}_{(k)}$ denotes the minimum level of ambiguity such that for all $b \geq b^{max}_{(k)}$ no probability mass is ever allocated to experts having an x_n that is less than or equal to the k'th order statistic of the x_n's. Conversely, in the case of problem (11), $b^{min}_{(k)}$ denotes the minimum level of ambiguity such that for all $b \geq b^{min}_{(k)}$ no probability mass is allocated to experts having an x_n that is greater than or equal to the $(k_N - k + 1)$'th order statistic of the x_n's. Lemma 3 formalizes an intuitive result with regard to these levels of ambiguity and the optimal solutions of (10) and (11).

Lemma 3 *Ambiguity levels $b^{max}_{(k)}$ and $b^{min}_{(k)}$, defined in Eqs. (13)-(14), are strictly increasing in k.*

Proof. See Appendix. ■

We are now ready to prove the paper's first main result. Theorem 1 formally establishes the existence of a first derivative for functions $V_{max}(b)$ and $V_{min}(b)$ and provides a set of differential equations that they must satisfy.

Theorem 1 *The function $V_{max}(b)$ is differentiable on $[0, b^*_{max}) \cup (b^*_{max}, (N-1)/N]$. Consider $b \in (0, b^*_{max})$ and let $\mathbf{p^{max}}(b)$ denote the unique optimal solution of $V_{max}(b)$. Suppose that $n_k \in \mathcal{N}_j$. $V_{max}(b)$ satisfies the following differential equation:*

$$2\frac{\mathrm{d}}{\mathrm{d}b}V_{max}(b)\left(p^{max}_{n_k}(b) - \frac{1}{N} - b\right) = x_{n_k} - V_{max}(b), \ \ b \in \left[0, b^{max}_{(j)}\right). \tag{15}$$

Equivalent results apply for the function $V_{min}(b)$.

Proof. See Appendix. ■

Proposition 1, the argument in the proof of Lemma 2, and Theorem 1 lead to an immediately corollary.

Corollary 1 *(a) Consider the optimization problem (10) for $b \in (0, b^*_{max})$, and let $\mathbf{p^{max}}(b)$ denote the unique optimal solution of $V_{max}(b)$. For any $n \in \mathcal{N}$ we have*

$$x_n > V_{max}(b) \;\Leftrightarrow\; p_n^{max}(b) > \frac{1}{N} + b. \tag{16}$$

(b) Consider the optimization problem (11) for $b \in (0, b^*_{min})$, and let $\mathbf{p^{min}}(b)$ denote the unique optimal solution of $V_{min}(b)$. For any $n \in \mathcal{N}$

$$x_n > V_{min}(b) \;\Leftrightarrow\; p_n^{min}(b) < \frac{1}{N} + b. \tag{17}$$

Now, recall the notation we introduced earlier regarding the sets $\mathcal{N}_k, \mathcal{N}_k^+$, and $\mathcal{N}_k^-$ and their cardinalities. Consider the following two systems of nonlinear equations, which play a central role in the subsequent analysis.

System 1 $(Variables: \boldsymbol{C}^+, \boldsymbol{b}^+)$

Case 1: $k_N \geq 3$.

$$\frac{\sum_{n\in\mathcal{N}} x_n}{N} + C_1^+\sqrt{b_1^+} = \frac{\sum_{n\in\mathcal{N}_2^+} x_n}{N_2^+} + C_2^+\sqrt{N_2^+ b_1^+ - \frac{N_1^-}{N}} \tag{18}$$

$$\frac{C_1^+}{\sqrt{b_1^+}} = \frac{C_2^+ N_2^+}{\sqrt{N_2^+ b_1^+ - \frac{N_1^-}{N}}} \tag{19}$$

$$\frac{\sum_{n\in\mathcal{N}_k^+} x_n}{N_k^+} + C_k^+\sqrt{N_k^+ b_k^+ - \frac{N_{k-1}^-}{N}} = \frac{\sum_{n\in\mathcal{N}_{k+1}^+} x_n}{N_{k+1}^+} + C_{k+1}^+\sqrt{N_{k+1}^+ b_k^+ - \frac{N_k^-}{N}}, \quad k = 2, 3, ..., k_N - 2 \tag{20}$$

$$\frac{C_k^+ N_k^+}{\sqrt{N_k^+ b_k^+ - \frac{N_{k-1}^-}{N}}} = \frac{C_{k+1}^+ N_{k+1}^+}{\sqrt{N_{k+1}^+ b_k^+ - \frac{N_k^-}{N}}}, \quad k = 2, 3, ..., k_N - 2 \tag{21}$$

$$\frac{\sum_{n\in\mathcal{N}_{k_N-1}^+} x_n}{N_{k_N-1}^+} + C_{k_N-1}^+\sqrt{N_{k_N-1}^+ b_{k_N-1}^+ - \frac{N_{k_N-2}^-}{N}} = \max_{n\in\mathcal{N}} x_n \tag{22}$$

$$b_{k_N-1}^+ = b^*_{max} = \frac{1}{N_{k_N}^+} - \frac{1}{N} \tag{23}$$

Case 2: $k_N = 2$.

$$\frac{\sum_{n\in\mathcal{N}} x_n}{N} + C_1^+\sqrt{b_1^+} = \max_{n\in\mathcal{N}} x_n \tag{24}$$

$$b_1^+ = b^*_{max} = \frac{1}{N_2^+} - \frac{1}{N} \tag{25}$$

System 2 $(Variables:\ \boldsymbol{C}^{-}, \boldsymbol{b}^{-})$
Case 1: $k_N \geq 3$.

$$\frac{\sum_{n\in\mathcal{N}} x_n}{N} + C_1^- \sqrt{b_1^-} = \frac{\sum_{n\in\mathcal{N}^-_{k_N-1}} x_n}{N^-_{k_N-1}} + C_2^- \sqrt{N^-_{k_N-1} b_1^- - \frac{N^+_{k_N}}{N}} \tag{26}$$

$$\frac{C_1^-}{\sqrt{b_1^-}} = \frac{C_2^- N^-_{k_N-1}}{\sqrt{N^-_{k_N-1} b_1^- - \frac{N^+_{k_N}}{N}}} \tag{27}$$

$$\frac{\sum_{n\in\mathcal{N}^-_{k_N-k+1}} x_n}{N^-_{k_N-k+1}} + C_k^- \sqrt{N^-_{k_N-k+1} b_k^- - \frac{N^+_{k_N-k+2}}{N}} = \frac{\sum_{n\in\mathcal{N}^-_{k_N-k}} x_n}{N^-_{k_N-k}} + C_{k+1}^- \sqrt{N^-_{k_N-k} b_k^- - \frac{N^+_{k_N-k+1}}{N}}$$
$$k = 2, 3, ..., k_N - 2 \tag{28}$$

$$\frac{C_k^- N^-_{k_N-k+1}}{\sqrt{N^-_{k_N-k+1} b_k^- - \frac{N^+_{k_N-k+2}}{N}}} = \frac{C_{k+1}^- N^-_{k_N-k}}{\sqrt{N^-_{k_N-k} b_k^- - \frac{N^+_{k_N-k+1}}{N}}}, \quad k = 2, 3, ..., k_N - 2 \tag{29}$$

$$\frac{\sum_{n\in\mathcal{N}_{2^-}} x_n}{N_2^-} + C^-_{k_N-1} \sqrt{N_2^- b^-_{k_N-1} - \frac{N_3^+}{N}} = \min_{n\in\mathcal{N}} x_n \tag{30}$$

$$b^-_{k_N-1} = b^*_{min} = \frac{1}{N_1} - \frac{1}{N} \tag{31}$$

Case 2: $k_N = 2$.

$$\frac{\sum_{n\in\mathcal{N}} x_n}{N} + C_1^- \sqrt{b_1^-} = \min_{n\in\mathcal{N}} x_n \tag{32}$$

$$b_1^- = b^*_{min} = \frac{1}{N_1^-} - \frac{1}{N} \tag{33}$$

Proposition 3 *Systems 1 and 2 admit unique real-valued solutions* $(\boldsymbol{C}^+, \boldsymbol{b}^+)$ *and* $(\boldsymbol{C}^-, \boldsymbol{b}^-)$.

Proof. See Appendix. ■

We are now ready to state our second main result and provide a closed-form expression for functions $V_{max}(b)$ and $V_{min}(b)$.

Theorem 2 *Consider the optimization problems (10) and (11) and denote the unique solution of systems 1 and 2 by* $(\boldsymbol{C}^+, \boldsymbol{b}^+)$ *and* $(\boldsymbol{C}^-, \boldsymbol{b}^-)$*, respectively. The vectors* $\boldsymbol{b}^+$ *and* $\boldsymbol{b}^-$ *satisfy*

$$b_k^+ = b_{(k)}^{max} \quad k \in \{1, 2, ..., k_N - 1\}$$
$$b_k^- = b_{(k)}^{max}, \quad k \in \{1, 2, ..., k_N - 1\}$$

where $b_{(k)}^{max}$ and $b_{(k)}^{min}$ are defined in Eqs. (13)-(14). The functions $V_{max}(b)$ and $V_{min}(b)$ are equal to

$$V_{max}(b)=\begin{cases}\frac{\sum_{n\in\mathcal{N}}x_n}{N}+C_1^+\sqrt{b} & b\in[0,b_1^+)\\ \frac{\sum_{n\in\mathcal{N}_k^+}x_n}{N_k^+}+C_k^+\sqrt{N_k^+b-\frac{N_{k-1}^-}{N}} & b\in\left[b_{k-1}^+,b_k^+\right),\ k=2,3,...,k_N-1\\ \max_{n\in\mathcal{N}}x_n & b\in\left[b_{k_N-1}^+,\frac{N-1}{N}\right],\end{cases}\tag{34}$$

$$V_{min}(b)=\begin{cases}\frac{\sum_{n\in\mathcal{N}}x_n}{N}+C_1^-\sqrt{b} & b\in[0,b_1^-)\\ \frac{\sum_{n\in\mathcal{N}_{k_N-k+1}^-}x_n}{N_{k_N-k+1}^-}+C_k^-\sqrt{N_{k_N-k+1}^-b-\frac{N_{k_N-k+2}^+}{N}} & b\in\left[b_{k-1}^-,b_k^-\right),\ k=2,3,...,k_N-1\\ \min_{n\in\mathcal{N}}x_n & b\in\left[b_{k_N-1}^-,\frac{N-1}{N}\right].\end{cases}\tag{35}$$

Proof. See Appendix. ■

Theorem 2 shows that V_{max} and V_{min} will be concatenations of properly-defined square-root functions. These concatenations occur at levels of ambiguity $\boldsymbol{b}_{(\cdot)}^{\boldsymbol{max}},\boldsymbol{b}_{(\cdot)}^{\boldsymbol{min}}$ defined by Eqs. (13)-(14), which can be computed through Systems 1 and 2.

We verify and illustrate Theorem 2 for the simple case in which $k_N=2$. We focus on V_{max} as the argument for V_{min} is completely symmetric. That $V_{max}(b)=\max_{n\in\mathcal{N}}$ for $b\geq b_{max}^*$ follows by Lemma 1 so we proceed by considering $b\in[0,b_{max}^*)$. In this case, by first principles it is easy to see that the optimal solution of (10) will increase the probability share of all experts $n\in\mathcal{N}_2$ by an equal amount ϵ, which in turn will be offset by a uniform decrease in the probability shares of experts $n\in\mathcal{N}_1$. Since by Proposition 2 the quadratic ambiguity constraint will bind, ϵ must satisfy

$$N_2\frac{\epsilon^2}{N_2^2}+(N-N_2)\left(\frac{\epsilon N_2}{N_1}\right)^2=b\ \Rightarrow\ \epsilon=\sqrt{\frac{N_1b}{NN_2}}.$$

Thus, we may deduce that

$$\begin{aligned}V_{max}(b)&=N_2\left(\frac{1}{N}+\sqrt{\frac{N_1b}{NN_2}}\right)\max_n x_n+N_1\left(\frac{1}{N}-\frac{N_2}{N_1}\sqrt{\frac{N_1b}{NN_2}}\right)\min_n x_n\\&=\frac{\sum_{n\in\mathcal{N}}x_n}{N}+\sqrt{\frac{N_1N_2}{N}}\left(\max_{n\in\mathcal{N}}x_n-\min_{n\in\mathcal{N}}x_n\right)\sqrt{b},\quad b\in[0,b_{max}^*).\end{aligned}\tag{36}$$

On the other hand, solving for C_1^+ in Eqs. (24) and (25), we obtain

$$C_1^+=\sqrt{\frac{N_1N_2}{N}}\left(\max_{n\in\mathcal{N}}x_n-\min_{n\in\mathcal{N}}x_n\right),$$

so that Theorem 2 implies

$$V_{max}(b)=\frac{\sum_{n\in\mathcal{N}}x_n}{N}+\sqrt{\frac{N_1N_2}{N}}\left(\max_{n\in\mathcal{N}}x_n-\min_{n\in\mathcal{N}}x_n\right)\sqrt{b},\quad b\in[0,b_{max}^*),\tag{37}$$

which is consistent with Eq. (36).

4 Empirical Application to Solar R&D

During the course of 2010-2011, the ICARUS survey collected expert judgments on future costs and technological barriers of different Photovoltaic (PV) and Concentrated Solar Power (CSP) technologies.[5] Sixteen leading European experts from academia, the private sector, and international institutions took part in the survey. The elicitation collected probabilistic information on (1) the year-2030 expected cost of the technologies; (2) the role of public European Union R&D investments in affecting those costs; and (3) the potential for the deployment of these technologies (both in OECD and non-OECD countries). We refer readers interested in the general findings of the survey to Bosetti et al. [5] and we focus here on the data on future costs as they form the basis of our analysis.

Computing the subjective probability distributions. Current 5-year EU R&D investment in solar technology is estimated at 165 million US dollars. The ICARUS study elicited the probabilistic estimates of the 16 experts on the 2030 solar electricity cost (2005 c$/kWh) under three future *Scenarios*: (1) keeping current levels of R&D constant until 2030, (2) increasing them by 50%, and (3) increasing them by 100%. Experts were asked to provide values for the 10th, 50th, and 90th percentile of their distributions for the 2030 cost of solar technology conditional for all three scenarios. In addition, they were asked to provide values for the probability of this cost being less than or equal to the following three values: 11.3, 5.5, and 3c$/kWh. These "threshold" cost levels correspond to projections of the costs of electricity from fossil fuels or nuclear in 2030. The first (11.27 c$/kWh) corresponds to the 2030 projected cost of electricity from traditional coal power plants in the presence of a specific policy to control CO2 emissions (thus effectively doubling electricity costs from fossil sources). The second threshold cost (5.5 c$/kWh) is the projected cost of electricity from traditional fossil fuels in 2030, without considering any carbon tax. Finally, the third (3 c$/kWh) reflects a situation in which solar power becomes competitive with the levelized cost of electricity from nuclear power.

Asking experts the follow up question on the likelihood of reaching threshold cost targets allowed the survey authors to guard against the cognitive pitfalls associated with direct elicitation of subjective probabilities, to increase the amount of elicited information, and to deepen the discussion with the expert, hence improving their perception of the experts' beliefs. In cases where the two

[5] The survey is part of a 3-year ERC-funded project on innovation in carbon-free technologies (ICARUS - Innovation for Climate chAnge mitigation: a study of energy R&D, its Uncertain effectiveness and Spillovers www.icarus-project.org).

sets of answers (percentile values and threshold probabilities) were inconsistent, experts were asked follow-up questions in order to obtain coherent estimates. Finally, each expert was requested to give values for the upper and lower limits of his/her distribution's support in order to pinpoint the intervals over which his/her implied probability distributions range.

Such corrected estimates were obtained from 14 out of the original 16 experts, and therefore the analysis that follows focuses solely on them. Among the respondents, not all provided values on the left and right endpoints of their distributions' support. As a result, we deduced between 6 and 8 points of 14 experts' cumulative distribution functions (cdf) of the 2030 cost of solar electricity, given the aforementioned three R&D investment Scenarios. From these points a probability distribution function (pdf) was constructed using linear interpolation in the following way. First of all, and in accordance with the experts' answers, we considered cost levels c lying in [2c$/kWh, 30c$/kWh] and discretized this interval on a scale of 0.5. Now, suppose an expert reported the values of his/her cdf F_n at two successive points c_1 and c_2 where $c_2 > c_1$ and gave no further information on cost levels between c_1 and c_2. Assuming right-continuity of F_n we took the probability mass $F_n(c_2) - F_n(c_1)$ to be distributed uniformly among the cost levels $\{c_1 + .5, c_1 + 1, ..., c_2\}$. For experts who did not provide values for the lower limit of their distribution's support we assumed that whatever probability mass remained to be allocated (always less than .1) was distributed uniformly between the smallest argument of the cdf and two cost levels below it. For example, if an expert only indicated that c_l was his y'th percentile and gave no further points of the cdf below this, we assumed that a probability mass of y was distributed evenly across $\{c_l - 1, c_l - .5, c_l\}$. In the case of an unknown upper limit, if an expert only indicated that c_u was his yth percentile and gave no further arguments for the cdf above it, we assumed that a probability mass of $1 - y$ was distributed evenly across $\{c_u + .5, c_u + 1\}$.

Following this procedure we arrived at a probability distribution function (pdf) for each expert $n \in \{1, 2, ..., 14\}$, given the three relevant levels of R&D investment (Scenarios) denoted by $r \in \{r_1, r_2, r_3\}$ (here r_i refers to Scenario i). We use these pdfs as our subjective probability distributions of the cost of technology as denoted in Eq. (1). Figure 1 plots the cumulative distribution functions (cdfs) implied by these pdfs as well as the cdf that the aggregate pdf (2) leads to, under all three Scenarios.

[FIGURE 1 here]

As one can see in Figure 1 there is considerable disagreement between experts over the potential of solar technology. This disagreement is particularly acute under Scenario 1, and diminishes as R&D

levels increase. Nonetheless, the breakthrough nature of innovation and the need to cross certain firm cost thresholds, means that ambiguity in expert estimates remains an important concern, even under Scenario 3.

Determining the breakthrough cost level. Expected returns on solar technology R&D investments are quantified via a general equilibrium intertemporal model of the economy that can account for a range of macro-economic feedbacks and interactions. These include the effects of energy and climate change policies, the competition for innovation resources with other power technologies, the effect of growth, as well as a number of other factors.[6] The integrated assessment model (IAM) is run over the whole range of possible 2030-costs of solar power we are considering (recall that our cost domain is $D = \{2, 2.5, 3, ..., 29, 29.5, 30\}$). Subsequently, simulation results are compared to the benchmark case in which the cost of solar power is so high that the technology is not competitive with alternative production modes. For each possible 2030 solar-power cost, the payoff to the EU is quantified by the discounted EU-consumption improvement with respect to the case where solar technology is not competitive. Table 1 summarizes the results.

2030 solar-power cost (2005 USc$/kWh)	Payoff (US$ 10^9)
2	160
2.5	143
3	125
3.5	108
4	90
4.5	73
5	54
5.5	36
6	17
6.5	16
7	7
7.5	1
8	0.5
> 8	0

Table 1: EU discounted consumption improvement as a function of 2030 solar-power cost

[6]The analysis is carried out using the World Induced Technical Change Hybrid (WITCH) model (Bosetti et al. [6]), an energy-economy-climate model that has been used extensively for economic analysis of climate change policies. See www.witchmodel.org for a list of applications and papers.

Three important assumptions are at the basis of the numbers reported here. First, as the survey concentrated on public EU R&D investment and the effects of increasing it, we disregard spillovers and technological transfers to the rest of the world and consider only the consumption improvement for Europe. Second, we evaluate the benefit of alternative 2030 costs of solar power assuming that no carbon policy is in place and that no special constraints on other technologies are imposed (e.g., a partial ban on nuclear technology). Third, we are assuming a specific value for the discount rate. Although our choice is well in the range of discount rates adopted for large scale public projects, it is important to note that the cost threshold for positive payoffs is robust for a wide range of more myopic discount rate values.[7] In addition, the first two assumptions are in the direction of minimizing the benefits of R&D investments so that our results would be (most probably substantially) reinforced when one or both assumptions are relaxed.

Averaging across experts. We begin our analysis by focusing on the aggregate probability distribution described by Eq. (2) as applied to our context. We measure the effectiveness of R&D through a number of different metrics and their expected values: technology cost, probability of breakthrough, payoff $g(C)$ (measured as EU discounted consumption improvement, as per Table 1), and internal rate of return IRR(C). Table 2 summarizes our results.

Metric $f(C,r)$	Expected Value	Units	Scenario 1 (r_1)	Scenario 2 (r_2)	Scenario 3 (r_3)
C	$\mathbf{E}[C]$	USc$/kWh	11.67	9.58	7.76
$\mathbf{1}\{C \leq 7.5\}$	$\mathbf{Pr}[C \leq 7.5]$	pure number	.146	.341	.530
$g(C)$	$\mathbf{E}[g(C)]$	US$ 10^9	2.8	8.6	19.2
IRR(C)	$\mathbf{E}$[IRR(C)]	pure number	.056	.124	.235

Table 2: Expected R&D effectiveness under Scenarios 1,2, and 3, as measured by different metrics, using aggregate pdf (2).

Given that the maximum (discounted) R&D expenditures we consider (under Scenario 3) are approximately equal to 1.4 10^9 US$[8] it is clear that all three Scenarios imply a positive net return on investment. Doubling current R&D efforts results in an expected internal rate of return greater than 20%, much larger than the 5.6% return for the status quo level of investments. However, a word of caution is warranted. Although we decided to adopt a conservative set of assumptions in

[7]We discount cash flows using a 3% discount rate. Although using higher discount rates, say 5%, would obviously lead to lower cash flows, results would not change in qualitative terms as the threshold for positive payoffs remains 8c$/kWh.

[8]We arrive at this number by considering Scenario 3's implied 5-year levels of R&D expenditure (330 million 2005US$) and taking their discounted (at 3%) sum over 5 time periods.

evaluating these alternative investment prospects, we should stress that we are assessing them from a social planner, rather than firm level, perspective. Hence, it is relatively unsurprising that the long-term social benefits of a possible breakthrough in solar technology significantly outweigh their cost. That said, public R&D spending has to face tough competition from other public programs, as opportunity costs are large. In order to decide to double spending, policy makers might decide not to trust their average effectiveness as captured by (2), but rather consider alternative possibilities, e.g. by weighting more conservative or pessimistic experts, in order to avoid planning fallacies. They may therefore wish to use a tool that helps them grasp the full picture of their investment decisions, rather than considering one single probabilistic estimate. This is what our simple decision-analysic framework aims at.

Modeling ambiguity. Up to this point we have simply averaged across experts and used the average distribution of Eq. (2) to obtain the results presented in Table 2. We now shift the focus of our analysis to explicitly account for ambiguity in expert opinion and adopt the decision-theoretic model introduced in Section 2. We focus on a specific metric, i.e. the probability of a breakthrough in the cost of solar electricity, as described by Eq. (5). However, we note that an equivalent analysis can obviously be undertaken for any other metric f, such as for example those quoted in Table 2.

As we observed earlier in Table 1, our IAM analysis suggests that, under a set of conservative assumptions, a breakthrough takes place once the cost of solar electricity reaches a threshold of z =7.5 c$/kWh. Once this cost target has been attained and surpassed, solar technology begins to be adopted in a big way, generating huge gains in EU welfare and consumption.

So let us see how different levels of R&D investment fare in helping the EU attain this breakthrough cost target, given the inherent ambiguity of experts' estimates.[9] To wit, Figure 2 plots best- and worst-case breakthrough probabilities, as given by Eqs. (7) and (8) applied to metric (5) for $z = 7.5$c$/kWh. We focus first on Scenario 1 and verify that, in agreeement with Table 2, pure aggregation of expert opinion (corresponding to an ambiguity $b = 0$) yields a breakthrough probability of approximately 0.15. Subsequently we see that the worst-case probability reaches its lowest point of 0 relatively quickly at $b \approx .2$ ($p^*(b;N) \approx 50\%$), whereas the best-case one peaks at 0.45 at $b \approx 0.3$ ($p^*(b;N) \approx 60\%$). This means that all uncertainty regarding the worst- and best-cases has been resolved once b exceeds 0.3: at that level of ambiguity the tradeoffs become completely clear and a decision-maker with preferences consistent to Eq. (9) will not care to consider higher levels of ambiguity. Under Scenario 2, the unambiguous probability of breakthrough rises to a value of about

[9]We perform all computations in Mathematica.

0.34. Subequently, we see that the worst-case probability of breakthrough drops sharply between $b = 0$ and $b = 0.15$ ($p^*(b; N) \approx 44\%$) from 0.34 to around .04, at which point it keeps decreasing at a small rate until it reaches zero at around $b \approx .4$. Conversely the best-case probability rises sharply to about 0.7 for $b \approx .2$ ($p^*(b; N) \approx 50\%$) at which point it continues to rise at a smaller rate until it reaches a value of 0.9 at the maximum ambiguity level of $b = 13/14$. Under Scenario 3 the unambiguous probability is around 0.53, significantly higher than both other Scenarios. The worst-case probability drops relatively smoothly to a minimum value of 0.05 for $b = 13/14$, whilst the best-case one rises to 0.9. It is evident that ambiguity is still important under Scenario 3, for both the worst- and best-case probability of breakthrough, albeit less so than under Scenarios 1 and 2. This fact is particularly interesting in light of Panel c in Figure 2, which shows experts' cdfs clustered relatively close to one another.

[FIGURE 2 here]

We bring attention to the three points of the best and worst-case curves of Figure 2 at which they intersect each other, denoted by P1, P2, and P3. Denote the x-coordinates of these points by $\hat{b}_1, \hat{b}_2$, and $\hat{b}_3$ respectively. At point P1 the best-case breakthrough probability under Scenario 1 is equal to the worst-case probability under Scenario 2. This implies that for levels of ambiguity satisfying $b \leq \hat{b}_1 \approx .015$ ($p^*(b; N) \approx 19\%$) the investment decision of Scenario 1 can **never** be preferable to that of Scenario 2. Conversely, at point P2 the best-case probability of a breakthrough under Scenario 2 is equal to the worst-case one under Scenario 3. Thus, for levels of ambiguity satisfying $b \leq b_2 \approx .012$ ($p^*(b; N) \approx 17\%$) investment under Scenario 3 is unambiguously preferred to that of scenario 2. Finally, point P3 suggests that investment under scenario 3 will always be preferable to that of 1 for ambiguity $b \leq b_3 \approx .089$ ($p^*(b; N) \approx 36\%$).

We briefly comment on the potential policy implications of Figure 2. Policy makers are interested in a breakthrough that would make solar technology competitive with most other power technologies. Let us suppose that probability of success of 20% represents a kind of a minimum cutoff point that any public investment should satisfy. Under the aggregate distribution (2), the investment decision of Scenario 1 would fail to meet this requirement. At this point one could conceivably claim that by only looking at the probability averaged across experts one is being too severe with Scenario 1. Looking at Figure 2, however, these reservations are rapidly dispelled. Even for an ambiguity level $b = .3$, which implies the possibility of granting to a single expert a weight of almost 60%, the best-case probability of breakthrough is only around 40%. Conversely, the worst-case probability drops to zero much more rapidly. Thus the decision to reject this R&D program on the basis of the 20% probability of success threshold seems robust. In Scenario 2, focusing on the simple average can indeed be misleading, as the aggregatte probability of breakthrough of around 35% could lead to overly optimistic conclusions. For this level of R&D expenditure, expert consensus is very small and allowing for even small levels of ambiguity leads to very different policy recommendations, depending on the decision maker's ambiguity attitude. On the other hand, the picture is much clearer under Scenario 3. Here, the worst-case breakthrough probability decreases much more slowly so that it falls below 20% only for $b \geq 0.27$ ($p^*(b; N) \approx 57\%$), at which point the respective best-case probability exceeds 75%. Hence the 20% probability of success threshold seems to be a relatively safe bet under Scenario 3.

We now consider the comprehensive effect of Scenarios 1, 2, and 3 given all possible levels of ambiguity b and ambiguity attitude α. Figure 3 plots the value function (9) for all three Scenarios over all $b \in [0, (N-1)/N]$ and $\alpha \in [0, 1]$. This allows policy makers to visualize the effects of all three R&D investment decisions on the probability of breakthrough over the entire range of possible ambiguity levels and ambiguity attitudes. The picture that emerges confirms our earlier claims. Scenario 3 fares much better than both 1 and 2 over a very wide range of b and α. Moreover, it results in significantly less variation across α than Scenario 2, suggesting that it represents a more robust policy decision.

[FIGURE 3 here]

We conclude this section with a necessary qualification of our results. Focusing purely on breakthrough probabilities as a metric of R&D effectiveness means that we are only considering the potential benefits of R&D investments, and not their costs. It is thus no surprise that Scenario 2 outperforms Scenario 1 and Scenario 3 outperforms both 1 and 2. In theory, different metrics that explicitly consider the cost of R&D investments could very well yield more complex tradeoffs and therefore more complex versions of Figures 2 and 3. However, for the application at hand, even considering metrics such as net payoff, as defined in Eq. (6), would yield results qualitatively similar to those presented above. The main reason behind this fact is that, as we mentioned earlier, the cost of the R&D programs, even under the more aggressive Scenario 3, is greatly exceeded by the potential long-term benefits to society of reducing the cost of solar power (recall Table 1).

5 Conclusion

Determining the optimal portfolio of government R&D is an important task, especially at times of public funding scarcity. As R&D programs imply uncertain returns, it is important to assess these investments using probabilistic tools. Expert elicitation can play an important role in this process if used to capture in a transparent and objective way subjective probabilities that can be used as scientific data. At the same time, gathered information can vary substantially across experts. In particular, if the elicitation is designed correctly is should exactly aim at covering all prevailing "visions" about that specific technology. The different backgrounds and perspectives that experts bring to the elicitation process imply that collected subjective probability distributions could, more often than not, span a wide spectrum.

Averaging across experts' probability distributions is a widely-used procedure in addressing such divergence in expert opinion. Still, condensing all of the problem's uncertainty into one single average probability distribution, especially in cases where Bayesian learning methods cannot be applied, may conceal important imformation and yield policy recommendations that are not robust. To deal with this issue, in this paper we proposed and analyzed a simple decision-theoretic framework inspired by the economic-theory literature on ambiguity aversion. Our model helps decision makers visualize the full range of outcomes associated with a particular R&D investment and make informed decisions given their particular beliefs about, and attitude towards, ambiguous expert opinion. We applied our framework to original data from a recent expert elicitation survey on solar technology. The analysis suggested that more aggressive investment in solar technology R&D is likely to yield substantial benefits even (or rather especially) after ambiguity over expert opinion has been taken into account.

We conclude by noting that, while this paper has been motivated by the issue of R&D allocation, the model and analysis presented herein are general and can be applied to any context of decision making under ambiguity.

Appendix

Proof of Proposition 1. We prove the Proposition for $V_{max}(b)$ (the argument for $V_{min}(b)$ is exactly analogous). That $V_{max}(b)$ is increasing in b follows by definition. Consider the optimization problems given by the right-hand-side of Eq. (10) for $b_1 \in [0, \frac{N-1}{N}]$ and $b_2 \geq b_1$ and denote their optimal solutions by $\mathbf{p^{max}}(b_1)$ and $\mathbf{p^{max}}(b_2)$ respectively. By feasibility we may note the following:

$$\sum_{n=1}^{N}\left(p_n^{max}(b_1)-\frac{1}{N}\right)^2 \leq b_1, \quad \sum_{n=1}^{N}\left(p_n^{max}(b_2)-\frac{1}{N}\right)^2 \leq b_2. \tag{38}$$

Consider a convex combination of b_1 and b_2 given by $b(\lambda) = \lambda b_1 + (1-\lambda)b_2$ for some $\lambda \in [0,1]$ and the optimization problem

$$V_{max}(b(\lambda)) = \max_{\mathbf{p}\in \mathrm{P}(b(\lambda))} \sum_{n=1}^{N} p_n x_n. \tag{39}$$

To prove concavity of V_{max} in b it suffices to show that

$$V_{max}(b(\lambda)) \geq \lambda V_{max}(b_1) + (1-\lambda)V_{max}(b_2).$$

To this end, consider the probability vector given by

$$\mathbf{p}(\lambda) = \lambda \mathbf{p^{max}}(b_1) + (1-\lambda)\mathbf{p^{max}}(b_2).$$

Dy feasibility of $\mathbf{p^{max}}(b_1)$ and $\mathbf{p^{max}}(b_2)$ we immediately deduce that $\mathbf{p}(\lambda) \geq \mathbf{0}$ and that $\sum_{n=1}^{N} p_n(\lambda) = 1$. Now we may write

$$\begin{aligned}
\sum_{n=1}^{N}\left(p_n(\lambda)-\frac{1}{N}\right)^2 &= \sum_{n=1}^{N}\left(\lambda\left(p_n^{max}(b_1)-\frac{1}{N}\right)+(1-\lambda)\left(p_n^{max}(b_2)-\frac{1}{N}\right)\right)^2 \\
&\overset{\text{triangle ineq.}}{\leq} \left[\lambda\left(\sum_{n=1}^{N}\left(p_n^{max}(b_1)-\frac{1}{N}\right)^2\right)^{\frac{1}{2}}+(1-\lambda)\left(\sum_{n=1}^{N}\left(p_n^{max}(b_2)-\frac{1}{N}\right)^2\right)^{\frac{1}{2}}\right]^2 \\
&\overset{(38)}{\leq} \left[\lambda\sqrt{b_1}+(1-\lambda)\sqrt{b_2}\right]^2 \overset{\text{concavity of } \sqrt{\cdot}}{\leq} \left[\sqrt{\lambda b_1+(1-\lambda)b_2}\right]^2 = b(\lambda).
\end{aligned} \tag{40}$$

By Eq. (40) and the observations immediately preceding it we can conclude that $\mathbf{p}(\lambda)$ is feasible for optimization problem (39). Thus we may write

$$\begin{aligned}
V_{max}(b(\lambda)) \geq \sum_{n=1}^{N} p(\lambda)_n x_n &= \lambda\sum_{n=1}^{N} p_n^{max}(b_1)x_n + (1-\lambda)\sum_{n=1}^{N} p_n^{max}(b_2)x_n \\
&= \lambda V_{max}(b_1) + (1-\lambda)V_{max}(b_2),
\end{aligned}$$

where the last equality follows from the assumed optimality of $\mathbf{p^{max}}(b_1)$ and $\mathbf{p^{max}}(b_2)$. We now proceed to show continuity. By concavity $V_{max}(b)$ will be continuous on the open interval $(0, \frac{N-1}{N})$ so we need only consider the endpoints 0 and $\frac{N-1}{N}$. Since $V_{max}(b)$ is increasing in b we must have $\lim_{b\to(\frac{N-1}{N})^-} V_{max}(b) \leq V_{max}(\frac{N-1}{N})$. However, if $\lim_{b\to(\frac{N-1}{N})^-} V_{max}(b) < V_{max}(\frac{N-1}{N})$ then we reach a contradiction if we apply concavity to $(N-1)/N$ and other values of b.

To prove continuity at $b = 0$ consider an $\epsilon > 0$. Now let $\delta > 0$ and write

$$\begin{aligned} |V_{max}(\delta) - V_{max}(0)| &= V_{max}(\delta) - V_{max}(0) = \sum_{n=1}^{N} \left(p_n^{max}(\delta) - \frac{1}{N} \right) x_n \\ &\leq \max_{n\in\mathcal{N}} |x_n| \sum_{n=1}^{N} \left| p_n^{max}(\delta) - \frac{1}{N} \right| \\ &\overset{\text{Hölder's ineq.}}{\leq} \max_{n\in\mathcal{N}} |x_n| \left[\sum_{n=1}^{N} \left(p_n^{max}(\delta) - \frac{1}{N} \right)^2 \right]^{\frac{1}{2}} \leq \max_{n\in\mathcal{N}} |x_n| \sqrt{\delta}. \end{aligned}$$

Thus, any choice of $0 < \delta < \frac{\epsilon^2}{(\max_{n\in\mathcal{N}} |x_n|)^2}$ will ensure that $|V_{max}(\delta) - V_{max}(0)| < \epsilon$, completing the proof.

■

Proof of Lemma 1. The function $V_{max}(b)$ is bounded above by x_n for any $n \in \mathcal{N}_{k_N}$. This upper bound is attained by a probability vector $\mathbf{p}$ if and only if it satisfies

$$\sum_{n\in\tilde{\mathcal{N}}} p_n = 1, \quad \text{for some } \tilde{\mathcal{N}} \subseteq \mathcal{N}_{k_N} \tag{41}$$

Denote by $\tilde{P}$ the set of all probability vectors satisfying (41). Consider the vector $\mathbf{p}^* \in \tilde{P}$ satisfying (recall that $|\mathcal{N}_k| = N_k$)

$$p_n^* = \begin{cases} \frac{1}{N_{k_N}} & n \in \mathcal{N}_{k_N} \\ 0 & \text{otherwise,} \end{cases} \tag{42}$$

By convexity of the ambiguity constraint it is clear that $\mathbf{p}^* = \operatorname{argmin}_{p\in\tilde{P}} \sum_{n=1}^{N} \left(p_n - \frac{1}{N} \right)^2$, so that

$$\begin{aligned} \min_{p\in\tilde{P}} \sum_{n=1}^{N} \left(p_n - \frac{1}{N} \right)^2 &= \sum_{n\in\mathcal{N}_{k_N}} \left(\frac{1}{N_{k_N}} - \frac{1}{N} \right)^2 + \sum_{n\notin\mathcal{N}_{k_N}} \frac{1}{N^2} \\ &= N_{k_N} \left(\frac{1}{N_{k_N}} - \frac{1}{N} \right)^2 + (N - N_{k_N}) \frac{1}{N^2} = \frac{1}{N_{k_N}} - \frac{1}{N} \equiv b_{max}^*. \end{aligned}$$

It is clear that for $b > b_{max}^*$ this same vector $\mathbf{p}^*$ will remain feasible for optimization problem (10), and thus optimal. Hence $V_{max}(b)$ will be equal to $\max_{n\in\mathcal{N}} x_n$ on $b \geq b_{max}^*$. Now consider $b < b_{max}^*$ and the optimal solution $\mathbf{p^{max}}(b)$. As $b < b_{max}^*$ there must exist a $j \neq \mathcal{N}_{k_N}$ such that $p_j^{max}(b) > 0$. Now consider increasing b by an amount ϵ. For $\delta > 0$ small enough the solution $\mathbf{\tilde{p}^{max}}$ in which $\tilde{p}_j = p_j^{max} - \delta$ and $\tilde{p}_k = p_k^{max} + \delta$ for some $k \in \mathcal{N}_{k_N}$ will be feasible and result in a strictly greater objective value, so that $V_{max}(b + \epsilon) > V_{max}(b)$. Equivalent reasoning applies to the V_{min} case. ■

Proof of Proposition 2. Suppose first that $b = b^*_{max}$. It is clear here that the unique optimal solution is given by $\mathbf{p^{max}}$ such that $p^{max}_n = 1/N_{k_N}$ for all $n \in \mathcal{N}_{k_N}$ and $p^{max}_n = 0$ otherwise. The quadratic ambiguity constraint binds by the definition of b^*_{max}.

Consider now the case $b < b^*_{max}$ and suppose there exists an optimal solution $\mathbf{p^{max}}$ such that the quadratic ambiguity constraint is slack. As $b < b^*_{max}$ there must exist an $j \neq \mathcal{N}_{k_N}$ such that $p^{max}_j(b) > 0$. For $\epsilon > 0$ small enough the solution $\mathbf{\tilde{p}^{max}}$ in which $\tilde{p}_j = p^{max}_j - \epsilon$ and $\tilde{p}_k = p^{max}_k + \epsilon$ for some $k \in \mathcal{N}_{k_N}$ will be feasible and result in a strictly greater objective value, contradicting $\mathbf{p^{max}}$'s optimality. Thus, all optimal solutions must satisfy the quadratic ambiguity constraint with equality.

We now prove uniqueness. Suppose there exist two optimal solutions $\mathbf{p^{max,1}}$ and $\mathbf{p^{max,2}}$. By the preceding argument they must bind the quadratic ambiguity constraint. Consider the set of probability vectors given by their convex combinations

$$\mathbf{p}(\lambda) = \lambda \mathbf{p^{max,1}} + (1-\lambda)\mathbf{p^{max,2}}, \quad \lambda \in [0,1].$$

For $\lambda \in (0,1)$, $\mathbf{p}(\lambda)$ will satisfy the ambiguity constraint with strict inequality, since:

$$\begin{aligned}
\sum_{n=1}^{N}\left(p_n(\lambda) - \frac{1}{N}\right)^2 \quad &= \quad \sum_{n=1}^{N}\left(\lambda\left(p^{max,1}_n - \frac{1}{N}\right) + (1-\lambda)\left(p^{max,2}_n - \frac{1}{N}\right)\right)^2 \\
&\overset{\text{strict convexity}}{<} \quad \sum_{n=1}^{N}\left[\lambda\left(p^{max,1}_n - \frac{1}{N}\right)^2 + (1-\lambda)\left(p^{max,2}_n - \frac{1}{N}\right)^2\right] \\
&= \quad \lambda\sum_{n=1}^{N}\left(p^{max,1}_n - \frac{1}{N}\right)^2 + (1-\lambda)\sum_{n=1}^{N}\left(p^{max,2}_n - \frac{1}{N}\right)^2 \\
&= \quad \lambda b + (1-\lambda) b = b.
\end{aligned}$$

Thus all solutions $\mathbf{p}(\lambda)$ are feasible. That they are optimal follows trivially by the assumed optimality of $\mathbf{p^{max,1}}, \mathbf{p^{max,2}}$ and the linear objective function of (10). But this is a contradiction as all optimal solutions must satisfy the quadratic ambiguity constraint with equality. ■

Proof of Lemma 2. Suppose $b_1 \leq b^*_{max}$, consider $V_{max}(b_1)$'s unique optimal solution $\mathbf{p^{max}}(b_1)$, and define the set $\mathcal{N}_0(b_1) = \{n \in \mathcal{N} : \; p^{max}_n(b_1) = 0\}$. By optimality, we must have that

$$\max_{n \in \mathcal{N}_0(b_1)} x_n < \min_{n \notin \mathcal{N}_0(b_1)} x_n. \tag{43}$$

If the above were not true we could pick an expert $n \notin \mathcal{N}_0(b_1)$ satisfying $x_n < \max_{n \in N_0(b_1)} x_n$ and give his entire probability mass to the expert in $\mathcal{N}_0(b_1)$ with the maximum x_n. This would maintain feasibility and strictly increase the objective function value. Eq. (43) implies that for values $b > b_1$, which represent enlargements of the feasible set and thus admit solutions that assign zero probability to all $n \in \mathcal{N}_0(b_1)$, it will never be optimal to assign positive probability to experts in $\mathcal{N}_0(b_1)$. ∎

Proof of Lemma 3. Follows immediately from Eq. (43) and the reasoning in the proof of Lemma 2. ∎

Proof of Theorem 1. We prove the result for V_{max}; the argument for V_{min} is exactly analogous. For convenience, we begin by restating the optimization problem (10):

$$\begin{aligned} V_{max}(b) = \max_{\mathbf{p}} \quad & \sum_{n=1}^{N} x_n p_n \\ \text{subject to:} \quad & \sum_{n=1}^{N} \left(p_n - \frac{1}{N}\right)^2 \leq b \\ & \sum_{n=1}^{N} p_n = 1 \\ & \mathbf{p} \geq \mathbf{0}. \end{aligned} \tag{44}$$

Introducing Lagrangian multipliers we write the Karush-Kuhn-Tucker (KKT) conditions:

$$x_n - 2\lambda\left(p_n - \frac{1}{N}\right) + \mu + \nu_n = 0, \;\; n \in \{1, 2, ..., N\} \tag{45}$$

$$\lambda\left(\sum_{n=1}^{N}\left(p_n - \frac{1}{N}\right)^2 - b\right) = 0, \;\; \lambda \geq 0 \tag{46}$$

$$\sum_{n=1}^{N}\left(p_n - \frac{1}{N}\right)^2 \leq b, \;\; \sum_{n=1}^{N} p_n = 1, \;\; \mathbf{p} \geq \mathbf{0} \tag{47}$$

$$\nu_n p_n = 0, \;\; \nu_n \geq 0, \;\; n \in \{1, 2, ..., N\}. \tag{48}$$

Since our problem is concave with affine equality constraints and satisfies Slater's condition (see section 5.2.3 in [7]), strong duality holds and the KKT conditions (45)-(48) will be necessary and sufficient for both primal and dual optimality. In other words, the duality gap is zero and the vector $(\mathbf{p}^*, \boldsymbol{\nu}^*, \lambda^*, \mu^*)$ satisfies (45)-(48) if and only $\mathbf{p}^*$ and $\lambda^*, \boldsymbol{\nu}^*, \mu^*$ are primal and dual optimal respectively (see section 5.5.3 in [7]).

From Proposition 2 we know that there exists a unique primal optimal solution $\mathbf{p}^*$. Since strong duality holds, the Lagrangean dual problem admits an optimal solution, and we refer to it by $\lambda^*, \boldsymbol{\nu}^*, \mu^*$. We now argue that this dual optimal solution will also be unique. Since $b < b^*_{max}$

there must exist at least two experts n_1 and n_2 such that $\min\{p^*_{n_1}, p^*_{n_2}\} > 0$ and $p^*_{n_1} \neq p^*_{n_2}$. Thus, the complementary slackness conditions (48) immediately imply $\nu^*_{n_1} = \nu^*_{n_2} = 0$. Applying Eq. (45) to $n = n_1$ and $n = n_2$ yields a nondegenerate linear system of two equations with two unknowns, which uniquely define the optimal dual variables λ^* and μ^*.[10] Subsequently, applying Eqs. (45) and (48) to all $n \notin \{n_1, n_2\}$ uniquely determines ν^*_n for all $n \notin \{n_1, n_2\}$.

With the above in mind, consider $b \in (0, b^*_{max})$ and let $(\mathbf{p^{max}}(b), \boldsymbol{\nu^{max}}(b), \lambda^{max}(b), \mu^{max}(b))$ denote the unique primal and dual optimal solutions for problem (44). Since the primal problem is a quadratically constrained quadratic program (QCQP) its Lagragean dual will admit a similar structure as the one presented in Equation 5.29 in [7] and its objective function is linear and thus convex and differentiable in b. The above facts, in combination with the uniqueness of the optimal dual variables for any choice of $b \in (0, b^*_{max})$, imply that we can apply Danskin's theorem (see Proposition B.25 in Bertsekas [4]) to deduce that the optimal cost of the Lagrangean dual is differentiable in b in the open interval $(0, b^*_{max})$.[11] Thus, by strong duality $V_{max}(b)$ will also be differentiable at all $b \in (0, b^*_{max})$ (that it is differentiable at the left endpoint of the domain $b = 0$ follows by continuity). Now let us restrict ourselves to the open interval $(0, b^*_{max})$. Since $V_{max}(b)$ is differentiable and strong duality holds we follow Section 5.6.3 in Boyd and Vandenberghe [7] to deduce the following simple relation:

$$\frac{\mathrm{d}}{\mathrm{d}b} V_{max}(b) = \lambda^{max}(b), \quad b \in (0, b^*_{max}). \tag{49}$$

Eq. (49) means that we can now focus on calculating the Lagrange multiplier $\lambda^{max}(b)$. Before we do so we note the following useful identity

$$\begin{aligned} \sum_{n=1}^{N} \left(p^{max}_n(b) - \frac{1}{N} \right)^2 &= \sum_{n=1}^{N} p^{max}_n(b) \left(p^{max}_n(b) - \frac{1}{N} \right) - \frac{1}{N} \underbrace{\sum_{n=1}^{N} p^{max}_n(b)}_{=1} + \sum_{n=1}^{N} \frac{1}{N^2} \\ &= \sum_{n=1}^{N} p^{max}_n(b) \left(p^{max}_n(b) - \frac{1}{N} \right). \end{aligned} \tag{50}$$

Multiplying both sides of Eq. (45) by $p^{max}_n(b)$ and then summing over all $n = 1, 2, .., N$ obtains

$$\sum_{n=1}^{N} x_n p^{max}_n(b) - 2\lambda^{max}(b) \sum_{n=1}^{N} p^{max}_n(b) \left(p^{max}_n(b) - \frac{1}{N} \right) + \mu^{max}(b) \sum_{n=1}^{N} p^{max}_n(b) = 0$$

[10] Note how this argument fails in the case of $b \in \{0, b^*_{max}\}$.

[11] The attentive reader will note that the Lagrangean dual's domain as given in Equation 5.29 of [7] is unbounded and thus not compact, so that we cannot directly invoke Danskin's theorem as stated in Bertsekas [4]. But the absence of compactness here is not a problem because of the special structure of the set of primal-dual optimal solutions that must satisfy (45)-(48), so that the proof of Bertsekas [4] carries through with only one or two minor modifications. Details available upon request.

$$\overset{(50)}{\Rightarrow} \quad \sum_{n=1}^{N} x_n p_n^{max}(b) - 2\lambda^{max}(b) \sum_{n=1}^{N} \left(p_n^{max}(b) - \frac{1}{N} \right)^2 + \mu^{max}(b) = 0$$

$$\overset{\text{Prop. 2}}{\Rightarrow} \quad \mu^{max}(b) = 2\lambda^{max}(b) \cdot b - \sum_{n=1}^{N} x_n p_n^{max}(b) \tag{51}$$

Now we consider Eq. (45) for expert $n_k \in \mathcal{N}_j$. By Eq. (13) and Lemma 2 we must have $p_{n_k}^{max}(b) > 0$ for $b \in \left[0, b_{(j)}^{max}\right)$. Substituting the value of $\mu^{max}(b)$ obtained in Eq. (51), and applying the complementary slackness condition (48) we obtain

$$x_{n_k} - 2\lambda^{max}(b) \left(p_{n_k}^{max}(b) - \frac{1}{N} \right) = \sum_{n=1}^{N} x_n p_n^{max}(b) - 2\lambda^{max}(b) \cdot b$$

$$\overset{(49)}{\Rightarrow} \quad 2\frac{\mathrm{d}}{\mathrm{d}b} V_{max}(b) \left(p_{n_k}^{max}(b) - \frac{1}{N} - b \right) = x_{n_k} - V_{max}(b), \quad b \in \left[0, b_{(j)}^{max}\right). \tag{52}$$

■

Proof of Proposition 3. We focus System 1; the argument for System 2 is exactly analogous. The result is trivially true for $k_N = 2$ so we focus on the case of $k_N \geq 3$. We prove the result by backwards induction. The base case of $j = k_N - 1$ is immediate as Eqs. (22) and (23) uniquely determine $C_{k_N-1}^{+}$ and $b_{k_N-1}^{+}$. Now, suppose that the result is true for all $j \geq k_N - k + 1$ and consider $j = k_N - k$. We distinguish between two cases.

Case 1: $k < k_N - 1$**.** In this case, $k_N - k > 1$ and Eqs. (20) and (21) apply. By the induction hypothesis $C_{k_N-k+1}^{+}$ is uniquely determined. Focusing on Eq. (21) for $k_N - k$ and solving for $C_{k_N-k}^{+}$ yields:

$$C_{k_N-k}^{+} = \frac{\sqrt{N_{k_N-k}^{+} b_{k_N-k}^{+} - \frac{N_{k_N-k-1}^{-}}{N}}}{N_{k_N-k}^{+}} \frac{C_{k_N-k+1}^{+} N_{k_N-k+1}^{+}}{\sqrt{N_{k_N-k+1}^{+} b_{k_N-k}^{+} - \frac{N_{k_N-k}^{-}}{N}}}. \tag{53}$$

Plugging (53) into Eq. (20) for $k_N - k$ yields:

$$C_{k_N-k+1}^{+} \sqrt{N_{k_N-k+1}^{+} b_{k_N-k}^{+} - \frac{N_{k_N-k}^{-}}{N}} \left(1 - \frac{N_{k_N-k+1}^{+}}{N_{k_N-k}^{+}} \cdot \frac{N_{k_N-k}^{+} b_{k_N-k}^{+} - \frac{N_{k_N-k-1}^{-}}{N}}{N_{k_N-k+1}^{+} b_{k_N-k}^{+} - \frac{N_{k_N-k}^{-}}{N}} \right) = \frac{\sum_{n \in \mathcal{N}_{k_N-k}^{+}} x_n}{N_{k_N-k}^{+}} - \frac{\sum_{n \in \mathcal{N}_{k_N-k+1}^{+}} x_n}{N_{k_N-k+1}^{+}}. \tag{54}$$

After some algebra Eq. (54) leads to the following equation, which uniquely determines $b_{k_N-k}^{+}$:

$$\frac{C_{k_N-k+1}^{+} N_{k_N-k+1}^{+}}{N} \frac{\frac{N_{k_N-k-1}^{-}}{N_{k_N-k}^{+}} - \frac{N_{k_N-k}^{-}}{N_{k_N-k+1}^{+}}}{\sqrt{N_{k_N-k+1}^{+} b_{k_N-k}^{+} - \frac{N_{k_N-k}^{-}}{N}}} = \frac{\sum_{n \in \mathcal{N}_{k_N-k}^{+}} x_n}{N_{k_N-k}^{+}} - \frac{\sum_{n \in \mathcal{N}_{k_N-k+1}^{+}} x_n}{N_{k_N-k+1}^{+}}. \tag{55}$$

Note that Eq. (55) is well-defined since

$$\frac{\sum_{n\in\mathcal{N}^+_{k_N-k}} x_n}{N^+_{k_N-k}} - \frac{\sum_{n\in\mathcal{N}^+_{k_N-k+1}} x_n}{N^+_{k_N-k+1}} < 0, \quad \frac{N^-_{k_N-k-1}}{N^+_{k_N-k}} - \frac{N^-_{k_N-k}}{N^+_{k_N-k+1}} < 0.$$

Moreover, Eq. (55) implies that

$$b^+_{k_N-k} > \frac{N^-_{k_N-k}}{NN^+_{k_N-k+1}} \quad \Rightarrow \quad b^+_{k_N-k} > \frac{N^-_{k_N-k-1}}{NN^+_{k_N-k}}. \tag{56}$$

Solving Eq. (55) for $b^+_{k_N-k}$ and plugging in the result to (53) yields a unique real-valued solution also for $C^+_{k_N-k}$.[12]

Case 2: $k = k_N - 1$. In this case, $k_N - k = 1$ and Eqs. (20) and (21) apply. An identical argument as the one for Case 1 establishes the uniqueness of b^+_1 and C^+_1. ■

Proof of Theorem 2. We focus on on V_{max} and System 1; the argument for V_{min} and System 2 is exactly analogous. Recall the definition of $b^{max}_{(k)}$ of Eq. (13). Consider first $b \in (0, b^{max}_{(1)})$ so that $p^{max}_n(b) > 0$ for all $b \in (0, b^{max}_{(1)})$ and $n \in \mathcal{N}$. Adding Eqs. (52) for all $n \in \mathcal{N}$ yields the following differential equation

$$-2Nb\frac{\mathrm{d}V_{max}(b)}{\mathrm{d}b} = -NV_{max}(b) + \sum_{n\in\mathcal{N}} x_n, \quad b \in (0, b^{max}_{(1)}). \tag{57}$$

Solving differential equation (57) leads to the following expression:

$$V_{max}(b) = C_1\sqrt{b} + \frac{\sum_{n\in\mathcal{N}} x_n}{N}, \quad b \in [0, b^{max}_{(1)}), \tag{58}$$

where C_1 is a constant. Consider now $b \in [b^{max}_{(k-1)}, b^{max}_{(k)})$ for $k \in \{2, 3, ..., k_N - 1\}$. In this range of b we will have $p^{max}_n(b) > 0$ if and only $n \in \mathcal{N}^+_k$. Adding Eqs. (52) for all such $n \in \mathcal{N}^+_k$ yields the following differential equation

$$2\left(\frac{N^-_{k-1}}{N} - N^+_k b\right)\frac{\mathrm{d}V_{max}}{\mathrm{d}b} = \sum_{n\in\mathcal{N}^+_k} x_n - N^+_k V_{max}(b), \quad b \in [b^{max}_{(k-1)}, b^{max}_{(k)}) \tag{59}$$

Solving differential equation (59) gives the following:

$$V_{max}(b) = \frac{\sum_{n\in\mathcal{N}^+_k} x_n}{N^+_k} + C_k\sqrt{N^+_k b - \frac{N^-_{k-1}}{N}}, \quad b \in \left[b^{max}_{(k-1)}, b^{max}_{(k)}\right), \tag{60}$$

[12] Note that (56) ensures that $C^+_{k_N-k}$ will be real-valued.

where C_k is a constant, for $k \in \{2, 3, ..., k_N - 1\}$. Finally since $b^{max}_{(k_N-1)} = b^*_{max}$ we use Lemma 1 to conclude

$$V_{max}(b) = \max_{n \in \mathcal{N}} x_n, \quad b \in \left[b^{max}_{(k_N-1)}, \frac{N-1}{N} \right]. \tag{61}$$

Putting together Eqs. (58) (60) and (61) we see that V_{max} will equal

$$V_{max}(b) = \begin{cases} \frac{\sum_{n \in \mathcal{N}} x_n}{N} + C_1 \sqrt{b} & b \in \left[0, b^{max}_{(1)}\right) \\ \frac{\sum_{n \in \mathcal{N}^+_k} x_n}{N^+_k} + C^+_k \sqrt{N^+_k b^+_k - \frac{N^-_{k-1}}{N}} & b \in \left[b^{max}_{(k-1)}, b^{max}_{(k)}\right), \ k = 2, 3, ..., k_N - 1 \\ \max_{n \in \mathcal{N}} x_n & b \in \left[b^{max}_{(k_N-1)}, \frac{N-1}{N}\right] \end{cases} \tag{62}$$

for appropriately chosen constants $(C_1, C_2, ..., C_{k_N-1})$ and $(b_{(1)}, b_{(2)}, ..., b_{(k_N-1)}$. By Proposition 1 and Theorem 1, V_{max} is continuous everywhere and differentiable in $\left[0, b^{max}_{(k_N-1)}\right)$. Thus, the vectors $(C_1, C_2, ..., C_{k_N-1})$ and $(b_{(1)}, b_{(2)}, ..., b_{(k_N-1)}$ must fulfill these criteria of continuity and differentiability and are thus uniquely determined by the system of equations (18)-(23).

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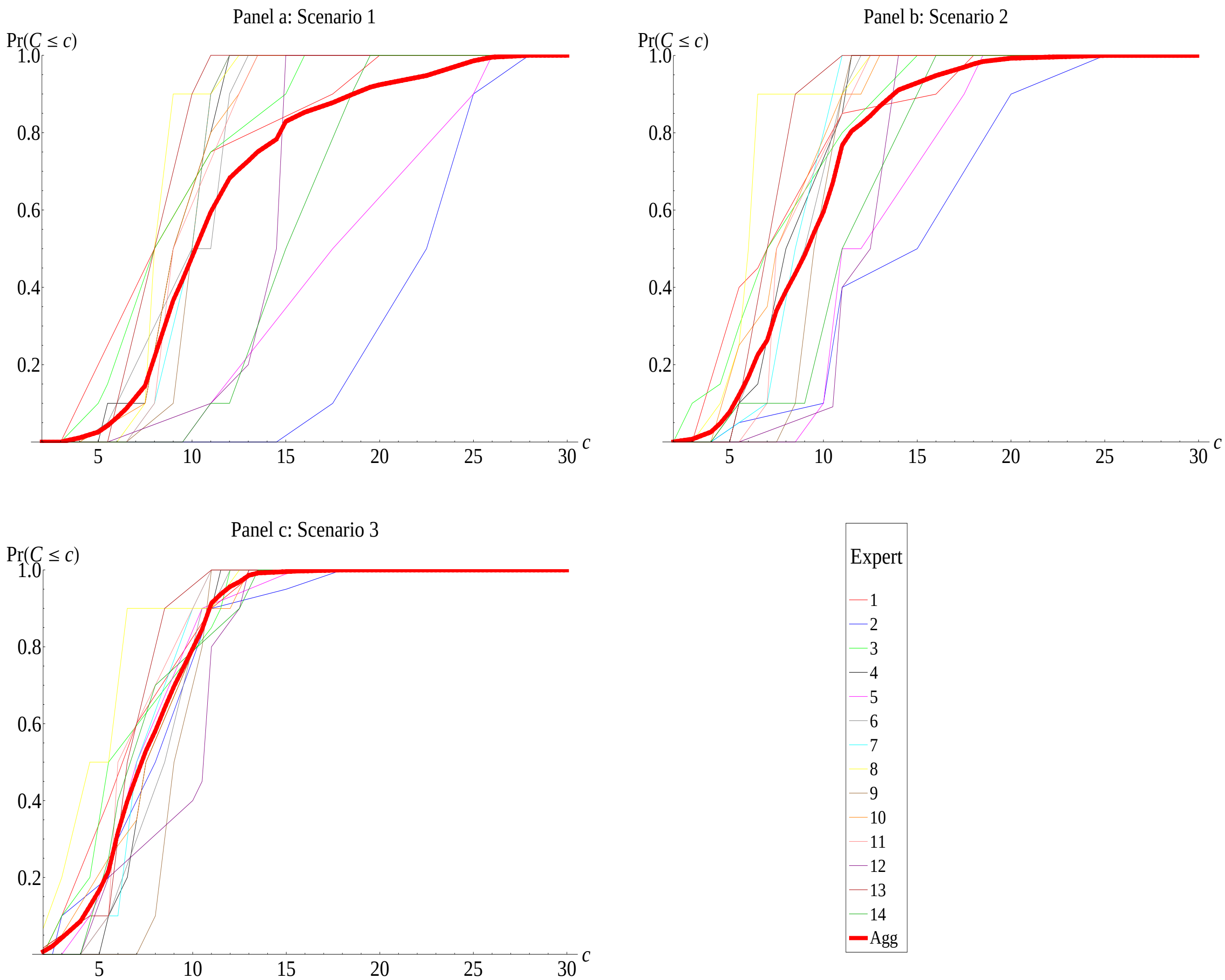

Figure 1: Expert and aggregate cdfs of the 2030 cost of solar technology under the three R&D Scenarios. Recall that the cdf's domain is $\{2, 2.5, ..., 29, 29.5, 30\}$

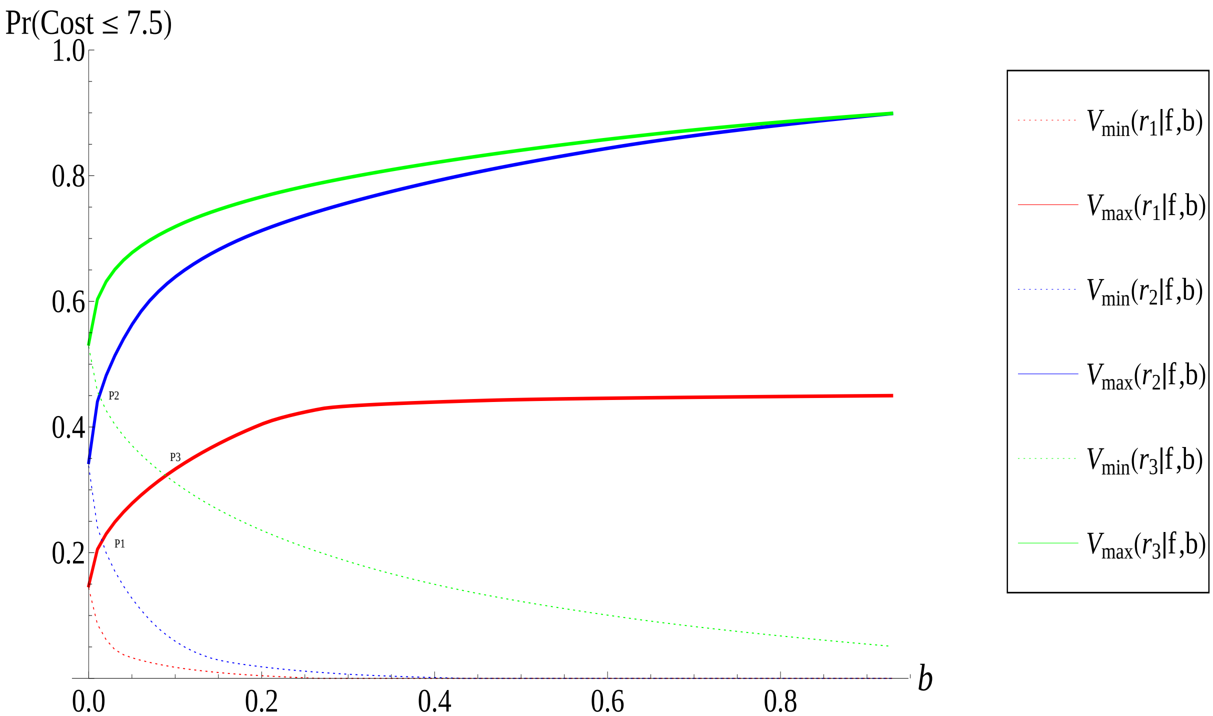

Figure 2: Worst and Best-Case Breakthrough Probabilities for the three R&D scenarios.

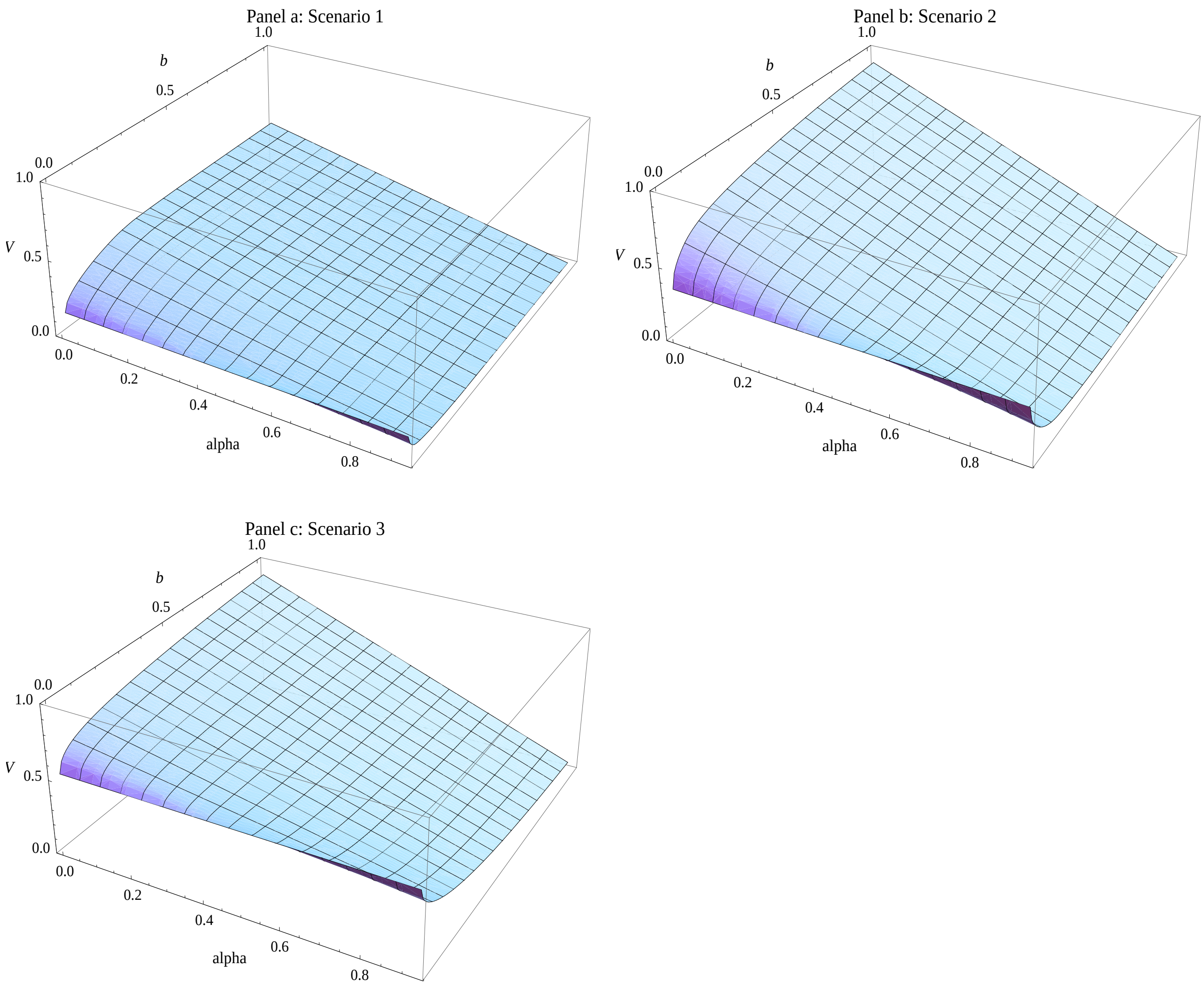

Figure 3: Plotting Eq. (9), applied to breakthrough probability under the three R&D Scenarios, as a function of ambiguity b and ambiguity attitude α.

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