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By **Rahmi İlkiliç,** Department of Economics, Maastricht University

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Summary
Suppose markets and firms are connected in a bi-partite network, where firms can only supply to the markets they are connected to. Firms compete *a la Cournot* and decide how much to supply to each market they have a link with. We assume that markets have linear demand functions and firms have convex quadratic cost functions. We show there exists a unique equilibrium in any given network of firms and markets. We provide a formula which expresses the quantities at an equilibrium as a function of a network centrality measure.

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Cournot Competition on a Network of Markets and Firms[*]

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Abstract

Suppose markets and firms are connected in a bi-partite network, where firms can only supply to the markets they are connected to. Firms compete *a la Cournot* and decide how much to supply to each market they have a link with. We assume that markets have linear demand functions and firms have convex quadratic cost functions. We show there exists a unique equilibrium in any given network of firms and markets. We provide a formula which expresses the quantities at an equilibrium as a function of a network centrality measure.

Keywords: Cournot markets, networks, Nash equilibrium, centrality measures.
JEL Classification: C62, C72, D85, L11

1 Introduction

Thanks to the decreased costs of transportation and communication, the world became a densely connected network. This is particularly true for markets and natural resources. International trade connects world markets to the extent that the world prices are determined by the demand and supply conditions of countries that take part in it. How and how much a country affects the market depend both on the size and the position of the country in the network.

One example is the market for crude oil. The price is determined by many factors from different regions of the world. As the petrol is relatively easy to transport, we observe a single global price which does not change much from region to region. Any difference between regional prices would be offset through trade. Market power of an oil exporting country is determined by the capacity and efficiency of its production. The Organization of Petroleum Exporting Countries use their combined market share to influence the price of oil.

The market for natural gas presents a much more complex example. It requires an infrastructure to be carried to consumers. It is carried mainly through pipelines[1]. Other forms of transportation are not economical when compared with pipelines. Which countries can trade natural gas is determined by the structure of the network formed by the natural gas pipelines. This leads to the formation of regional prices. The price for a thousand cubic meters of natural gas ranges almost from zero to 300 (EU Commission Staff Working Document (2006)), depending on the location. An importing country with a single supplier faces a monopoly and pays a higher price. An importing country which has alternative suppliers will pay a lower price thanks to the competition between the latter. The market power of exporting countries are determined both by their production and their position in the market. The recent attempt of natural gas exporting countries to mimic OPEC will potentially create a cartel which can decide both the quantity and the destination of supply. Moreover, the transit countries which transport the gas from producers to consumers become strategic actors, independent of whether they produce natural gas or not.

To understand how the market for the natural gas functions we need to go into the details

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[1]More than 90 percent of the natural gas imports of the European Union are through pipelines (EU Commission Staff Working Document (2006)). The ratio for global gas imports is around 80 percent (Victor et. al. 2006). The three countries which depend most on maritime transportation of natural gas are Japan, Taiwan and South Korea. It is due to the infeasibility of building long distance pipelines in the ocean.

of the network that connects suppliers with consumers. A structural analysis is required to understand the patterns of interaction and to quantify the influence that countries or regions have on each other. We need to understand how each link affects the countries that it connects and measure how much it changes the prices. We can ask how the addition of a new link changes the market and whether it would be profitable to construct it. Then we can see how a country (e.g. Russia) or a group of countries (e.g. the European Union) can improve their market power by coordinating their policies on infrastructure and consumption.

We model a bipartite network, where links connect firms with markets. We look at the Cournot game, where firms decide how much to sell at each market they are connected to. We assume that firms have convex quadratic costs and markets have linear inverse demand functions.

We show that there exists a unique the Cournot equilibrium. We write the equilibrium conditions as a linear complementarity problem and provide an interpretation of the equilibrium flows using the Katz-Bonacich centrality (Katz 1953, Bonacich 1987). We then study the effects of a cartel and the strategic complementarities between links.

We bridge two branches of the literature. On one side we study Cournot competition. We extend the basic to a network of firms and markets. Given a network, we show how the structure of connections determines firms' supply levels.

The closest line of literature is the analysis of behavior on networks. Ballester et al. (2006) analyzes the equilibrium activities at each node of a simple (i.e. not bipartite) non-directed network. Players create externalities on their neighbors. A player has a single level of activity. Her payoff depends on her activity level and of her neighbors'. They show that the equilibrium levels are given by a network centrality index, which is similar to the Katz-Bonacich centrality. Ballester and Calvó-Armengol (2006) shows that the first order equilibrium conditions of games which exhibit cross influences between agents' actions are linear complementarity problems. They study some interesting classes of such games which have a unique equilibrium. In both of these papers, the agents' strategy spaces are subsets of the real line. A link between two agents shows that they impose externalities on each other. In our model, agents' strategy spaces are multidimensional and a link is not only a qualitative object, but also carries a value which is determined endogenously.

As in Corominas-Bosch (2004) we study a bipartite network. She studies the equilibria of a bargaining game in a network of buyers and sellers. Her model differs from ours in two basic points. First, both buyers and sellers are active agents, where we model only the firms

as strategic. Second, in Corominas-Bosch (2004) buyers and sellers are bargaining over a single indivisible good. In contrast we assume that the good transferred through the links is perfectly divisible, allowing a firm to supply to many markets.

The basic notation, some of which we borrow from Corominas-Bosch (2004), is introduced in Section 2. Section 3 defines the payoffs. We define the Cournot game in Section 4 and solve for its equilibrium. In Section 5 we demonstrate some applications of the model. Section 6 concludes. The proofs are given in the Appendix.

2   Notation

There are n markets $v_1, ..., v_n$, and m firms $f_1, ..., f_m$. They are embedded in a network that links markets with firms, and firms can supply to the markets they are connected to. We will represent the network as a graph.

A non-directed *bipartite graph* $g = \langle V \cup F, L \rangle$ consists of a set of *nodes* formed by markets $V = \{v_1, ..., v_n\}$, and firms $F = \{f_1, ..., f_m\}$ and a set of *links* L, each link joining a market with a firm. A link from v_i to f_j will be denoted as (i, j). We say that a market v_i is *linked* to a firm f_j if there is a link joining the two. We will use $(i, j) \in g$ and $(i, j) \in L$ interchangeably, meaning that v_i and f_j are connected in g. Let $r(g)$ be the number of links in g.

A graph g is *connected* if there exists a path linking any two nodes of the graph. Formally, a path linking nodes v_i and f_j will be a collection of t firms and t markets, $t \geq 0$, $v_1, ...v_t, f_1, ..., f_t$ among $V \cup F$ (possibly some of them repeated) such that

$$\{(i, 1), (1, 1), (1, 2), ..., (t, t), (t, j)\} \in g$$

A *subgraph* $g_0 = \langle V_0 \cup F_0, L_0 \rangle$ of g is a graph such that $V_0 \subseteq V, F_0 \subseteq F, L_0 \subseteq L$ and such that each link in L that connects a market in V_0 with a firm in F_0 is a member of L_0. Hence a node of g_0 will continue to have the same links it had with the other nodes in g_0. We will write $g_0 \subseteq g$ to mean that g_0 is a subgraph of g. For a subgraph g_0 of g, we will denote by $g - g_0$, the subgraph of g that results when we remove the set of nodes $V_0 \cup F_0$ from g.

Given a subgraph $g_0 = \langle V_0 \cup F_0, L_0 \rangle$ of g, let $\overleftrightarrow{g_0}$ be the complete bipartite graph with nodes $V_0 \cup F_0$. We call $\overleftrightarrow{g_0}$ *the completed graph of* g_0.

$N_g(v_i)$ will denote the set of firms linked with v_i in $g = \langle V \cup F, L \rangle$, more formally:

$$N_g(v_i) = \{f_j \in F \text{ such that } (i,j) \in g\}$$

and similarly $N_g(f_j)$ stands for the set of markets linked with f_j.

For a set A, let $|A|$ denote the number of elements in A. For v_i in S, we denote $|N_g(v_i)|$ by $m_i(g)$. Similarly for $f_j \in F$, let $|N_g(f_j)| = n_j(g)$, be the number of markets connected to f_j.

Labeling of pairs (i,j) We will first order all possible links such that the links of a firm j are assigned a lower number than any firm i for $i > j$, and the links of a firm are ordered according to the indices of the markets they connect. The label of a possible link (i,j) will be denoted by $\tau(i,j)$. For example for 2 firms and 2 markets, we will order the links starting from firm f_1 and market m_1, $\tau(1,1) = 1$. The second link is between f_1 and m_2, $\tau(2,1) = 2$. Now, as all links of firm f_1 are ranked, τ will next rank the link between f_2 and m_1, $\tau(1,2) = 3$. Then comes the link between firm f_2 and market m_2, $\tau(2,2) = 4$.

For a network g, let $Y(g) = \{y \in \mathbb{N}_+ : y = \tau(i,j) \text{ for some } (i,j) \notin g\}$ be the set of indices that τ assigns to links which are not in g. Assume, without loss of generality that $|Y(g)| = m \times n - r(g)$, for some $1 \leq r(g) \leq m \times n$, where $r(g)$ is the number of links in graph g. For 2 firms and 2 markets, for a graph g, if the only missing link is $(1,2)$, then $Y(g) = \{3\}$ and $r(g) = 3$.

τ orders all possible links, independent of g, where as $Y(g)$ does depend on g. We can see how this works on an example. Suppose that 2 cities and 2 sources, form a completely connected bipartite graph g_1. For graph g_1, $Y(g_1) = \emptyset$.
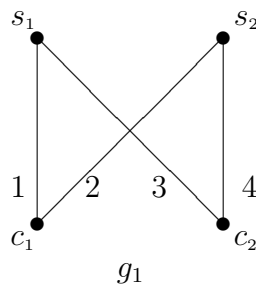

g_1

Figure 1

Now we cut the link between c_2 and s_1, to obtain g_2.

$$s_1 \qquad s_2$$
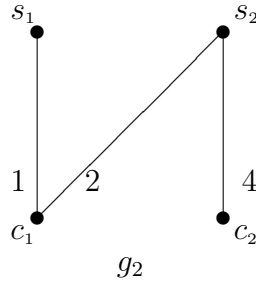
$$1 \quad 2 \qquad 4$$

$$c_1 \qquad c_2$$

$$g_2$$

Figure 2

Although link $(1, 2)$ does not exist in g_2 it is still labeled equally by τ. $\tau(1, 2) = 3$, meaning that $Y(g_2) = \{3\}$.

We will make use of graphs g_1 and g_2 in many examples throughout the paper.

Now we define the column vector that shows the quantities flowing at each link. Let $Q = [e_z]$ be the column vector of quantities extracted such that for q_{ij}, the quantity extracted from market v_i by f_j, $e_{\tau(i,j)} = q_{ij}$. For 2 firms and 2 markets:

$$Q = \begin{bmatrix} q_{11} \\ q_{21} \\ q_{12} \\ q_{22} \end{bmatrix}$$

Let Q_{-j} be the vector obtained by deleting row j from Q. For $J \subset \mathbb{N}_+$, let Q_{-J} be the vector obtained deleting each row $j \in J$ and column $j \in J$ from Q. For $Y(g) \subset \mathbb{N}$, let Q_g be the matrix obtained by deleting each row $y \in Y(g)$ from Q. Then Q_g has size r. Q_g is the link by link profile of supplies. For the two graphs given above:

$$Q_{g_1} = \begin{bmatrix} q_{11} \\ q_{21} \\ q_{12} \\ q_{22} \end{bmatrix} \qquad\qquad Q_{g_2} = \begin{bmatrix} q_{11} \\ q_{21} \\ q_{22} \end{bmatrix}$$

For $j \in \mathbb{N}_+$, let Q_{g-j} be the vector obtained from Q_g by deleting row j. For $J \subset \mathbb{N}_+$, let Q_{g-J} be the vector obtained from Q_g by deleting each row $j \in J$.

Let $\mathbb{Q}_{(m \times n)}$ be the set of all non-negative real valued column vectors of size $(m \times n)$. Let $\mathbb{Q}_r$ be the set of all non-negative real valued column vectors of size r.

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Given a vector of flows Q_g, for a firm f_j, we will denote by $S_j(Q_g)$ the total production by f_j. For a market v_i we will denote by $D_i(Q_g)$ the total demand at v_i.

3 Demand and Cost Functions

We assume that markets have linear inverse demand functions. Given a market v_i and a flow vector Q_g the price at v_i is

$$p_i(Q_g) = \alpha_i - \beta_i D_i(Q_g)$$

where $\alpha_i, \beta_i > 0$.

We assume that firms have quadratic costs of production. For firm f_j the total cost of production is

$$T_j(Q_g) = \frac{\gamma_j}{2}(S_j(Q_g))^2$$

where $\gamma_j > 0$

For $\alpha, \beta, \gamma > 0$, the profit of firm f_j is:

$$\pi_j(Q_g) = \sum_{v_i \in N_g(f_j)} \alpha_i q_{ij} - \frac{\gamma_j}{2}(S_j(Q_g))^2 - \sum_{v_i \in N_g(f_j)} \beta_i q_{ij}(D_i(Q_g))$$

Marginal profit is not separable with respect to each market. The marginal profit from q_{ij} does depend on the supply from f_j to markets other than v_i.

4 The Cournot Game

Given a network g, each firm f_j maximizes its profit by supplying a non-negative amount to the markets in $N_g(f_j)$. So, the set of players are the set of firms F. The set of strategies of a firm f_j is $\mathbb{Q}_j = \mathbb{Q}_{N_g(f_j)}$. We denote a representative strategy of f_j by $Q_j \in \mathbb{Q}_j$. Given that there are $r(g)$ links in g, the strategy space of the game is $\mathbb{Q}_g = \prod_{f_j \in F} \mathbb{Q}_j = \mathbb{Q}_{r(g)}$.

The best response Q'_j of firm f_j to $Q_g \in \mathbb{Q}_g$ is such that,

$$\text{for all links } (i,j), \ q'_{ij} = \begin{cases} \frac{\alpha - \gamma \sum\limits_{v_l \in N_g(f_j)\setminus\{v_i\}} q_{lj} - \beta \sum\limits_{f_k \in N_g(v_i)\setminus\{f_j\}} q_{ik}}{2\beta + \gamma}, & \text{if } \frac{\partial u_j}{\partial q_{ij}}|_{Q_g} \geq 0 \\ 0 & \text{, if } \frac{\partial u_j}{\partial q_{ij}}|_{Q_g} < 0 \end{cases}$$

The first order equilibrium conditions of the Cournot game constitutes a linear complementarity problem. Given a matrix $M \in \mathbb{R}^{t \times t}$ and a vector $p \in \mathbb{R}^t$, the linear complementarity problem $LCP(p; M)$ consists of finding a vector $z \in \mathbb{R}^t$ satisfying:

$$z \geq 0, \tag{1}$$
$$p + Mz \geq 0, \tag{2}$$
$$z^T(p + Mz) \geq 0 \tag{3}$$

Samelson *et al.* (1958) shows that a linear complementarity problem $LCP(p; M)$ has a unique solution for all $p \in \mathbb{R}^t$ if and only if all the principal minors of M are positive. We prove this to be true for the linear complementarity problem formed by the first order equilibrium conditions of the Cournot game.

We further check for the second order conditions for each agent, which reveals that the solution of the linear complementarity problem is indeed the equilibrium of the game.

Theorem 1 *The Cournot game has a unique Nash equilibrium.*

Example Suppose we have the graph g_1. Let $\alpha = \beta = \gamma = 1$. Then the link supplies at equilibrium are $q_{11}^* = q_{21}^* = q_{12}^* = q_{22}^* = 0.2$. The prices and the profits are $p_1 = p_2 = 0.6$ and $\pi_1 = \pi_2 = 0.16$, respectively.

Suppose the graph was g_2. Now at equilibrium, $q_{11}^* = 0.2857$, $q_{21}^* = 0.1429$, and $q_{22}^* = 0.2857$. The deletion of the link $(1, 2)$ changes the supply to market v_2, and moreover firm f_1 supplies less to the market she shares with firm f_2. The prices and the profits are $p_1 = 0.7125$, $p_2 = 0.5696$ and $\pi_1 = 0.1936$, $\pi_2 = 0.1224$, respectively.

Let Q_g^* be an equilibrium of the Cournot game. There might be some links in g which carry zero flow at equilibrium Q_g^*. Marginal profits of supply via those links need not be zero at Q_g^*.

$$q_{ij}^* > 0 \Rightarrow \frac{\partial u_j}{\partial q_{ij}} = 0$$
$$q_{ij}^* = 0 \Rightarrow \frac{\partial u_j}{\partial q_{ij}} \leq 0$$

To calculate the equilibrium quantities, first we need to weed out the links with zero flow. Let $\rho : L \to \mathbb{N}_+$ be a lexicographic order on L respecting τ such that ρ relabels the (i, j)

pairs from 1 to $r(g)$ by skipping those links which are not in g.[2] Now we delete from Q_g^*, the entries that correspond to links with no flow.

Let $Z\left(Q_g^*\right) = \{z \in \mathbb{N}_+ : z = \rho(i,j)$ for some (i,j) s.t. $q_{ij}^* = 0\}$. Let $\left|Z\left(Q_g^*\right)\right| = t^*$, then $Q_{g-Z\left(Q_g^*\right)}^*$ is a vector of size $r(g) - t^*$ obtained from Q_g^* by deleting the zero entries. It is the vector of equilibrium quantities for links over which there is a strictly positive flow from a firm to a market.

Let Q_g^* be the equilibrium of the Cournot game at network g. We denote by $g - Z(Q_g^*)$ the network obtained from g by deleting the links which have zero flow at Q_g^*.

Theorem 2 *Given two networks g and g'. Let Q_g^* and $Q_{g'}^*$ be the equilibrium of the Cournot game in g and g', respectively. If $g - Z(Q_g^*) = g' - Z(Q_{g'}^*)$, then $Q_{g-Z(Q_g^*)}^* = Q_{g'-Z(Q_{g'}^*)}^*$.*

At equilibrium there might be links which carry no flows. For the firms of such links, the marginal profits of supplying via them are not positive. They are indifferent between having such a link or not. Theorem 2 tells us such links with zero flow play no role in determining the equilibrium. They are strategically redundant. Take graph g_3. Let $\alpha = \beta = \gamma = 1$. Then at equilibrium,

g_3
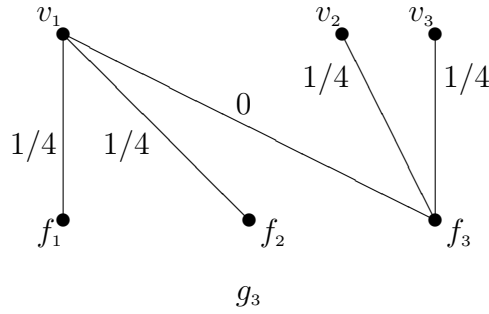
Figure 3

[2]Explicitly, $\rho : L \to \mathbb{N}_+$ is such that:

(*i*) $\exists(i,j) \in L$ such that $\rho(i,j) = 1$,

(*ii*) $(i,j) \neq (k,l) \Rightarrow \rho(i,j) \neq \rho(k,l)$,

(*iii*) $j < l \Rightarrow \rho(i,j) < \rho(k,l)$ for all $(i,j), (k,l) \in L$,

(*iv*) $i < k \Rightarrow \rho(i,j) < \rho(k,j)$ for all $(i,j), (k,j) \in L$,

(*v*) if $\exists(i,j)$ s.t. $\rho(i,j) = z > 1$ then $\exists(k,l) \in L$ s.t. $\rho(k,l) = y - 1$.

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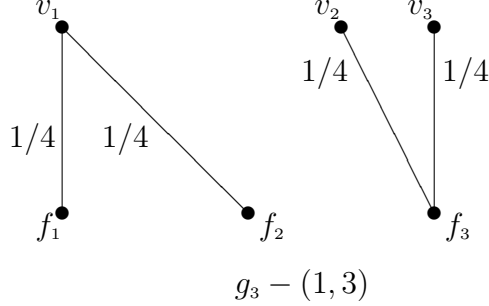Now we cut the link $(1,3)$ and denote the new graph by $g_3 - (1,3)$.

Figure 4

For $\alpha = \beta = \gamma = 1$, according to Theorem 2 the supplies at equilibrium are $q_{11}^* = q_{12}^* = \frac{1}{4}$ and $q_{23}^* = q_{33}^* = \frac{1}{4}$. At the equilibrium in g_3, the marginal profit to firm f_3 from supplying via $(1,3)$ was negative. Deleting it does not change the equilibrium quantities on other links, because the marginal profits from them are the same as in graph g_3.

We will use the marginal profit argument employed in this example to give a network interpretation for the quantities at equilibrium $Q^*_{g-Z(Q^*_g)}$ on any given graph g.

Definition 1 *Given a graph g, a line graph $I(g)$ of g is a graph obtained by denoting each link in g with a node in $I(g)$ and connecting two nodes in $I(g)$ if and only if the corresponding links in g meet at one endpoint.*

Given a network g, let $r^*(g) = r(g) - t^*$. Let $G^* = [g_{ij}]_{r^*(g) \times r^*(g)}$ be the weighted adjacency matrix of the line graph of $g - Z(Q^*_g)$ such that

$$g_{ij} = \begin{cases} \gamma_l, & \text{if } \rho^{-1}(i) \text{ and } \rho^{-1}(j) \text{ share firm } f_l \\ \beta_l, & \text{if } \rho^{-1}(i) \text{ and } \rho^{-1}(j) \text{ share market } v_l \\ 0, & \text{otherwise} \end{cases}$$

For example for graph g_2 all links have positive flows at equilibrium. Then,

$$G^*_{g_2} = \begin{bmatrix} 0 & \gamma_1 & 0 \\ \gamma_1 & 0 & \beta_2 \\ 0 & \beta_2 & 0 \end{bmatrix}$$

For any graph g, G^* has diagonal entries as 0 and non-diagonal entries are either $0, \gamma$ or β. We will use G^* to denote both the line graph of $g - Z(Q^*_g)$ and the weighted adjacency

matrix of this graph. Similarly, we define A, a diagonal matrix with the same size as G^* such that

$$a_{kl} = \begin{cases} \frac{1}{2\beta_i + \gamma_j}, & \text{if } k = l \text{ and } \rho^{-1}(k) = (i,j) \\ 0, & \text{otherwise} \end{cases}$$

For $a \geq 0$, and a network adjacency matrix G^*, let

$$M(G^*, a) = [I - aG^*]^{-1} = \sum_{k=0}^{\infty} (aG^*)^k$$

If $M(a, G^*)$ is non-negative, its entries $m_{ij}(G^*, a)$ counts the number of paths in the network, starting at node i and ending at node j, where paths of length k are weighted by a^k.

Definition 2 *For a network adjacency matrix G, and for scalar $a > 0$ such that $M(G, a) = [I - aG]^{-1}$ is well-defined and non-negative, the vector Katz-Bonacich centralities of parameter a in G is:*

$$\boldsymbol{b}(G, a) = [I - aG]^{-1} . \mathbf{1}$$

In a graph with z nodes, the Katz-Bonacich centrality of node i,

$$b_i(G, a) = \sum_{j=1}^{z} m_{ij}(G, a)$$

counts the total number of paths in G starting from i.

Theorem 3 *Given a network of Cournot markets and firms g, the Nash equilibrium flow vector is*

$$Q_{g-Z(Q_g^*)}^* = \left[\sum_{k=0}^{\infty} (AG^*)^{2k} - \sum_{k=0}^{\infty} (AG^*)^{2k+1} \right] A\boldsymbol{\alpha}$$

where $\boldsymbol{\alpha}$ is a column vector such that for $t = \rho(i, j)$, $\alpha_t = \alpha_i$.

The first summation counts the total number of even paths that start from the corresponding node in G^*, and the second summation counts the total number of odd paths that start from it.

The first sum tells that the equilibrium flows from a link is positively related with the number of even length paths that start from it. The links which have an even distance

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between them are complements. In contrast, the negative sign on the second summation means the equilibrium supply from a link is negatively related with the number of odd length paths that start from it. The links which have an odd distance between them are substitutes.

For example, in graph g_1,

g_1
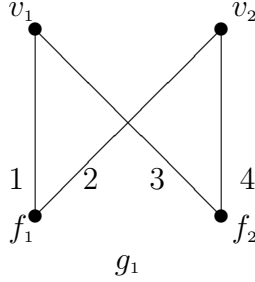
Figure 5

links $(1,1)$ and $(2,2)$ are complements. The supply to market v_2 by firm f_2 increases incentives for firm f_1 to supply more to market v_1, because the former decreases the marginal revenue on v_2. This makes v_1 a better option. Links $(1,1)$ and $(2,1)$ are substitutes, because supply through one decreases the marginal revenue to firm f_1. This decreases firm's incentives to supply more.

In general, the links of a firm are substitutes for each other (e.g. $(1,1)$ and $(2,2)$ at graph g_1). Similarly, the links of a market are substitutes for each other, too (e.g. $(1,1)$ and $(1,2)$ at graph g_1). If two firms are sharing a market, then their links to markets they don't share are complements (e.g. $(1,1)$ and $(2,2)$ at graph g_1). Moreover, if a link (i_1, j_1) is a substitute of a link (i_2, j_2) and (i_2, j_2) is a substitute of (i_3, j_3), then (i_1, j_1) and (i_3, j_3) are complements. Therefore, the effect depends on the parity of the distance between two links.

In the Cournot game the adjacency matrix G^* does not necessarily have binary entries, neither its non-zero entries are all equal. Each link in G^* has a weight. While counting the number of paths, these weights are taken into account as well. The total supply a firm f_j is calculated by summing up the link centralities of the elements in $N_g(f_j)$.

5 Applications

5.1 Cartel

Let there be 2 markets and 5 firms connected as in the graph g_5 below.

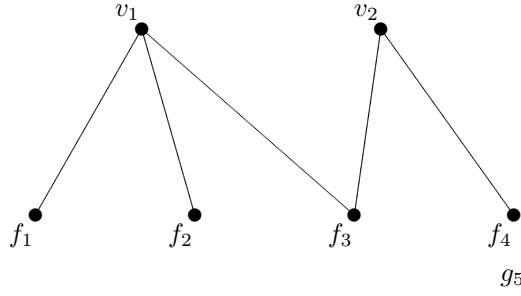$$v_1 \qquad v_2$$

$$f_1 \qquad f_2 \qquad f_3 \qquad f_4 \qquad f_5$$

$$g_4$$
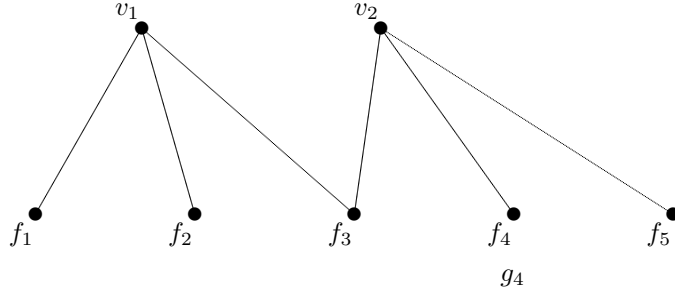
Figure 6

Let $\alpha = \beta = \gamma = 1$. Then the equilibrium quantities, prices and profits are

$$(q_{11}, q_{21}, q_{31}, q_{32}, q_{42}, q_{52}) = \left(\tfrac{3}{14}, \tfrac{3}{14}, \tfrac{1}{7}, \tfrac{1}{7}, \tfrac{3}{14}, \tfrac{3}{14}\right)$$
$$(p_1, p_2) = \left(\tfrac{3}{7}, \tfrac{3}{7}\right)$$
$$(\pi_1, \pi_2, \pi_3, \pi_4, \pi_5) = (0.069, 0.069, 0.082, 0.069, 0.069)$$

Suppose firms 2 and 3 form a cartel. Now, the equilibrium quantities, prices and profits are

$$(q'_{11}, q'_{21}, q'_{31}, q'_{32}, q'_{42}) = \left(\tfrac{12}{49}, \tfrac{11}{49}, \tfrac{2}{49}, \tfrac{9}{49}, \tfrac{10}{49}, \tfrac{10}{49}\right)$$
$$(p'_1, p'_2) = \left(\tfrac{24}{49}, \tfrac{20}{49}\right)$$
$$(\pi'_1, \pi'_2, \pi'_3, \pi'_4) = (0.090, 0.085, 0.070, 0.062, 0.062)$$

The collusion benefited firms

Formation of a cartel in a standard market would benefit all producers which are not in the cartel and hurt all consumers. In a networked market the effects are not symmetric. The markets which are linked to more than one member of the cartel are worse off. Those producers whose production can substitute the cartel's supply reduction improve. The markets which are linked with a single member of the cartel are better off, while the producers of those markets are worse off. The effects of the cartel would diffuse through the network affecting all consumers and producers.

5.2 Strategic Complementarities

Let there be 2 markets and 4 firms connected as in the graph g_5 below.

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Figure 7

Let $\alpha = \beta = \gamma = 1$. Then the equilibrium quantities, prices and profits are

$$(q_{11}, q_{21}, q_{31}, q_{32}, q_{42}) = (\tfrac{15}{68}, \tfrac{15}{68}, \tfrac{2}{17}, \tfrac{7}{34}, \tfrac{9}{34})$$
$$(p_1, p_2) = (\tfrac{15}{34}, \tfrac{9}{17})$$
$$(\pi_1, \pi_2, \pi_3, \pi_4) = (0.073, 0.073, 0.067, 0.1)$$

Suppose firm 3 commits to supply zero to market 1. Now, the equilibrium quantities, prices and profits are

$$(q'_{11}, q'_{21}, q'_{31}, q'_{32}, q'_{42}) = (\tfrac{1}{2}, \tfrac{1}{2}, 0, \tfrac{1}{2}, \tfrac{1}{2})$$
$$(p'_1, p'_2) = (\tfrac{1}{2}, \tfrac{1}{2})$$
$$(\pi'_1, \pi'_2, \pi'_3, \pi'_4) = (0.094, 0.094, 0.094, 0.094)$$

The commitment to zero supply increased the profits of firms 1,2 and 3, but hurt firm 4. The consumers in market 1 are worse off, but the consumers in market 2 are better off.

This is the type of strategic complementarities analyzed in Bulow *et al.* (1985). The model they studied had two markets, while our model shows that their results carry through in a setup with multiple markets and firms.

6 Conclusion

We have analyzed a situation where firms embedded in a network with markets compete *a la* Cournot. We have shown that the equilibrium flows will depend on the whole structure. The quantity supplied by a firm to a market depends on the centrality of the links it has. The centrality index which determines the quantities is calculated using the line graph of the positive flow network. The quantity flowing through a link is positively proportional with the number of even paths and negatively proportional with the number of odd paths starting from it.

The network flows studied in graph theory and operations research literature[3] do not parallel the economic model studied in this paper. The analyzes of strategic behavior requires the introduction of decision making nodes to the model of flow networks. This distinguishes our approach from the existing literature on flow networks.

Although the network in our model is fixed, the analysis paves way for further research on strategic network formation in competitive markets. The results we provide can be used to calculate the benefit of each potential link to a firm. Once players know the payoff they would obtain in each network, they could manipulate their connections to maximize their profits.

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Appendix

Proof of Theorem 1 Given a graph g, at any the equilibrium of the Cournot game the flows cannot be negative

$$Q_g^* \geq 0 \tag{4}$$

For each link $(i, j) \in g$, at equilibrium $\frac{\partial \pi_j}{\partial q_{ij}}|_{q_{ij}^*} \leq 0$. More explicitly

$$\frac{\partial \pi_j}{\partial q_{ij}}|_{q_{ij}^*} = \alpha_i - \beta_i q_{ij}^* - \gamma_j \sum_{v_k \in N_g(f_j)} q_{kj}^* - \beta_i \sum_{f_k \in N_g(v_i)} q_{ik}^* \leq 0$$

These set of equations can be written in matrix form

$$-\boldsymbol{\alpha} + D_g Q_g^* \geq 0 \tag{5}$$

where $\boldsymbol{\alpha} = [\alpha_t]_r$ such that for $t = \tau(i, j)$, $\alpha_t = \alpha_i$ and $D_g = [d_{tz}]_{r \times r}$ such that

$$d_{tz} = \begin{cases} 2\beta_i + \gamma_j, & \text{if } t = z = \tau(i, j) \text{ for some } v_i \in V, f_j \in F \\ \gamma_j & , \text{ if } t \neq z, t = \tau(i, j), z = \tau(k, j) \text{ for some } v_i, v_k \in V, f_j \in F \\ \beta_i & , \text{ if } t \neq z, t = \tau(i, j), z = \tau(i, k) \text{ for some } v_i \in V, f_j, f_k \in F \\ 0 & , \text{ otherwise} \end{cases}$$

Lastly, for each link $(i, j) \in g$, at equilibrium $\frac{\partial \pi_j}{\partial q_{ij}}|_{q_{ij}^*} q_{ij}^* \leq 0$. In matrix form

$$(Q_g^*)^T(-\boldsymbol{\alpha} + D_g Q_g^*) \geq 0 \tag{6}$$

The first order equilibrium conditions $(4), (5), (6)$ of the Cournot game constitute a $LCP(-\boldsymbol{\alpha}; D_g)$.

Samelson *et al.* (1958) shows that a linear complementarity problem $LCP(p; M)$ has a unique solution for all $p \in \mathbb{R}^t$ if and only if all the principal minors of M are positive. Positive definite matrices satisfy this condition and we will now that $D_g{}^4$ is positive definite for any graph g.

We show that for any matrix D_g we can find a matrix R with independent columns such that $D_g = R^T R$. [5]

For example for graph g_1,

$$D_{g_1} = \begin{pmatrix} 2\beta_1 + \gamma_1 & \gamma_1 & \beta_1 & 0 \\ \gamma_1 & 2\beta_2 + \gamma_1 & 0 & \beta_2 \\ \beta_1 & 0 & 2\beta_1 + \gamma_2 & \gamma_2 \\ 0 & \beta_2 & \gamma_2 & 2\beta_2 + \gamma_2 \end{pmatrix}$$

We write R as

$$R = \begin{bmatrix} \sqrt{\beta_1} & 0 & 0 & 0 \\ 0 & \sqrt{\beta_2} & 0 & 0 \\ 0 & 0 & \sqrt{\beta_1} & 0 \\ 0 & 0 & 0 & \sqrt{\beta_2} \\ \sqrt{\gamma_1} & \sqrt{\gamma_1} & 0 & 0 \\ 0 & 0 & \sqrt{\gamma_2} & \sqrt{\gamma_2} \\ \sqrt{\beta_1} & 0 & \sqrt{\beta_1} & 0 \\ 0 & \sqrt{\beta_2} & 0 & \sqrt{\beta_2} \end{bmatrix}$$

Then clearly $D_{g_1} = R^T R$. Given a graph g, the same technique can be used to show that D_g is positive definite. Hence, for any g and any $\boldsymbol{\alpha}$, $LCP(-\boldsymbol{\alpha}; D_g)$ has a unique solution.

Now, let's check that the second order conditions are satisfied. For firm f_k with n connections we first label the connections from 1 to n. Hence, $N_g(f_k) = \{v_1, ..., v_n\}$. Then the Hessian of the profit function π_k is $H = [h_{ij}]_{n \times n}$ where

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[4]The interpretation, when we use it to find the equilibrium quantities flowing from markets to firms, is that the column z and the row z in D_g corresponds to the link (i, j) in g such that $\tau(i, j) = z$. Hence, column 1 and row 1 corresponds to the link $(1, 1)$, column 2 and row 2 corresponds to the link $(2, 1)$, column 3 and row 3 corresponds to the link $(1, 2)$, and column 4 and row 4 corresponds to the link $(2, 2)$.

[5]This is equivalent to checking that D is positive definite. For other characterizations of positive definiteness see Strang (1988).

$$h_{ij} = \begin{cases} -2\beta_i - \gamma_k, & \text{if } i = j \\ -\gamma_k, & \text{otherwise} \end{cases}$$

Let $H' = -H$. We can use the same technique applied for D_g to show that H' is positive definite. Hence, H is negative definite. The solution of $LCP(-\boldsymbol{\alpha}; D_g)$ is the equilibrium of the Cournot game. ∎

Proof of Theorem 2 Assume $Q^*_{g-Z(Q_g)}, Q^*_{g-Z(Q''_g)}$ are equilibria of the game at g and g', respectively. Let

$$g - Z(Q^*_g) = g' - Z(Q^*_{g'})$$

Then,

$$D_{g-Z(Q^*_g)}.Q^*_{g-Z(Q^*_g)} = \alpha.\mathbf{1} = D_{g'-Z(Q^*_{g'})}.Q^*_{g'-Z(Q^*_{g'})} = D_{g-Z(Q^*_g)}.Q^*_{g'-Z(Q^*_{g'})}$$

As we showed in proposition 6 $D_{g-Z(Q^*_g)}$ is positive definite, hence invertible.

$$Q^*_{g-Z(Q_g)} = Q^*_{g-Z(Q''_g)}$$

∎

Proof of Theorem 3

$$\begin{aligned} D_{g-Z(Q^*_g)}.Q^*_{g-Z(Q^*_g)} &= \left[A^{-1} + G^*\right].Q^*_{g-Z(Q^*_g)} \\ &= A^{-1}\left[I + AG^*\right].Q^*_{g-Z(Q^*_g)} \end{aligned}$$

Remember that Q^*_g is the solution to $LCP(-\alpha\mathbf{1}_r; D_g)$. Then, when we invert $D_{g-Z(Q^*_g)}$, the matrix multiplication $\left[D_{g-Z(Q^*_g)}\right]^{-1}\boldsymbol{\alpha}$ will give us a strictly positive vector.

$$\begin{aligned} \left[I + AG^*\right] &= \left[I - AG^*\right]^{-1}\left[I - (AG^*)^2\right] \\ \left[I + AG^*\right]^{-1} &= \left[I - (AG^*)^2\right]^{-1}\left[I - AG^*\right] \\ &\text{and} \\ \left[I - (AG^*)^2\right]^{-1} &= \sum_{k=0}^{\infty}(AG^*)^{2k} \end{aligned}$$

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Substituting this into $D_{g-Z(Q_g^*)}.Q_{g-Z(Q_g^*)}^* = \boldsymbol{\alpha}$,

$$
\begin{aligned}
Q_{g-Z(Q_g^*)}^* &= \left[I - (AG^*)^2\right]^{-1} \left[I - AG^*\right] A\boldsymbol{\alpha} \\
&= \sum_{k=0}^{\infty} (AG^*)^{2k} \left[I - AG^*\right] A\boldsymbol{\alpha} \\
&= \left[\sum_{k=0}^{\infty} (AG^*)^{2k} - \sum_{k=0}^{\infty} (AG^*)^{2k+1}\right] A\boldsymbol{\alpha} \\
&= \left[M((AG^*)^2, 1) - M((AG^*)^2, 1).(AG^*)\right] A\boldsymbol{\alpha}
\end{aligned}
$$

∎

NOTE DI LAVORO PUBLISHED IN 2009