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Summary

Is pollution a *dirty* word? To answer this question we develop an endogenous growth model *à la* Rebelo (1991) where dirtiness becomes a fundamental choice variable for the economy to grow. Conclusions to our analysis say that a positive sustainable economic growth is attainable only if polluting production activities are taken into account. Moreover, transitional dynamics points out that local stability and uniqueness of equilibria are also achieved.

Keywords: Environmental quality, Endogenous economic growth, Pollution-augmenting technology

JEL Classification: O41, Q01, Q32

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Introduction

Nowadays, pollution is still considered a *dirty* word. The basic question is whether or not a continued environmental degradation becomes necessary to the process of industrialisation of an economy of our times. It is commonly accepted in the literature of this field that a clear connection between growth and environmental quality is so much complex, and not so easy to be found.[1] In fact, although concentration in the environment of some pollutants seem to benefit from growth (see, for example, coliforms in river basins); others irremediably worsen (as for CO_2, SO_2); and still others do exhibit deterioration at a first stage followed by clear amelioration in a second phase of development.

Following Aghion-Howitt (1998), and Grimaud (1999), our scope is then to introduce environmental concerns as a fundamental choice variable for an economy to grow. To this end, the present paper is so aimed at describing how an economy — with depletion of environmental quality — performs by means of a Rebelo-type (1991) model. Why to choose this model? First, because it guarantees endogenous growth, although the simplicity of the structure, which is within the scope of our work. Second, because we do not need to endogenise the technological sector which is simply maintained constant, thus simplifying the analysis. In other words, this version of the model considers a production function close to Rebelo's (1991), and given by

$$y = Akz \tag{1}$$

where z represents a measure of dirtiness due to the existing production techniques (as pointed out by Aghion-Howitt, 1998), while A is a constant which captures the level of technology. Besides, y stands for output, and k is a measure for aggregate capital, respectively. As we do not distinguish any kind of specialisation among workers, from now on we will be dealing only with variables in per capita terms. Therefore, the level of new investments in physical capital can be expressed in the usual form

$$\dot{k} = y - c$$

We also borrow from Aghion-Howitt (1998) the assumption that pollution be a by-product of output. The flow of pollution loads P is then assumed to

[1] For a complete survey of the literature concerning environmental economics, sustainable development and endogenous growth, see Pittel (2003).

be proportional to the level of production, and to the use of cleaner technologies (which means low values of z) that reduce the pollution/output ratio

$$P = Yz^{\gamma} \qquad \gamma > 0 \tag{2}$$

Following the existing literature of the field we also assume that the structure of preferences be given by the following CIES utility function:

$$U(c, E) = \frac{(cE)^{1-\sigma} - 1}{1 - \sigma}$$

where c is per capita consumption, and E the usual environmental quality indicator (see, for example, Musu, 1995).[2] Moreover, we define

$$\phi(c, E) = \frac{E \cdot U_E}{c \cdot U_c}$$

as the ratio of the values of environmental quality and consumption, both evaluated at their marginal utilities (see Le Kama-Schubert, 2004). That is, $\phi(\cdot)$ reflects the "relative preference for the environment" of the representative agent. Therefore, the utility function we adopted so far allows us to deal with the useful property of unitarian "green preferences", that is $\phi = 1$.

On the other hand, environmental quality is supposed to evolve according to the law of motion

$$\dot{E} = \theta E - P \tag{3}$$

where θ represents the speed at which nature regenerates, and being now aware of the functional form assumed by the flow of pollution, when we substitute equation (1) into (2), such that $P = Yz^{\gamma} = Akz^{1+\gamma}$.

Finally, we focus on a centralised solution problem.[3]

[2] Both arguments c and E enter this utility function as two substitute goods. That is, as long as one increases, the second one must necessarily be reduced. Formally, this assumption requires

$$\frac{\partial^2 U}{\partial c \partial E} = \frac{1 - \sigma}{(cE)^{\sigma}} < 0$$

and consequently, $\sigma > 1$. Remember also that the higher σ, the less willing are households to accept deviations from a uniform pattern of consumption over time (see Barro-Sala-i-Martin, 2004).

[3] Appendix A provides a complete solution to the maximisation problem which will be discussed in the next section.

Social planner analysis

The social planner maximises the present discounted utility

$$\int_0^\infty \frac{(cE)^{1-\sigma}-1}{1-\sigma} e^{-\rho t} dt$$

subject to the following constraints on per capita physical capital, and environmental quality:

$$\begin{aligned} \dot{k} &= Akz - c \\ \dot{E} &= \theta E - Akz^{1+\gamma} \end{aligned}$$

and given initial conditions on the state variables

$$k(0) = k_0 \qquad E(0) = E_0$$

The current value Hamiltonian then looks like

$$H_C = \frac{(cE)^{1-\sigma}-1}{1-\sigma} + \lambda \left[Akz - c\right] + \mu \left[\theta E - Akz^{1+\gamma}\right]$$

where λ, and μ represent the shadow prices of physical capital, and environmental quality, respectively.

First order condition for a maximum requires the discount Hamiltonian function to be maximised with respect to its control variables (c, and z)

$$\frac{\partial H_C}{\partial c} = 0 \qquad \Longrightarrow \qquad \lambda = c^{-\sigma} E^{1-\sigma} \tag{4}$$

$$\frac{\partial H_C}{\partial z} = 0 \qquad \Longrightarrow \qquad \lambda = \mu(1+\gamma) z^{1+\gamma} \tag{5}$$

though the canonical system provides also the law of motion of each costate variable,

$$\begin{aligned} \frac{\dot{\lambda}}{\lambda} &= -\left(\frac{\gamma}{1+\gamma}\right) Az + \rho \\ \frac{\dot{\mu}}{\mu} &= -\frac{c}{E}(1+\gamma) z^{\gamma} - \theta + \rho \end{aligned} \tag{6}$$

this leading to a balanced growth rate given by

$$g = \frac{\varphi\left(1+\gamma\right)\tilde{z}^{\gamma}+\theta-\rho}{2\sigma-1} \tag{7}$$

where φ represents the share of consumption to environmental quality.[4]

A comparative static exercise might show that, depending the growth rate, g, on the level of dirty emissions, $\tilde{z}$, it follows necessarily that when $\tilde{z}$ increases, g raises accordingly. Hence, polluting emissions seem necessary for the economy to grow in the long-run. Formally,

$$\frac{\partial g}{\partial \tilde{z}} = \frac{\varphi\gamma\left(1+\gamma\right)\tilde{z}^{\gamma-1}}{2\sigma-1} > 0$$

since we assumed $\sigma > 1$.

Local stability and uniqueness of equilibria

Which path will this economy follow while converging to the steady state? Is our system stable or unstable? If it is stable, do solutions describe uniqueness or multiplicity of equilibria, or might we face indeterminacy problems? To answer these questions, we ought to investigate the local stability properties of the BGP found in the previous section and describe the reasons for why an indeterminate equilibrium could possibly arise. To this end, we have to analyse the Jacobian matrix of the reduced system, and check for the sign of the associated eigenvalues.

First of all, to reduce the system, we introduce the following convenient variable substitution:

$$\begin{aligned} x &= \frac{c}{E} \\ y &= \frac{c}{k} \end{aligned} \tag{8}$$

[4] Solution to this model requires consumption and environmental quality to grow in balanced growth at the same rate, that is

$$\frac{\dot{c}}{c} = \frac{\dot{E}}{E} = g$$

thus being the share of consumption to environmental quality constant along the BGP,

$$\frac{c}{E} = \varphi.$$

and make the weak sustainability condition, that is we assume the environmental quality to grow over time at a constant rate ($\frac{\dot{E}}{E} = \xi$);[5] thus driving to a system of three equations in three unknowns

$$\begin{aligned}
\frac{\dot{x}}{x} &= \left[\left(\frac{1-2\sigma}{\sigma}\right)\xi - \frac{\rho}{\sigma}\right] + \left(\frac{\gamma}{1+\gamma}\right)\frac{A}{\sigma}z \\
\frac{\dot{y}}{y} &= \left[\left(\frac{1-\sigma}{\sigma}\right)\xi - \frac{\rho}{\sigma}\right] + \left[\frac{\gamma-\sigma(1+\gamma)}{1+\gamma}\right]\frac{A}{\sigma}z + y \\
\frac{\dot{z}}{z} &= \frac{\theta}{\gamma} + \left(\frac{1+\gamma}{\gamma}\right)xz^{\gamma} - \frac{A}{1+\gamma}z
\end{aligned}$$

with the following steady state values

$$\begin{aligned}
\tilde{x} &= \left(\frac{\gamma}{1+\gamma}\right)\frac{1}{\tilde{z}^{\gamma}}\left[\frac{A\tilde{z}}{1+\gamma} - \frac{\theta}{\gamma}\right] \\
\tilde{y} &= A\tilde{z} - \xi \\
\tilde{z} &= \left(\frac{\Theta+\sigma\xi}{A\gamma}\right)(1+\gamma)
\end{aligned}$$

where $\Theta = \rho - (1-\sigma)\,\xi > 0$.

The Jacobian matrix, evaluated at the steady state, then becomes

$$J^* = J_{(\tilde{x},\tilde{y},\tilde{z})} = \begin{bmatrix} 0 & 0 & \left(\frac{\gamma}{1+\gamma}\right)\frac{A}{\sigma} \\ 0 & \frac{\Theta}{\sigma} - \left[\frac{\gamma-\sigma(1+\gamma)}{1+\gamma}\right]\frac{A}{\sigma}\tilde{z} & \left[\frac{\gamma-\sigma(1+\gamma)}{1+\gamma}\right]\frac{A}{\sigma}\tilde{y} \\ \left(\frac{1+\gamma}{\gamma}\right)\tilde{z}^{1+\gamma} & 0 & -\frac{A\tilde{z}}{1+\gamma} \end{bmatrix}$$

with the following associated signs[6]

$$J^* = \begin{bmatrix} 0 & 0 & + \\ 0 & + & - \\ + & 0 & - \end{bmatrix}$$

Proposition 1 *Let assume the following parameters' restrictions: $\gamma > 1$, $\sigma > 1$, $\xi > 0$, $\varphi > 0$, and $\rho > \theta$; then the equilibrium is locally unique: J^* has one negative eigenvalue and two eigenvalues with positive real parts.*

[5] Remember that under the *weak sustainability* version, environmental quality is not constrained to be constant over time ($\dot{E} = 0$), thanks to technological progress which permits to substitute natural capital with phisical capital continuously.

[6] Since both parameters γ and σ are constrained to be greater than unity ($\gamma > 1$, $\sigma > 1$).

Proof. For completeness, see also the Appendix B. ■

Briefly, we are able to derive the characteristic equation of the system, defined as

$$-\kappa^3 + trJ^*\kappa^2 - BJ^*\kappa + DetJ^* = 0$$

being κ the auxiliary variable (the eigenvalue of the system). Provided that $trJ^* > 0$, $BJ^* < 0$, and $DetJ^* < 0$, we can thus check for local stability of the system around the steady state by means of the neat Routh-Hurwitz theorem, which can be summarised as

> *The number of roots of the characteristic polynomial with positive real parts is equal to the number of variations of sign in the scheme*

$$-1 \quad trJ^* \quad -BJ^* + \frac{DetJ^*}{trJ^*} \quad DetJ^*$$

that we can briefly synthesise for our model as

$$- \quad + \quad + \quad -$$

that is, we have two changes of sign, hence J^* has one negative eigenvalue and two eigenvalues with positive real parts. As a consequence, the equilibrium is *locally unique.*

Trying to simulate the system numerically, we can solve for it by substituting out some reasonable parameter values that can be found across the literature on the field (See, for example, Stokey, 1998).[7] Therefore, the characteristic equation of the system now becomes

$$f(\kappa) = -\kappa^3 + 0.09\kappa^2 + 0.013\kappa - 0.0007 \tag{9}$$

---

[7] With the following parameters' scheme:

γ	σ	A	ξ	ρ
2	1.5	1	0.03	0.02

the Jacobian matrix now becomes:

$$J^* = \begin{bmatrix} 0 & 0 & 0.44 \\ 0 & 0.15 & 0.09 \\ 0.01 & 0 & -0.06 \end{bmatrix}$$

since we assume that the level of technology (A) is normalised to one, for simplicity, while γ and σ are strictly greater than one. Moreover, natural capital is assumed to grow at a 3% annual rate $\left(\frac{\dot{E}}{E} = \xi = 0.03\right)$. Finally, we assume a small, but still positive, level of the social discount rate, ρ.

which can be solved through Cardano's formula, and represented as follows

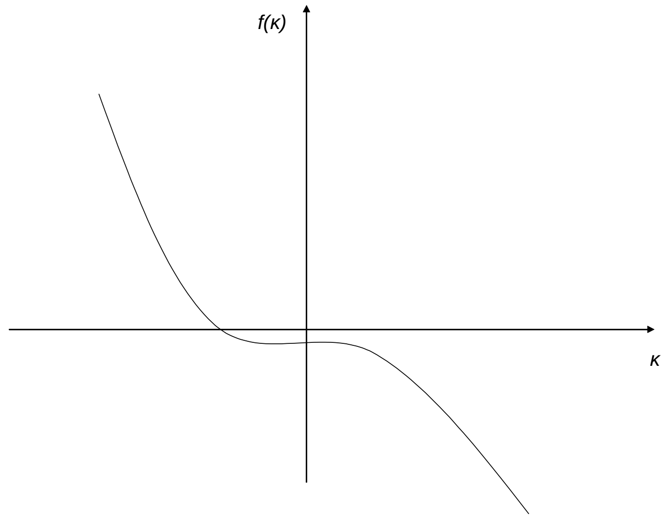

Figure 1: Characteristic Function

that is to say, there is a double change of sign, and there are one negative eigenvalue and two complex conjugate eigenvalues with positive real part.[8] With a three-dimensional phase space, motion close to an equilibrium can be studied on the basis of local linearised equations. In our case, graphic

[8] Cardano's formula to solve cubic equations in basic form:

$$x^3 + ax^2 + bx + c = 0$$

can be obtained through the convenient substitution $y = x + a/3$, that leads to the reduced form:

$$y^3 + py + q = 0$$

with $p = \frac{3b-a^2}{3}$ and $q = c + \frac{2a^3}{27} - \frac{ab}{3}$, thus deriving the following associated roots:

$$\begin{aligned} x_1 &= -\frac{a}{3} + u + v \\ x_{2,3} &= -\frac{a}{3} - \frac{u+v}{2} \pm \sqrt{3}\frac{u-v}{2} i \end{aligned}$$

where $i = \sqrt{-1}$ is the imaginary root, $u = \sqrt[3]{-\frac{q}{2} + \sqrt{D}}$ and $v = \sqrt[3]{-\frac{q}{2} - \sqrt{D}}$, whereas $D = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2$ is the discriminant.

representation of the solution might be depicted as

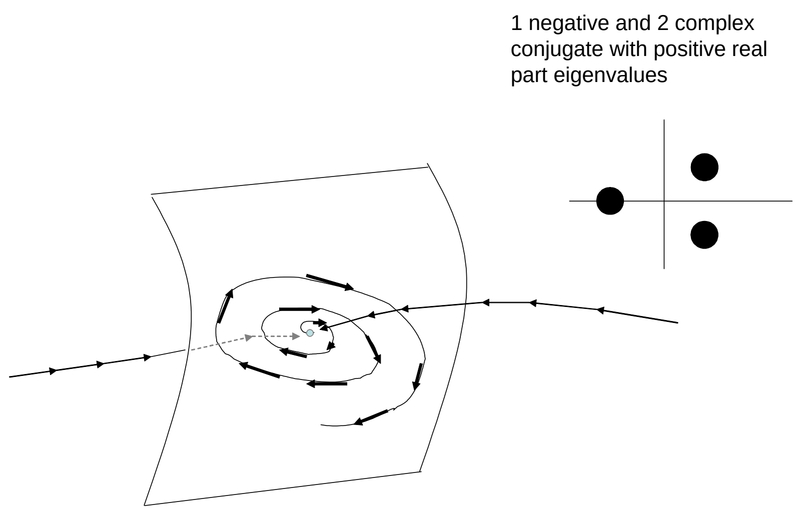

Figure 2: Liapunov's Saddle

As pointed out in the Argand diagram above the picture, we can think of it as a so-called Liapunov's "saddle of index 2", where the index stands for the number of positive eigenvalues. Hence, for any starting value of our *state-like* variables — capital (k) and environmental quality (E) — the corresponding initial values of our *control-like* variables — consumption (c) and the level of dirtiness (z) — must be those which lay along the stable manifold that drives the system towards the stable equilibrium point. The general idea is that, for any positive initial level of physical and natural capital, k_0 and E_0, there is a unique initial level of consumption and *dirtiness*, c_0 and z_0, that is consistent with households' intertemporal optimisation. Obviously, if the economy does not start with these initial values, that is we are off the stable manifold, we will never reach the equilibrium, thus being balanced growth amongst variables irremediably compromised.

Remarks

Although some attempts have been made to extend traditional endogenous growth models, they mostly lead to indeterminacy, that is multiple equilibria might arise.[9] Traditional explanations of indeterminacy arising in endogenous growth models is explained through two identical economies with identical initial conditions that might consume, produce new goods though polluting the environment, and exploit natural resources at completely different rates. Only in the long run are these economies supposed to converge to the same growth rate, but not to the same level of output, consumption, and human and physical capital.[10] Conclusions to our analysis say that we might face determinacy instead, and transitional dynamics confirms that when environmental issues are introduced into a Rebelo (1991) type model a unique stable equilibrium can be reached (that is, BGP is determinate), depending on the initial values the economy starts up with.

Appendix

A The Social Planner Maximisation Problem

The current value Hamiltonian for the maximisation problem is given by

$$H_c = \frac{(cE)^{1-\sigma} - 1}{1-\sigma} + \lambda \left[Akz - c\right] + \mu \left[\theta E - Akz^{1+\gamma}\right] \tag{A.1}$$

where λ and μ denote the costate variables associated with the accumulation of physical and natural capital, respectively.

1. First order conditions can be written as:

1.a $\left[\frac{\partial H_c}{\partial c} = 0\right]$:

$$\frac{\partial H_c}{\partial c} = c^{-\sigma} E^{1-\sigma} - \lambda = 0 \quad \Longrightarrow \quad c^{-\sigma} E^{1-\sigma} = \lambda \tag{A.2}$$

[9] For example, Benhabib and Perli (1994) study the dynamics of endogenous growth in a generalised version of Lucas (1988) that incorporates a labour-leisure choice; while Scholz and Ziemes (1999) try to explain exhaustible resource use by means of a Romer (1990) type model. They both conclude that equilibrium trajectories are indeterminate, and a continuum of equilibria is very likely to happen.

[10] It is usually assumed the presence of cultural and non-economic factors affecting fundamentals like technology or preferences to greenery, as a possible explanation for equilibria to differ along the transition paths.

1.b $\left[\frac{\partial H_c}{\partial z}=0\right]$:

$$\frac{\partial H_c}{\partial z}=\lambda Ak-\mu(1+\gamma)Akz^{\gamma}=0 \tag{A.3}$$

that is simply[11]

$$\lambda=\mu(1+\gamma)z^{\gamma} \tag{A.4}$$

2. Equation of motion for each costate variable is given by

$$\dot{\lambda}=-\frac{\partial H_c}{\partial k}+\lambda\rho \tag{A.5}$$

$$\dot{\mu}=-\frac{\partial H_c}{\partial E}+\mu\rho \tag{A.6}$$

and we can simply derive, by means of the conditions obtained above:

$$\begin{aligned}\frac{\dot{\lambda}}{\lambda} &= -\left(\frac{\gamma}{1+\gamma}\right)Az+\rho \\ \frac{\dot{\mu}}{\mu} &= -\frac{c}{E}(1+\gamma)z^{\gamma}-\theta+\rho\end{aligned} \tag{A.7}$$

2.b whereas, taking logs in (A.1) and differentiating, we have

$$\frac{\dot{\lambda}}{\lambda}=-\sigma\frac{\dot{c}}{c}+(1-\sigma)\frac{\dot{E}}{E} \tag{A.8}$$

since we assume that, in balanced growth, c and E must grow at the same rate, g, it is indeed true that

$$\frac{\dot{\lambda}}{\lambda}=(1-2\sigma)g \tag{A.9}$$

[11] Necessary condition for a maximum can be checked by studying the sign of all principal minors of the Hessian matrix for the control variables of the problem, whose determinant is formed by the following signs:

$$|H|=\begin{vmatrix} - & 0 \\ 0 & - \end{vmatrix}$$

thus obtaining, $|H_1|<0$, $|H_2|=|H|>0$, that is to say, the system is maximised.

2.c Moreover, the law of motion of the shadow price of the environment, the costate variable μ, is

$$\dot{\mu} = -c^{1-\sigma}E^{-\sigma} - \mu\theta + \mu\rho \tag{A.10}$$

or, alternatively,

$$\frac{\dot{\mu}}{\mu} = -\frac{U_E}{\mu} - \theta + \rho \tag{A.11}$$

given that $\frac{\partial U(\cdot)}{\partial E} = c^{1-\sigma}E^{-\sigma} = U_E$. But substituting out μ in the RHS, by means of equation (A.4), we obtain

$$\frac{\dot{\mu}}{\mu} = -\frac{U_E}{\lambda}(1+\gamma)z^{\gamma} - \theta + \rho \tag{A.12}$$

Since $\lambda = U_c$, from FOC, and given constancy of z in balanced growth at some value $\tilde{z}$, we have

$$\frac{\dot{\mu}}{\mu} = -\frac{U_E}{U_c}(1+\gamma)\tilde{z}^{\gamma} - \theta + \rho \tag{A.13}$$

and finally, since equilibrium requires that $\frac{U_E}{Uc} = \frac{c}{E} = \varphi$, it follows

$$\frac{\dot{\mu}}{\mu} = -\varphi(1+\gamma)\tilde{z}^{\gamma} - \theta + \rho \tag{A.14}$$

2.d Equation (A.4) says that

$$\lambda = \mu(1+\gamma)z^{\gamma} \tag{A.15}$$

but we are assuming that, for balanced growth to be achieved, z must be held constant at some value called $\tilde{z}$. In fact, from equation (1) we can derive

$$\frac{\dot{y}}{y} = \frac{\dot{k}}{k} + \frac{\dot{z}}{z}$$

but since balanced growth requires both output, y, and physical capital, k, to grow at the same rate, it follows consequently that z must be held constant at some value called $\tilde{z}$. Hence, λ and μ must be equal, so it is their growth rate:

$$\frac{\dot{\lambda}}{\lambda} = \frac{\dot{\mu}}{\mu} \tag{A.16}$$

On the other hand, from (A.16), by means of (A.9), follows that

$$\lambda_t = \mu_t = \tilde{\lambda} e^{(1-2\sigma)gt} \tag{A.17}$$

where $(1-2\sigma)g < 0$, since we assumed that $\sigma > 1$.

It is easy to note that as long as $t \to \infty$ all Lagrange multipliers converge to zero (with $\tilde{\lambda}$ being a constant value assumed by both shadow prices in BGP).

4. Transversality conditions for a free terminal state hold for all shadow prices, and are given by

$$\begin{aligned} \lim_{t\to\infty} \lambda k e^{-\rho t} &= \tilde{\lambda} e^{(1-2\sigma)gt} \tilde{k} e^{gt} e^{-\rho t} = \tilde{\lambda}\tilde{k} e^{-(2\sigma g+\rho)t} = 0 \qquad \text{(A.18)} \\ \lim_{t\to\infty} \mu E e^{-\rho t} &= \tilde{\mu} e^{(1-2\sigma)gt} \tilde{E} e^{gt} e^{-\rho t} = \tilde{\mu}\tilde{E} e^{-(2\sigma g+\rho)t} = 0 \end{aligned}$$

   where $\tilde{\lambda}$, $\tilde{\mu}$, and $\tilde{k}$, $\tilde{E}$, are the shadow prices and the state-values on the balanced growth path, respectively.

5. Moreover, for free time t, we need to show that $\lim_{t\to\infty} H = 0$, which is always verified due to convergence towards zero of both the discounted utility function, $\lim_{t\to\infty} U(\cdot)e^{-\rho t} = 0$, and all the multipliers, as proved above.

B Dynamics of a Rebelo-type model with *dirtiness*

Transitional dynamics of the problem can be derived through the law of motion of the state variables:

$$\begin{aligned} \dot{k} &= Akz - c \qquad \text{(B.1)} \\ \dot{E} &= \theta E - Akz^{1+\gamma} \end{aligned}$$

with the stated equations for the multipliers:

$$\begin{aligned} \frac{\dot{\lambda}}{\lambda} &= -\left(\frac{\gamma}{1+\gamma}\right) Az + \rho \qquad \text{(B.2)} \\ \frac{\dot{\mu}}{\mu} &= -\frac{c}{E}(1+\gamma)\tilde{z}^{\gamma} - \theta + \rho \end{aligned}$$

and being aware that the law of motion for z can be derived from first order condition (5), by taking logs to both sides, and then substituting out the law of motion of each multiplier, as defined in (6).

To make things simpler, we adopt the following convenient substitutions:

$$\begin{aligned} \frac{c}{E} &= x \\ \frac{c}{k} &= y \end{aligned} \tag{B.3}$$

and derive the system of autonomous equations:

$$\begin{aligned} \frac{\dot{x}}{x} &= \left[\left(\frac{1-2\sigma}{\sigma}\right)\xi - \frac{\rho}{\sigma}\right] + \left(\frac{\gamma}{1+\gamma}\right)\frac{A}{\sigma}z \\ \frac{\dot{y}}{y} &= \left[\left(\frac{1-\sigma}{\sigma}\right)\xi - \frac{\rho}{\sigma}\right] + \left[\frac{\gamma-\sigma(1+\gamma)}{1+\gamma}\right]\frac{A}{\sigma}z + y \\ \frac{\dot{z}}{z} &= \frac{\theta}{\gamma} + \left(\frac{1+\gamma}{\gamma}\right)xz^{\gamma} - \frac{A}{1+\gamma}z \end{aligned} \tag{B.4}$$

with the following steady-states equilibria:

$$\begin{aligned} \tilde{x} &= \left(\frac{\gamma}{1+\gamma}\right)\frac{1}{\tilde{z}^{\gamma}}\left[\frac{A\tilde{z}}{1+\gamma} - \frac{\theta}{\gamma}\right] \\ \tilde{y} &= A\tilde{z} - \xi \\ \tilde{z} &= \left(\frac{\Theta+\sigma\xi}{A\gamma}\right)(1+\gamma) \end{aligned} \tag{B.5}$$

where $\Theta = \rho - (1-2\sigma)\xi > 0$.

Stability analysis can be checked through the signs of the Jacobian matrix of the system

$$J^*_{(\tilde{x},\tilde{y},\tilde{z})} = \begin{bmatrix} J^*_{11} & J^*_{12} & J^*_{13} \\ J^*_{21} & J^*_{22} & J^*_{23} \\ J^*_{31} & J^*_{32} & J^*_{33} \end{bmatrix}$$

evaluated at the steady state $(\tilde{x}, \tilde{y}, \tilde{z})$, thus obtaining,

$$J^*_{(\tilde{x},\tilde{y},\tilde{z})} = \begin{bmatrix} 0 & 0 & \left(\frac{\gamma}{1+\gamma}\right)\frac{A}{\sigma} \\ 0 & \frac{\Theta}{\sigma} - \left[\frac{\gamma-\sigma(1+\gamma)}{1+\gamma}\right]\frac{A}{\sigma}\tilde{z} & \left[\frac{\gamma-\sigma(1+\gamma)}{1+\gamma}\right]\frac{A}{\sigma}\tilde{y} \\ \left(\frac{1+\gamma}{\gamma}\right)\tilde{z}^{1+\gamma} & 0 & -\frac{A\tilde{z}}{1+\gamma} \end{bmatrix} \tag{B.6}$$

where we assume $\frac{\dot{E}}{E} = \xi > 0$, and $\sigma > 1$.

The associated determinant then becomes

$$DetJ^* = \begin{vmatrix} 0 & 0 & \left(\frac{\gamma}{1+\gamma}\right)\frac{A}{\sigma} \\ 0 & \frac{\Theta}{\sigma} - \left[\frac{\gamma-\sigma(1+\gamma)}{1+\gamma}\right]\frac{A}{\sigma}\tilde{z} & \left[\frac{\gamma-\sigma(1+\gamma)}{1+\gamma}\right]\frac{A}{\sigma}\tilde{y} \\ \left(\frac{1+\gamma}{\gamma}\right)\tilde{z}^{1+\gamma} & 0 & -\frac{A\tilde{z}}{1+\gamma} \end{vmatrix} \tag{B.7}$$

that can be reduced to

$$DetJ^* = -\frac{A}{\sigma}\tilde{z}^{1+\gamma}\left\{\frac{\Theta}{\sigma} - \left[\frac{\gamma-\sigma(1+\gamma)}{1+\gamma}\right]\frac{A}{\sigma}\tilde{z}\right\} \tag{B.8}$$

which is always negative ($DetJ^* < 0$), as long as all parameters are constrained to be positive and, particularly, either $\gamma > 1$ or $\sigma > 1$, thus determining the following sign sequence for each matrix element:

$$J^* = \begin{bmatrix} 0 & 0 & + \\ 0 & + & - \\ + & 0 & - \end{bmatrix} \tag{B.9}$$

2. Following Benhabib and Perli (1994), we need to check for the sign of the real part of the roots (the eigenvalues of the Jacobian matrix), and study the stability of the system by means of the Routh-Hurwitz criterion. To this end, we derive the characteristic equation of the Jacobian matrix:

$$-\kappa^3 + trJ^*\kappa^2 - BJ^*\kappa + DetJ^* = 0$$

and examine the variation of signs in the following sequence:

$$-1 \quad trJ^* \quad -BJ^* + \frac{DetJ^*}{trJ^*} \quad DetJ^*$$

where the trace of the determinant is given by

$$trJ^* = J^*_{11} + J^*_{22} + J^*_{33}$$

that is explicitly

$$trJ^* = \frac{\Theta}{\sigma} - \left(\frac{\gamma}{1+\gamma}\right)\frac{(1-\sigma)A\tilde{z}}{\sigma} \tag{B.10}$$

which is clearly positive ($trJ^* > 0$), since $\sigma > 1$.

Furthermore, the cross determinant of the minors, is given by

$$BJ = \begin{vmatrix} J^*_{11} & J^*_{12} \\ J^*_{21} & J^*_{22} \end{vmatrix} + \begin{vmatrix} J^*_{22} & J^*_{23} \\ J^*_{32} & J^*_{33} \end{vmatrix} + \begin{vmatrix} J^*_{11} & J^*_{13} \\ J^*_{31} & J^*_{33} \end{vmatrix}$$

or, explicitly,

$$BJ^* = -\frac{A}{\sigma}\tilde{z}\left\{\frac{\Theta}{\sigma} - \left[\frac{\gamma - \sigma(1+\gamma)}{1+\gamma}\right]\frac{A}{\sigma}\tilde{z}\right\} - \frac{A}{\sigma}\tilde{z}^{1+\gamma} \tag{B.11}$$

which is clearly always strictly negative, $BJ < 0$.

And since,

$$-BJ^* + \frac{DetJ^*}{trJ^*} = \frac{A}{\sigma}\tilde{z}\left\{\frac{\Theta}{\sigma} - \left[\frac{\gamma - \sigma(1+\gamma)}{1+\gamma}\right]\frac{A}{\sigma}\tilde{z}\right\} + \\ + \frac{A}{\sigma}\tilde{z}^{1+\gamma} - \frac{\frac{A}{\sigma}\tilde{z}^{1+\gamma}\left\{\frac{\Theta}{\sigma} - \left[\frac{\gamma-\sigma(1+\gamma)}{1+\gamma}\right]\frac{A}{\sigma}\tilde{z}\right\}}{\frac{\Theta}{\sigma} - \left(\frac{\gamma}{1+\gamma}\right)\frac{(1-\sigma)A\tilde{z}}{\sigma}} > 0 \tag{B.12}$$

it is indeed true that the necessary condition

$$-BJ^* + \frac{DetJ^*}{trJ^*} > 0$$

always holds. It can be so proved that there are two change of sign in the characteristic roots, with one negative eigenvalue and two eigenvalues with positive real part. That is, there is always a continuum of equilibria.

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