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Environmental Pollution Risk and Insurance

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Summary

We consider environmental risks that are evaluated too heavy for a single insurance company, but that can be insured by *n* companies which a premium is assigned to.
This is precisely the Italian scenario where a pool of companies co-insures these risks. Under a game theoretic approach we start by analyzing how they should split the risk and the premium in order to be better off. Under suitable hypotheses, there exists an optimal decomposition of the risk, that allows us to define a cooperative game whose properties and some particular solutions are analyzed.

Keywords: Environmental risk, cooperative game

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1 Introduction

In this paper we consider environmental pollution risks depending on firms that in their production processes may have as a side effect the release of polluting wastes, that damage the environment.

Firms may interact with the environment in different ways; more precisely they can alter the standard environment, influence the possibilities of using it damaging public or private goods, compromise directly or indirectly the human health, contaminate biological resources and ecosystems.

The consequences may be described as damages to persons and/or materials or interruption of various activities (industrial, agricultural or recreational).

We want to recall some environmental pollution risks according to a simple classification:

- *air pollution*, generated by emissions, harmful gases, exhaust fumes, stenches, waste disposals, chemical productions;
- *water pollution*, when the factories discharge effluents in the rivers;
- *soil pollution*, deriving from rubbish, solid wastes, industrial wastes, for example from farms that use chemical manures or pesticides or from factories that dispose their wastes in the soil;
- *marine pollution*, this happens in the case of coastal firms that get rid wastes and sewage in the sea or by oil tankers accidents;
- *acoustic pollution*, due to high noise level and vibrations with a risk for workers and neighboring inhabitants.

In order to limit the costs related to environmental risk a firm generally effects an insurance policy that cover both third party liability for any damage that may be caused to persons or materials and unpollution costs due to removal of pollutants (cfr. Bazzano, 1994).

The losses for environmental pollution can be very heavy as it is shown in Table 1.

Year	*Place*	*Cost* *	*Cause (pollutant)*
1976	*Seveso (Italy)*	103	*Chemical plant (dioxin)*
1978	*Los Alfaques (Spain)*	15	*Tanker truck explosion (propylene)*
1982	*Livingstone (USA)*	41	*Derailment (toxics)*
1984	*Denver (USA)*	20	*Tank (gasoline)*
1985	*Kenora (Canada)*	7	*Spill (PCB)*
1986	*Basilea (Switzerland)*	16	*Fire with river Rhein pollution*
1987	*Herborn (Germany)*	8	*Tanker truck*
1988	*Floreffe (USA)*	67	*Tank explosion (oil)*
1988	*S. Basile (Canada)*	39	*Fire (toxic wastes)*
1988	*Piper Alpha (Northern Sea)*	111	*Explosion (gas)*

* in millions of euros
SCOR NOTES (1989)

Table 1

Costs refunded after main accidents in OCSE countries 1976-1988
Naval transportation accidents are not included

Moreover we want just to mention that the Lloyd's alone were charged with about 15 billions of euros (cfr. "Il sole 24 ore" 27 September 2001), only to refund damages related to absestos in about twenty years (but some trials are still lasting). So it is not possible that a single insurance company accepts to assume this risk on her own; as a consequence it is insured by a pool of companies.

For example in Italy there is a pool of 61 insurance companies that is the unique responsible for all such kind of risks, where each company assumes a percentage of risk as in Table 2. Here we suppose that n companies all together have to insure a given risk. Two important intertwined practical questions that arise then read: Which premium should they charge? How should they split the risk and the premium in order to make the n companies as competitive as possible and obtain a fair division? This problem can be modelled as a cooperative game.

We want to remark that the first examples of applications of game theory to insurance were given by Borch (1962a, 1962b) and Lemaire (1977, 1991) and more recently by Suijs, Borm, De Waegenaere and Tijs (1999); we address to Suijs (2000) for a survey on these topics.

The organization of the paper is the following: In Section 2 we give the formal description of a co-insurance problem introducing suitable notations and hypotheses, and state some preliminary results; Section 3 deals with a class of co-insurance games, paying particular attention to the property of balancedness and analysing some classical game theoretical solutions; Section 4 is devoted to the case of optimal decomposition of the risk in constant quotas; finally in Section 5 we summarize our results applying them to a case study, using the data of the Italian situation.

2 Hypotheses and Notations

As already said in the Introduction we consider a problem in which a risk is evaluated too much heavy for a single insurance company, but it can be insured by n companies, where each company assumes a quota of the risk and receives a fraction of the premium.

We consider a fixed and suitable probability space; we denote the set of companies by $N = \{1, ..., n\}$ and we suppose that every company $i \in N$, expresses her valuation of a random variable X as the value $H_i(X)$, where H_i is a functional from a class $\mathcal{L}$ of random variables (the insurable risks) into the set of real numbers, $\mathbb{R}$; this means that, given a risk X, $H_i(X)$ provides a measure of X (expected claims and security considerations). In order to determine the commercial premium to be charged, each company have to take into account her evaluation of the risk and usual economic factors (commissions and expenses).

As in Deprez and Gerber (1985) (see also Gerber, 1980 and Goovaerts et al., 1984), we add the hypotheses that a loading for a degenerate risk is not justified (**a**) and that if a risk is increased by an additive constant, this constant has to be added to the evaluation of the risk (translation invariance **b**), so for each $i \in N$ we ask that:

Hypothesis 1 **a)** $H_i(w) = w, \ \forall \ w \in \mathbb{R}$;
b) $H_i(w + X) = w + H_i(X), \ \forall \ w \in \mathbb{R}, \forall \ X \in \mathcal{L}$.

Many classical principles satisfy this hypothesis, for example:

- the *net premium principle* $H(X) = E(X)$, where $E(X)$ is the expectation of X;
- the *variance principle* $H(X) = E(X) + aV(X)$, where $V(X)$ is the variance of X and $a > 0$;
- the *standard deviation principle* $H(X) = E(X) + \beta\sqrt{V(X)}$, where $\beta > 0$;

- the *zero utility principle* $H(X) = \bar{H}$, where $\bar{H}$ satisfies $E[u(z + \bar{H} - X)] = u(z)$ and u is the utility function of the insurance company which has an initial surplus z; in particular for exponential utility $u(x) = \frac{1}{a}(1 - e^{-ax})$, with $a > 0$ we have $\bar{H} = \frac{1}{a} ln E(e^{aX})$.

- the ε-*percentile principle* $H(X) = min\ \{x | F(x) \geq 1 - \varepsilon\}$, where F is the distribution function of X.

Now we suppose that the n companies have to insure a given risk R and receive a premium π. Each company can decide to co-insure the risk or not, i.e. she has to be considered a decision maker; as a consequence in order to define "fair" allocations of the pair (π, R) we introduce the following notations, referring to all subsets (coalitions) S of companies involved. For any subset of companies $S \subseteq N$ we denote by $\mathcal{D}(S)$ the set of feasible divisions of the premium π, i.e.:

$$\mathcal{D}(S) = \left\{ (d_i)_{i \in S} \in \mathbb{R}^{|S|}\ s.t.\ \sum_{i \in S} d_i = \pi \right\}$$

and by $\mathcal{A}(S)$ the set of the feasible decompositions of the risk R, i.e.:

$$\mathcal{A}(S) = \left\{ (X_i)_{i \in S} \in \mathcal{L}^{|S|}, s.t. \sum_{i \in S} X_i = R \right\}$$

and we suppose that $\mathcal{A}(S)$ is non-empty. According to the allocation $(d_i, X_i)_{i \in S} \in \mathcal{D}(S) \times \mathcal{A}(S)$, for each $i \in S$ the company i receives the amount d_i and pays the random variable X_i. Now we suppose that for each subset $S \subseteq N$ it is possible to compute an optimal decomposition of the risk, i.e. we introduce the hypothesis:

Hypothesis 2 $\forall\ S \subseteq N$ *there exists* $\displaystyle\min_{(X_i)_{i \in S} \in \mathcal{A}(S)} \left\{ \sum_{i \in S} H_i(X_i) \right\} = P(S)$

$P(S)$ can be seen as the evaluation that the companies in S (as a whole) give of the risk R.

Example 1 $\forall\ i \in N$ *the variance principle holds, i.e.:*

$$H_i(Y) = E(Y) + a_i Var(Y) \qquad \forall\ Y \in \mathcal{L},\ 0 < a_1 \leq ... \leq a_n$$

It is possible to prove (Deprez - Gerber, 1985) that $P(N) = \sum_{i \in N} H_i(q_i R)$*; moreover as in Fragnelli-Marina (2001) we have:*

$$P(S) = \sum_{i \in S} H_i \left(\frac{q_i}{q(S)} R \right) \qquad \forall\ S \subset N$$

where $\frac{1}{a(S)} = \sum_{i \in S} \frac{1}{a_i}$, $q_i = \frac{a(N)}{a_i}$ *and* $q(S) = \sum_{i \in S} q_i$ *and we can also write:*

$$P(S) = E(R) + a(S) Var(R) \qquad \forall\ S \subseteq N$$

$\diamondsuit$

Now we consider the allocations of $(\pi,\ R)$ that assign to each company a risk and an amount high enough to cover this risk. Formally for any subset of companies $S \subseteq N$ we define the set of individually rational allocations:

$$\mathcal{B}(S) = \{(d_i, X_i)_{i\in S} \in \mathcal{D}(S) \times \mathcal{A}(S)|\ d_i - H_i(X_i) \geq 0,\ \forall\ i \in S\}$$

Remark 1 $\mathcal{B}(S) \neq \emptyset \Leftrightarrow P(S) \leq \pi$, *i.e. if the premium is larger than the evaluation (of the set of companies in* S*) of the optimal decomposition of the risk, there not exist individually rational allocations and vice versa.*

Remark 2 $\mathcal{B}(S) \neq \emptyset \Rightarrow \mathcal{B}(T) \neq \emptyset,\ \forall\ T \supset S$. *In fact* $P(T) \leq P(S)$ *because* $H_i(0) = 0,\ \forall i \in N$.

Remark 3 *In order to avoid trivial situations we suppose that* $\pi > P(N)$, *i.e. the* n *companies all together may obtain a positive gain.*

For any subset of companies $S \subseteq N$ such that $\mathcal{B}(S) \neq \emptyset$ we may define the set of allocations of $(\pi,\ R)$ corresponding to optimal risk decompositions:

$$\mathcal{O}(S) = \left\{(d_i, X_i)_{i\in S} \in \mathcal{B}(S)\ |\sum_{i\in S} H_i(X_i) = P(S)\right\}$$

and the set of Pareto optimal allocations of $(\pi,\ R)$:

$$\begin{aligned}\mathcal{PO}(S) &= \{(d_i, X_i)_{i\in S} \in \mathcal{B}(S)\ | \\ &\quad \nexists\ (d'_i, X'_i)_{i\in S} \in \mathcal{B}(S), \text{s.t. } d'_i - H_i(X'_i) > d_i - H_i(X_i),\ \forall\ i \in S\}\end{aligned}$$

We can state the following theorem similar to Proposition 3.5 of Suijs and Borm (1999):

Theorem 1 $\mathcal{O}(S) = \mathcal{PO}(S)$.

The proof is close to that of Proposition 1 in Fragnelli-Marina (2001), with S in the role of N.

Remark 4 *If* $\mathcal{B}(S) \neq \emptyset$ *then if we take* $(X_i)_{i\in\mathcal{A}(S)}$ *s.t.* $\sum_{i\in S} H_i(X_i) = P(S)$ *and define* $d_i = H_i(X_i) + \frac{1}{|S|}(\pi - P(S)),\ \forall\ i \in S$ *then* $(d_i, X_i)_{i\in S} \in \mathcal{PO}(S)$.

Finally we define two subsets of the set of Pareto optimal allocations for the grand coalition N:

$$\begin{aligned}Q(N) = \{(d_i, X_i)_{i\in N} \in \mathcal{PO}(N)\ |\ \forall\ S \subset N \text{ s.t. } \mathcal{B}(S) \neq \emptyset, \\ \nexists\ (d'_i, X'_i)_{i\in S} \in \mathcal{PO}(S) \text{ s.t. } d'_i - H_i(X'_i) > d_i - H_i(X_i),\ \forall\ i \in S\}\end{aligned}$$

$$CO(N) = \left\{(d_i, X_i)_{i\in N} \in \mathcal{PO}(N)\ |\ \forall\ S \subseteq N\ \textstyle\sum_{i\in S}(d_i - H_i(X_i)) \geq max\ \{0, \pi - P(S)\}\right\}$$

The allocations in the set $Q(N)$, restricted to the subsets $S \subset N$, are such that there do not exist Pareto optimal allocations for the set S preferable to them (for those subsets S for which individually rational allocations exist), while the set $CO(N)$ contains those allocations for which the restriction to any subset $S \subseteq N$ is rational for that subset, i.e. guarantees that S cannot do better acting separately by itself. We have:

Theorem 2 $CO(N) = Q(N)$

Proof.

"$\subseteq$" Let $(d_i, X_i)_{i\in N} \in CO(N)$; if there exists a coalition S s.t. $\mathcal{B}(S) \neq \emptyset$ and there exists an allocation $(d'_i, X'_i)_{i\in S} \in \mathcal{PO}(S)$ s.t. $d'_i - H_i(X'_i) > d_i - H_i(X_i),\ \forall\ i \in S$ then $\pi - P(S) = \sum_{i\in S}(d'_i - H_i(X'_i)) > \sum_{i\in S}(d_i - H_i(X_i))$. Contradiction.

"$\supseteq$" Let $(d_i, X_i)_{i\in N} \in Q(N)$; as $(d_i, X_i)_{i\in N} \in \mathcal{B}(N)$ then $\forall\ S \subseteq N\ \sum_{i\in S}(d_i - H_i(X_i)) \geq 0$; suppose that there exists a coalition S with $\pi - P(S) > 0$ (so $\mathcal{B}(S) \neq \emptyset$) s.t. $\sum_{i\in S}(d_i - H_i(X_i)) < \pi - P(S)$. Let $(X'_i)_{i\in S} \in \mathcal{A}(S)$ s.t. $\sum_{i\in S} H_i(X'_i) = P(S)$. If we define, $\forall i \in S,\ d'_i = d_i - H_i(X_i) + H_i(X'_i) + \frac{1}{|S|}\left(\pi - P(S) - \sum_{j\in S}(d_j - H_j(X_j))\right)$ then $\sum_{i\in S} d'_i = \pi$ and $d'_i - H_i(X'_i) > d_i - H_i(X_i) \geq 0,\ \forall i \in S$. Contradiction. □

The argumentations of the previous proof are similar to those of Theorem 2 in Lari - Marina (2000).

3 Co-Insurance Games

All what we said in the previous sections can be reviewed under the light of game theory. We recall that a cooperative game in characteristic function form with transferable utility (TU-game) is a pair (N, v) where N is the set of players and v is a real valued function on 2^N, with $v(\emptyset) = 0$, where $v(S), S \subseteq N$ is the worth of coalition S.

Given a game (N, v) the core is the set $Core(v) = \{(x_i)_{i\in N} \in \mathbb{R}^{|N|}\ s.t.\ \sum_{i\in S} x_i \geq v(S), \forall\ S \subseteq N\ and\ \sum_{i\in N} x_i = v(N)\}$. The first condition is called *coalition rationality* and expresses that given an allocation $x \in Core(v)$ each coalition S get at least its worth; the second condition is called *efficiency* and says that the core allocations divide exactly the worth of the grand coalition N. When a game has non-empty core it is said to be *balanced*.

For our co-insurance problem we define a game whose characteristic function v is:

$$v(S) = max\ \{0, \pi - P(S)\} \qquad \forall\ S \subseteq N$$

The following theorem states a connection between the allocations of the co-insurance problem and the allocations of the co-insurance game.

Theorem 3 $CO(N) \neq \emptyset \Leftrightarrow Core(v) \neq \emptyset$

Proof.

"$\Rightarrow$" Let $(d_i, X_i)_{i\in N} \in CO(N)$; if we define $y_i = d_i - H_i(X_i)\ \forall\ i \in N$ then $(y_i)_{i\in N} \in Core(v)$.

"$\Leftarrow$" Let $(y_i)_{i\in N} \in Core(v)$, $(X^*_i)_{i\in N} \in \mathcal{A}(N)$ s.t. $\sum_{i\in N} H_i(X^*_i) = P(N)$ and define $d^*_i = y_i + H_i(X^*_i)\ \forall\ i \in N$; then $(d^*_i, X^*_i)_{i\in N} \in CO(N)$. □

Before studying the properties of the game we can reorder the players in such a way that:

$$P(N) \leq P(N \setminus \{n\}) \leq ... \leq P(N \setminus \{1\}) \tag{1}$$

As a consequence we have $\sum_{j \in N} P(N \setminus \{j\}) - (n-1)P(N) \geq P(N \setminus \{i\}), \forall\, i \in N$.

In these general hypotheses we can state the following results:

Lemma 1 $\pi \leq P(N \setminus \{1\}) \Rightarrow$ *the game is balanced.*

Proof. If $P(N) < \pi \leq P(N \setminus \{1\})$ a core-allocation is given by $x = (\pi - P(N), 0, ..., 0)$. In fact x is efficient as $\sum_{i \in N} x_i = \pi - P(N) = v(N)$; x is coalitionally rational because if $S \supseteq \{1\}$ then $\sum_{i \in S} x_i = \pi - P(N) \geq max\{0, \pi - P(S)\} = v(S)$ and if $S \not\supseteq \{1\}$ then $S \subseteq N \setminus \{1\} \Rightarrow P(S) \geq P(N \setminus \{1\}) \Rightarrow \pi - P(S) \leq \pi - P(N \setminus \{1\}) \Rightarrow v(S) = 0 = \sum_{i \in S} x_i$. □

Lemma 2 $\pi > \sum_{i \in N} P(N \setminus \{i\}) - (n-1)P(N) \Rightarrow$ *the game is not balanced.*

Proof. Note that by hypothesis it follows that $\pi > P(N \setminus \{i\})$ and then $v(N \setminus \{i\}) = \pi - P(N \setminus \{i\}), \forall\, i \in N$. For a balanced game we have that for each core allocation $(x_i)_{i \in N}$:

$$v(N) - x_i = \sum_{j \in N \setminus \{i\}} x_j \geq v(N \setminus \{i\}) \qquad \forall\, i \in N$$

and consequently:

$$(n-1)v(N) \geq \sum_{i \in N} v(N \setminus \{i\})$$

But we have:

$$\frac{1}{n-1} \sum_{i \in N} v(N \setminus \{i\}) > v(N) \Leftrightarrow \frac{n}{n-1}\pi - \frac{1}{n-1} \sum_{i \in N} P(N \setminus \{i\}) > \pi - P(N) \Leftrightarrow$$
$$\Leftrightarrow \frac{1}{n-1}\pi > \frac{1}{n-1} \sum_{i \in N} P(N \setminus \{i\}) - P(N)$$

that is equivalent to the hypothesis of the lemma, so the core is empty. □

Finally we can state the following theorem:

Theorem 4 *There exists* $\widehat{\pi}$ *such that the game is balanced if and only if* $\pi \leq \widehat{\pi}$

The proof is similar to that of Theorem 1 in Fragnelli *et al.* (2000).

Now we introduce a hypothesis that allows us to determine the value of $\widehat{\pi}$ defined in Theorem 4.

Hypothesis 3 *We ask that the cost function P satisfies the (reduced convexity) hypothesis:*

$$P(N \setminus \{i\}) - P(N) \leq P(S) - P(S \cup \{i\}), \quad \forall\, S \subseteq N \setminus \{i\} \;\; s.t.\; P(S) < \sum_{j \in N} P(N \setminus \{j\}) - (n-1)P(N)$$

Theorem 5 *Suppose that P satisfies Hypothesis 3, then* $\widehat{\pi} = \sum_{j\in N} P(N\setminus\{j\}) - (n-1)P(N)$.

Proof. In view of Lemma 2 and of Theorem 4 it is sufficient to prove that the game is balanced for $\pi = \sum_{j\in N} P(N\setminus\{j\}) - (n-1)P(N)$.
We will prove that the marginal solution $x = (P(N\setminus\{1\}) - P(N), ..., P(N\setminus\{n\}) - P(N))$ is a core allocation. Note that $x_i \geq 0, \forall\, i \in N$. x is efficient in fact:

$$\sum_{i\in N} x_i = \sum_{i\in N} P(N\setminus\{i\}) - nP(N) = \pi - P(N) = v(N)$$

To prove that x is coalitionally rational we consider first the case of $v(S) = 0$ that is trivial as $x \geq 0$; in the case of $v(S) > 0$ we have:

$$\sum_{i\in S} x_i \geq v(S) \Leftrightarrow \sum_{i\in S}(P(N\setminus\{i\}) - P(N)) \geq \sum_{i\in N} P(N\setminus\{i\}) - (n-1)P(N) - P(S) \Leftrightarrow$$
$$\Leftrightarrow \sum_{i\in N\setminus S}(P(N\setminus\{i\}) - P(N)) \leq P(S) - P(N)$$

Let $N\setminus S = \{t_1, ..., t_m\}$; the previous inequalities holds as a surrogate of the following relations:

$$\begin{array}{rcl} P(N\setminus\{t_1\}) - P(N) & \leq & P(S) - P(S\cup t_1) \\ P(N\setminus\{t_2\}) - P(N) & \leq & P(S\cup t_1) - P(S\cup\{t_1,t_2\}) \\ & ... & \\ P(N\setminus\{t_{m-1}\}) - P(N) & \leq & P(S\cup\{t_1,...,t_{m-2}\}) - P(S\cup\{t_1,...,t_{m-1}\}) \\ P(N\setminus\{t_m\}) - P(N) & = & P(S\cup\{t_1,...,t_{m-1}\}) - P(N) \end{array}$$

□

Now we want to analyse the particular case in which P satisfies Hypothesis 3 and the premium is precisely $\widehat{\pi} = \sum_{j\in N} P(N\setminus\{j\}) - (n-1)P(N)$, referring to classical game theoretical solution concepts.

We claim that the core is the singleton whose only element is the marginal solution $x = (P(N\setminus\{1\}) - P(N), ..., P(N\setminus\{n\}) - P(N))$. Suppose that there exists a different core allocation y; by efficiency there exists a player j such that $y_j > x_j$; in this case

$$\sum_{i\in N\setminus\{j\}} y_i < \sum_{i\in N\setminus\{j\}} x_i = \sum_{i\in N\setminus\{j\}} P(N\setminus\{i\}) - (n-1)P(N) = \widehat{\pi} - P(N\setminus\{j\}) = v(N\setminus\{j\})$$

so y is not coalitionally rational and does not belong to the core.

As a consequence x is also the nucleolus of the game. This solution concept corresponds to the unique allocation that minimize the maximum excess of the coalitions, according to a lexicographic order and it lies in the core if it is non-empty (for more details see Schmeidler 1969).

In 1981 Tijs introduced as a solution for a TU-Game the τ-value, the first of a series of compromise values; it is defined as follows:
Let $M_i = v(N) - v(N\setminus\{i\})$, i.e. the marginal contribution of player i and let $m_i = max\left\{v(S) - \sum_{j\in S\setminus\{i\}} M_j \mid S \subseteq N, S \ni i\right\}$; if the game is quasi-balanced ($M_i \geq m_i, \forall i \in N$

and $\sum_{i\in N} m_i \leq v(N) \leq \sum_{i\in N} M_i)$ the τ-value is the unique convex combination of M and m s.t. $\sum_{i\in N}\tau_i = v(N)$. In our situation, for each player i we have:

$$M_i = v(N) - v(N\setminus\{i\}) = \widehat{\pi} - P(N) - \widehat{\pi} + P(N\setminus\{i\}) = x_i$$

As the game is balanced then it is also quasi-balanced and so $M_i \geq m_i$; on the other hand by definition of m_i we have:

$$m_i \geq v(N) - \sum_{j\in N\setminus\{i\}} M_j = M_i$$

and finally $m_i = M_i = \tau_i$.

Remark 5 *If $\pi > \sum_{j\in N} P(N\setminus\{j\}) - (n-1)P(N)$ then not only the game is not balanced but it is neither quasi-balanced in fact:*

$$v(N) = \pi - P(N) > \sum_{j\in N}(P(N\setminus\{j\}) - P(N)) = \sum_{j\in N} M_j$$

Moreover we have also:

$$m_i \geq v(N) - \sum_{j\in N\setminus\{i\}} M_j > M_i$$

Referring to Example 1 we have that the cost function P satisfies the reduced convexity hypothesis 3 and the optimal decomposition consists of constant quotas. In the next section we study this more general situation.

4 Constant Quotas

In this section we suppose that there exist a convex function H and n real numbers $q_1 \geq \ldots \geq q_n > 0, \sum_{i\in N} q_i = 1$ s.t.:

$$H_i(Y) = q_i H\left(\frac{Y}{q_i}\right), \qquad \forall\, i \in N, \forall\, Y \in \mathcal{L}$$

(In Example 1 $H(Y) = E(Y) + a(N)Var(Y)$).

If the function H verifies Hypothesis 1 and is strictly convex (i.e. $H(sY+tZ) < sH(Y) + tH(Z)$ for $s+t=1$, $s \in]0,1[$, $\forall\, Y,\ Z \in \mathcal{L}$, unless $Y-Z$ is a constant) and is Gâteaux differentiable, we have (cfr. Deprez - Gerber (1985) and Lari - Marina (2000)) that, for each $S \subseteq N$:

$$P(S) = q(S)H\left(\frac{R}{q(S)}\right) = \sum_{i\in S} H_i\left(\frac{q_i}{q(S)}R\right) \leq \sum_{i\in S} H_i(X_i) \qquad \forall\, (X_i)_{i\in S} \in \mathcal{A}(S)$$

(where $q(S) = \sum_{i\in S} q_i$)

Remark 6 *The equality holds only if there exist $(\gamma_i)_{i\in S} \in \mathbb{R}^{|S|}$ s.t. $\sum_{i\in S}\gamma_i = 0$ and $X_i = \frac{q_i}{q(S)}R + \gamma_i,\ \forall i \in S$.*

Now we have:

Proposition 1 *The function P verifies Hypothesis 3.*

Proof. We define the functions $g(z) = H(zR)$ and $h(z) = zg\left(\frac{1}{z}\right)$ for each $z > 0$.

By the convexity of H we have that g is a convex function. Moreover let $0 < z_1 < z_2$ and let $\lambda \in]0,1[$; we have

$$g\left(\frac{1}{\lambda z_1 + (1-\lambda)z_2}\right) = g\left(\frac{\lambda z_1}{\lambda z_1 + (1-\lambda)z_2}\frac{1}{z_1} + \frac{(1-\lambda)z_2}{\lambda z_1 + (1-\lambda)z_2}\frac{1}{z_2}\right) \leq$$
$$\leq \frac{\lambda z_1}{\lambda z_1 + (1-\lambda)z_2}\, g\left(\frac{1}{z_1}\right) + \frac{(1-\lambda)z_2}{\lambda z_1 + (1-\lambda)z_2}\, g\left(\frac{1}{z_2}\right)$$

and so

$$h(\lambda z_1 + (1-\lambda)z_2) = (\lambda z_1 + (1-\lambda)z_2)\; g\left(\frac{1}{\lambda z_1 + (1-\lambda)z_2}\right) \leq$$
$$\leq \lambda z_1\; g\left(\frac{1}{z_1}\right) + (1-\lambda)z_2\; g\left(\frac{1}{z_2}\right) = \lambda h(z_1) + (1-\lambda)h(z_2)$$

So we have that h is a convex function and then P verifies Hypothesis 3. □

As a consequence of Theorem 5 also in this more general case if the value of the premium is exactly $\sum_{j\in N} P(N \setminus \{j\}) - (n-1)P(N)$, the core is non empty and the only core allocation is the marginal solution, that assigns to each player exactly his marginal contribution; it corresponds to the co-insurance problem allocation $(q_iP(N) + P(N \setminus \{i\}) - P(N), q_iR)_{i\in N}$ that belongs to $CO(N)$.

Before concluding the paper we want to analyze the widely used proportional (problem) allocation $(q_i\pi, q_iR)_{i\in N}$; more precisely we are interested if this allocation belongs to $CO(N)$. First we check that this solution belongs to $\mathcal{B}(N)$; we have:

$$q_i\pi - H_i(q_iR) \geq 0 \Leftrightarrow q_i\pi \geq q_iH(R) \Leftrightarrow q_i\pi \geq q_iP(N) \Leftrightarrow \pi \geq P(N)$$

where the last inequality is true according to our hypothesis of non-trivial situation.

The previous result guarantees that $\sum_{i\in S}(q_i\pi - H_i(q_iR)) \geq 0$, so if $\pi - P(S) \leq 0\ \forall S \neq N$ then trivially the proportional allocation belongs to $CO(N)$.

Otherwise at least $\pi - P(N \setminus \{n\}) > 0$ and for those $S \neq N$ such that $\pi - P(S) > 0$ the condition is:

$$\begin{aligned}\sum_{i\in S} q_i(\pi - P(N)) \geq \pi - P(S) \quad &\Leftrightarrow \quad q(S)(\pi - P(N)) \geq \pi - P(S) \Leftrightarrow \\ &\Leftrightarrow \quad \pi(1 - q(S)) \leq q(S)\left(H\left(\tfrac{R}{q(S)}\right) - H(R)\right) \Leftrightarrow \\ &\Leftrightarrow \quad \pi \leq \frac{H\left(\frac{R}{q(S)}\right) - H(R)}{\frac{1}{q(S)} - 1}\end{aligned}$$

By the convexity of H the last condition holds for all $S \neq N$ s.t. $\pi - P(S) > 0$ if and only if it holds for $S = N \setminus \{n\}$, so for the proportional allocation we have:

$$(q_i\pi, q_iR)_{i\in N} \in CO(N) \Leftrightarrow \pi \leq \frac{1-q_n}{q_n}\left(H\left(\frac{R}{q(N \setminus \{n\})}\right) - H(R)\right) = \widetilde{\pi}$$

If the premium is exactly $\widetilde{\pi}$ as above, reverting to the game we have:

$$v(N) = \widetilde{\pi} - P(N) = \frac{1}{q_n} P(N \setminus \{n\}) - \frac{q(N \setminus \{n\})}{q_n} P(N) - P(N) = \frac{1}{q_n}(P(N \setminus \{n\}) - P(N))$$

In this case the game solution related to the previous proportional allocation is $\left(\frac{q_i}{q_n}(P(N \setminus \{n\}) - P(N)\right)_{i \in N}$; this means that player n gets exactly its marginal contribution, while each player $i \in N \setminus \{n\}$ gets the marginal contribution of player n times the ratio among q_i and q_n. Note that these amounts are non increasing.

We can also investigate the relationship of the proportional solution with the marginal solution of the previous section, when the function P satisfies the hypothesis of convexity and the premium is $\pi = \sum_{j \in N} P(N \setminus \{j\}) - (n-1)P(N)$. In this case the proportional solution assigns to player i the amount $q_i \left(\sum_{j \in N} P(N \setminus \{j\}) - P(N)\right)$, while the marginal solution assigns the amount $P(N \setminus \{i\}) - P(N)$. This means that the proportional solution divides each marginal contribution proportionally among all the players (that get also the "refund" of the risk assumed) while, as we said above, the marginal solution assigns to each player exactly his marginal contribution (besides the "refund" of the risk assumed).

5 Case Study

In this concluding section we want to apply our results to the data of the Italian case (see Section 1). We lack of suitable real data so we make some assumptions. We suppose that the 61 companies, as in Example 1, express their evaluation of a random variable X according to the variance principle. Next we suppose that $q_i, i \in N$ are the quotas of the risk as in Table 2 and $a(N) = 0.1$; finally we suppose that the distribution function of the risk R is $F(x) = 1 - e^{-\mu x}$, so we have $E(R) = \frac{1}{\mu}$ and $Var(R) = \frac{1}{\mu^2}$.

We make the assumption that $E(R)$ is 1.05 ($\mu = \frac{1}{1.05}$ and $Var(R) = 1.1025$) and compute (in millions of euros):

$P(N) = 1.160250$

$\widehat{\pi} = 1.274612$

$\widetilde{\pi} = 1.270816$

and some allocations for the problem and for the game, as in Table 3.

These allocations clearify why if $\pi = \sum_{j \in N} P(N \setminus \{j\}) - (n-1)P(N)$ the proportional allocation is not in the core of the game: It assigns too much to last players, w.r.t. what is assigned to the first players. This means that the unique core allocation, i.e. the marginal solution, is favourable to the first players, i.e. those who assume larger quotas of risk; on the other hand the proportional solution is more "levelled" so that in order to have a core allocation it is necessary to have a lower premium that produces an amount assigned to the last player by the proportional allocation at most equal to the amount of the marginal one.

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	COMPANY	QUOTA %
1	ALLIANZ SUBALPINA	1.286
2	LE ASSICURAZIONI DI ROMA	0.286
3	ASSICURAZIONI GENERALI	5.263
4	ASSIMOCO	0.429
5	ASSITALIA-LE ASSICURAZIONI D'ITALIA	5.263
6	AUGUSTA ASSICURAZIONI	0.717
7	AURORA ASSICURAZIONI	1.071
8	AXA ASSICURAZIONI	2.460
9	BAYERISCHE RUCK (*)	2.857
10	BERNESE ASS.NI-COMP. ITALO-SVIZZERA	0.429
11	BNC ASSICURAZIONI	0.286
12	COMPAGNIA ASSICURATRICE UNIPOL	2.231
13	COMPAGNIA DI ASSICURAZIONE DI MILANO	5.263
14	IL DUOMO	0.574
15	ERC - FRANKONA AG (*)	5.714
16	F.A.T.A.	1.429
17	LA FONDIARIA ASSICURAZIONI	5.263
18	GAN ITALIA	0.791
19	GENERAL & COLOGNE RE (*)	2.714
20	GIULIANA ASSICURAZIONI	0.286
21	ITALIANA ASSICURAZIONI	0.857
22	ITAS ASSICURAZIONI	0.529
23	ITAS SOC. DI MUTUA ASSICURAZIONE	0.529
24	LEVANTE NORDITALIA ASSICURAZIONI	1.029
25	LIGURIA	0.429
26	LLOYD ADRIATICO	1.340
27	LLOYD ITALICO ASSICURAZIONI	0.429
28	MAECI - SOC. MUTUA DI ASS.NI E RIASS.NI	0.286
29	MAECI ASSICURAZIONI E RIASSICURAZIONI	0.429
30	LA MANNHEIM	0.429
31	MEDIOLANUM ASSICURAZIONI	0.429
32	METE ASSICURAZIONI	1.143
33	MUNCHENER RUCK ITALIA (*)	3.286
34	LA NATIONALE	0.429
35	NATIONALE SUISSE	0.429
36	NAVALE ASSICURAZIONI	0.963
37	NEW RE (*)	2.571
38	NUOVA MAA ASSICURAZIONI	0.429
39	NUOVA TIRRENA	1.743
40	PADANA ASSICURAZIONI	2.143
41	LA PIEMONTESE SOC. MUTUA DI ASS.NI	0.429
42	LA PIEMONTESE ASSICURAZIONI	0.429
43	RISPARMIO ASSICURAZIONI	0.286
44	RIUNIONE ADRIATICA DI SICURTA'	5.263
45	ROYAL & SUN ALLIANCE	0.857
46	SAI	5.263
47	SARA ASSICURAZIONI	0.429
48	SASA	0.429
49	SCOR ITALIA RIASSICURAZIONI (*)	2.571
50	S.E.A.R.	0.286
51	SIAT-SOCIETA' ITALIANA ASS.NI E RIASS.NI	0.429
52	SOCIETA' CATTOLICA DI ASSICURAZIONE	1.186
53	SOCIETA' REALE MUTUA DI ASSICURAZIONI	1.429
54	SOREMA (*)	2.571
55	SWISSE RE - ITALIA	7.714
56	TICINO	0.306
57	TORO ASSICURAZIONI	2.857
58	UNIASS ASSICURAZIONI	0.686
59	UNIVERSO ASSICURAZIONI	0.429
60	VITTORIA ASSICURAZIONI	0.840
61	WINTERTHUR ASSICURAZIONI	0.857
TOTAL		100.000

(*) Reinsurance company

Table 2

Division Plan for the Italian Pool for Environmental Risk Insurance

			PROBLEM			GAME		
Comp.(i)	%	q_i	$q_i\widetilde{\pi}$	$q_i\widehat{\pi}$	marg	$q_i(\widetilde{\pi} - P(N))$	$q_i(\widehat{\pi} - P(N))$	marg
55	7.714	0.07714	98,031	98,324	98,717	8,529	8,822	9,216
15	5.714	0.05714	72,614	72,831	72,978	6,318	6,535	6,681
3	5.263	0.05263	66,883	67,083	67,189	5,819	6,019	6,125
5	5.263	0.05263	66,883	67,083	67,189	5,819	6,019	6,125
13	5.263	0.05263	66,883	67,083	67,189	5,819	6,019	6,125
17	5.263	0.05263	66,883	67,083	67,189	5,819	6,019	6,125
44	5.263	0.05263	66,883	67,083	67,189	5,819	6,019	6,125
46	5.263	0.05263	66,883	67,083	67,189	5,819	6,019	6,125
33	3.286	0.03286	41,759	41,884	41,872	3,633	3,758	3,746
9	2.857	0.02857	36,307	36,416	36,391	3,159	3,267	3,242
57	2.857	0.02857	36,307	36,416	36,391	3,159	3,267	3,242
19	2.714	0.02714	34,490	34,593	34,565	3,001	3,104	3,076
37	2.571	0.02571	32,673	32,770	32,739	2,843	2,940	2,909
49	2.571	0.02571	32,673	32,770	32,739	2,843	2,940	2,909
54	2.571	0.02571	32,673	32,770	32,739	2,843	2,940	2,909
8	2.460	0.02460	31,262	31,355	31,323	2,720	2,813	2,781
12	2.231	0.02231	28,352	28,437	28,401	2,467	2,551	2,516
40	2.143	0.02143	27,234	27,315	27,279	2,369	2,451	2,414
39	1.743	0.01743	22,150	22,216	22,179	1,927	1,993	1,956
16	1.429	0.01429	18,160	18,214	18,178	1,580	1,634	1,598
53	1.429	0.01429	18,160	18,214	18,178	1,580	1,634	1,598
26	1.340	0.01340	17,029	17,080	17,045	1,482	1,532	1,497
1	1.286	0.01286	16,343	16,392	16,357	1,422	1,471	1,436
52	1.186	0.01186	15,072	15,117	15,084	1,311	1,356	1,323
32	1.143	0.01143	14,525	14,569	14,536	1,264	1,307	1,275
7	1.071	0.01071	13,610	13,651	13,620	1,184	1,225	1,194
24	1.029	0.01029	13,077	13,116	13,085	1,138	1,177	1,146
36	0.963	0.00963	12,238	12,275	12,245	1,065	1,101	1,072
21	0.857	0.00857	10,891	10,923	10,896	948	980	953
45	0.857	0.00857	10,891	10,923	10,896	948	980	953
61	0.857	0.00857	10,891	10,923	10,896	948	980	953
60	0.840	0.00840	10,675	10,707	10,680	929	961	934
18	0.791	0.00791	10,052	10,082	10,057	875	905	879
6	0.717	0.00717	9,112	9,139	9,115	793	820	796
58	0.686	0.00686	8,718	8,744	8,721	758	785	762
14	0.574	0.00574	7,294	7,316	7,296	635	656	636
22	0.529	0.00529	6,723	6,743	6,724	585	605	586
23	0.529	0.00529	6,723	6,743	6,724	585	605	586
4	0.429	0.00429	5,452	5,468	5,452	474	491	475
10	0.429	0.00429	5,452	5,468	5,452	474	491	475
25	0.429	0.00429	5,452	5,468	5,452	474	491	475
27	0.429	0.00429	5,452	5,468	5,452	474	491	475
29	0.429	0.00429	5,452	5,468	5,452	474	491	475
30	0.429	0.00429	5,452	5,468	5,452	474	491	475
31	0.429	0.00429	5,452	5,468	5,452	474	491	475
34	0.429	0.00429	5,452	5,468	5,452	474	491	475
35	0.429	0.00429	5,452	5,468	5,452	474	491	475
38	0.429	0.00429	5,452	5,468	5,452	474	491	475
41	0.429	0.00429	5,452	5,468	5,452	474	491	475
42	0.429	0.00429	5,452	5,468	5,452	474	491	475
47	0.429	0.00429	5,452	5,468	5,452	474	491	475
48	0.429	0.00429	5,452	5,468	5,452	474	491	475
51	0.429	0.00429	5,452	5,468	5,452	474	491	475
59	0.429	0.00429	5,452	5,468	5,452	474	491	475
56	0.306	0.00306	3,889	3,900	3,889	338	350	338
2	0.286	0.00286	3,635	3,645	3,635	316	327	316
11	0.286	0.00286	3,635	3,645	3,635	316	327	316
20	0.286	0.00286	3,635	3,645	3,635	316	327	316
28	0.286	0.00286	3,635	3,645	3,635	316	327	316
43	0.286	0.00286	3,635	3,645	3,635	316	327	316
50	0.286	0.00286	3,635	3,645	3,635	316	327	316

Table 3

Some allocations for the problem and for the game (in euros)

For the problem marg $= q_iP(N) + P(N \setminus \{i\}) - P(N)$; *for the game* marg $= P(N \setminus \{i\}) - P(N)$