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Abstract

The paper considers the problem of evaluating the probability of investing in a capital-investment project as a measure of the uncertainty-investment relationship in a real option model. By the use of the contingent claims analysis the opportunity to invest is modeled as an American call option with expiring time. We show that an increase in uncertainty of the project may actually have positive or negative effects on the probability of investing depending on which market parameters are called to restore the asset price equilibrium condition.

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1 Introduction[1]

The real option model to investment decision proposed by McDonald and Siegel (1986) considers an expenditure on a capital-investment project that shares two important characteristics: the expenditure is irreversible (at least partially), and the investment decision can be delayed, i.e. the investor has the opportunity of waiting for better information concerning the profitability of the project. Under these assumptions, the investment opportunity is seen as a contingent claim (a perpetual call option) on the value of the project which, providing that the project does not enlarge the opportunity set available to investors in the capital markets, can be priced using a replication argument (Cox, Ingersoll and Ross, 1985).

Operatively, the investment option will be exercised when the value of the project, say V, exceeds a critical threshold V^* determined endogenously in the model. However, this trigger value is greater than the direct cost of the project and depends, among other things, on the uncertainty of the asset and satisfies the condition that it is higher the higher is the variance of the project. This has led to the conclusion that an increased uncertainty has a negative effect on the investment (Dixit and Pindyck, 1994, ch. 5).

However, since a higher volatility not only increases the trigger value but always results in wider variations of the asset value, Sarkar (2000) proposes computing the "probability that investment will take place", i.e. the probability that V^* will be reached in a specified time, as a measure of the overall effect of uncertainty on the investment decision. Varying the instantaneous variance of the project, an increase (decrease) in this probability has a positive (negative) effect on investment (Sarkar, 2000, pag. 220). He shows that starting

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from low asset volatility values, a "moderate" increase in uncertainty might increase the probability of investing, thus reversing the general prediction of the real option theory.

Although enticing, the probability measure evaluated by Sarkar seems to suffer two shortcomings. Firstly, while the probability of reaching the critical value V^* is evaluated within a specified "expiration" time, the trigger value computed by Sarkar refers to a perpetual investment opportunity, i.e. it is computed considering an optimal exercise policy which allows, by assumption, the investors to delay their decision indefinitely. Secondly, although the real option approach does not rely on market-equilibrium assumptions, Sarkar assumes that a Capital Asset Pricing Model (CAPM) equilibrium holds, so that both V^* and the probability of investing depend on market parameters like the market price of risk and the correlation between the project and the market portfolio.

The aim of this paper is to generalize Sarkar's analysis in such a way as to highlight the role played by these hypotheses. With this objective in mind we evaluate the option to invest as an American call option with expiring time to explicitly account for the limited ability to delay the investment decision. Moreover, we assume that for a non-traded asset, such as a capital-investment project, there is a (constant) rate of return shortfall between the equilibrium return on a similar traded financial asset and the project's actual growth rate.

In the following section, after recalling the real option model and stating our hypotheses (subsection 2.1), we solve the model by computing the option and the probability of investing in subsection 2.2. Section 3 presents some numerical results, which allow us to compare our approach to that of Sarkar. Finally some conclusions are stated in section 4.

2 Restatement of the problem

2.1 The Assumptions

Our starting point is the real option model of investment developed by McDonald and Siegel (1986), with some differences. In particular we make the following assumptions.

1. The investment project value V_t evolves over time according to a geometric Brownian motion with instantaneous expected return $\mu \geq 0$ and instantaneous volatility $\sigma > 0$

$$dV_t = \mu V_t dt + \sigma V_t dz_t \quad \text{with } V_0 = V \tag{1}$$

where dz_t is the increment of a standard Brownian process.

2. The investment project must be accepted by the investor before a date T, the expiry time.

3. Since the investment project is not traded, its expected rate of return μ falls below the equilibrium total expected rate of return, say $\hat{\mu}$, required in the market by investors from an equivalent-risk traded financial security.

By the latter assumption, the resulting rate of return shortfall between the equilibrium return on a similar financial security and the project's rate of return is analogous to a constant "dividend" yield $\delta \equiv \hat{\mu} - \mu > 0$ [2]. Furthermore, since this equivalent traded financial security must satisfy the asset price equilibrium relationship $\hat{\mu} - r = \lambda\sigma$, where r is the risk-free rate

[2] In complete markets we can hypothesize the existence of a traded asset x that maintains the same price as V while it pays a constant dividend yield δ, with $\hat{\mu} - \delta = \mu$ (McDonald and Siegel , 1984; Cox, Ingersoll and Ross, 1985).

and λ is the asset market price of risk, we can write the expected risk-adjusted rate of return of the project as

$$\mu - \lambda\sigma = r - \delta$$

where $r - \delta$ will be referred to hereafter as the cost of carry.

It is worthwhile to note that this characterization applies in general whether or not the CAPM holds. The option value can have a positive, negative or zero risk premium, depending on the asset's risk premium without any association with the option or the asset comouvement with the market..

2.2 The option and the probability of investing

The above assumptions make the investment opportunity analogous to an American call option on a constant dividend-paying asset, and it will certainly be exercised before the expiry date if the project value V_t increases sufficiently. This results in an optimal investment threshold, V_t^*, above which investment should be taken. In the region where investment is not optimal ($V_t \leq V_t^*$), the value of the option to invest $F(V_t, t)$ satisfies the usual second order differential equation

$$\frac{1}{2}\sigma^2 V^2 F_{VV} + (r - \delta)VF_V - rF - F_t = 0 \tag{2}$$

subject to the payoff conditions

$$F(V_t, t) \geq \sup\left[(V_t - I)^+, 0\right] \quad \text{and} \quad F(V_T, T) = \sup\left[(V_T - I)^+, 0\right] \tag{3}$$

where I is the fixed investment cost. On the optimal trigger function V_t^* we get the matching value condition and smooth pasting condition

$$F(V_t^*, t) = V_t^* - I \quad \text{and} \quad F_V(V_t^*, t) = 1, \tag{4}$$

while the optimal exercise time is defined as

$$\tau = \min\left(t \geq 0 \mid F(V_t^*, t) = \ (V_t^* - I)^+\right) \leq T. \tag{5}$$

We solve (2) by making use of the time homogeneous analytical approximation suggested by Barone-Adesi and Whaley (1986) for an American call option, given by

$$F(V) = \begin{cases} f(V) + AV^\alpha & \text{for } V < V^* \\ \\ V - I & \text{for } V \geq V^* \end{cases} \tag{6}$$

with the boundary conditions (4) reduced to

$$F(V^*) = V^* - I \quad \text{and} \quad F'(V^*) = 1. \tag{7}$$

In (6), $f(V)$ represents the correspondent European option to invest at time T, i.e. $f(V) = e^{-\delta T}\Phi(d_1)V - e^{-rT}\Phi(d_2)I$, where

$$d_1(V) = \frac{\ln(V/I) + (r - \delta + \sigma^2/2)T}{\sigma\sqrt{T}}, \quad d_2(V) = d_1(V) - \sigma\sqrt{T}$$

and $\Phi(x)$ is the area under the standard normal distribution. Furthermore, $\alpha > 1$ is the positive root of the quadratic equation $\frac{1}{2}\sigma^2\alpha(\alpha - 1) + (r - \delta)\alpha - rh^{-1} = 0$, i.e.

$$\alpha \equiv \frac{1}{2} - \frac{r - \delta}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{r - \delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2 h}} > 1$$

with $h = 1 - e^{-rT}$.

Applying conditions (7), the optimal trigger V^* is the solution of the following equation

$$V^* = \frac{\alpha}{\alpha - 1}\left[I + f(V^*) - e^{-\delta T}\frac{V^*}{\alpha}\Phi(d_1(V^*))\right] \tag{8}$$

while the constant A is given by

$$A = \left[1 - e^{-\delta T}\Phi(d_1(V^*))\right]\frac{V^{*1-\alpha}}{\alpha} > 0. \tag{9}$$

It is worthwhile to show that for $T \to \infty$ the solution corresponds to the one obtained by McDonald and Siegel (1986) for the special case of a perpetual investment opportunity. As T becomes larger $h \to 1$, $d_1 \to \infty$ and $d_2 \to \pm\infty$; the European option price $f(V) \to 0$ and $\Phi(d_1) \to 1$, so that (6), (8) and (9) reduce to (Barone-Adesi and Whaley, 1986; Dixit and Pindyck, 1994)

$$F^\infty(V) = \begin{cases} AV^\alpha & \text{for } V < V^\infty \\ \\ V - I & \text{for } V \geq V^\infty \end{cases} \tag{10}$$

with

$$V^\infty = \frac{\alpha}{\alpha - 1}I \quad \text{and} \quad A = \frac{(\alpha - 1)^{\alpha - 1}I^{1-\alpha}}{\alpha^\alpha} > 0. \tag{11}$$

Finally, note that (11) satisfies the optimal trigger value restriction

$$I < V^* < V^\infty.$$

According to (5), at time zero, the probability of investing in the project is the probability of having an optimal exercise time τ lower than (or equal to) the expiration time T. In other words, this is the probability of the geometric Brownian motion V_t reaching the critical value V^* within $[0,\ T]$ starting from an initial condition $V < V^*$. This can be expressed as

(Harrison, 1985, pp. 11-14)

$$\Pr(\tau \le T) = \Phi(s_1) + \left(\frac{V^*}{V}\right)^{2\mu/\sigma^2-1} \Phi(s_2) \tag{12}$$

where

$$s_1(V,V^*) = \frac{\ln(V/V^*) + (\mu - \sigma^2/2)T}{\sigma\sqrt{T}} \quad \text{and} \quad s_2(V,V^*) = s_1 - \left(\frac{2\mu}{\sigma^2} - 1\right)\sigma\sqrt{T}.$$

Although (12) is apparently identical to that of Sarkar (2000, pag. 222), it differs in two aspects:

1. the trigger value V^* obtained by condition (8) takes account of the expiration time T and depends on δ;
2. the cost of carry $r - \delta$ does not rely on the CAPM.

As for the investment opportunity value, it is interesting to analyze the limit for $T \to \infty$: as T becomes larger the probability (12) converges to (Harrison, 1985, Corollary 30, pag.43)

$$\Pr(\tau \le +\infty) = \begin{cases} 1 & \text{if } 2\mu/\sigma^2 \ge 1 \\ \left(\frac{V^\infty}{V}\right)^{(2\mu/\sigma^2)-1} & \text{if } 2\mu/\sigma^2 < 1 \end{cases} \tag{13}$$

Starting at V in the interior of the range $(0, V^\infty]$, after a "sufficient" long interval of time the process is sure to hit the trigger V^∞ if the trend is positive and sufficiently large with respect to the uncertainty. However, if μ is positive but low with respect to the uncertainty or it is negative, the process may drift away and never hit V^∞. Furthermore, a higher level of uncertainty will increase the trigger value V^∞ postponing, *caeteris paribus*, the investment decision. Indeed, the derivative of (13) with respect to σ is unambiguously negative

$$\frac{d\,\Pr}{d\sigma} = sign\left[-\frac{4\mu}{\sigma^3}\ln\left(\frac{V^\infty}{V}\right) + \left(\frac{2\mu}{\sigma^2} - 1\right)\frac{V^\infty}{V}\frac{dV^\infty}{d\sigma}\right] < 0.$$

Thus, the overall effect of a higher V^∞ and a lower probability imply a smaller chance of project acceptance.

3 A numerical evaluation of the probability

3.1 A constant "dividend" yield

Since for an American option with expiry date the sign of the derivative of (12) with respect to σ does not give clear-cut results, we conduct some numerical exercises. For practical convenience we use the following parameter values, which are kept constant in all computations: $I = 1$, $V_0 = 1$ $r = 0.1$, $\mu = 0$ (no growth effects) and $T = 5$ years. Note that while for the option value (6) and the trigger (8) we do not need to know μ, the probability of investing depends on the growth rate. For the constant dividend yield δ we consider three values: $\delta = (0.15, 0.1, 0.05)$, so that the cost of carry $r - \delta$ is negative, equal to zero and positive. We solve the full model (6), (8) and (9) using the algorithm proposed by Barone-Adesi and Whaley (1986) for determining V^*, and compute $\Pr(\tau \leq T)$ via (12) for several values of σ. The results are presented in figure 1, where graphs A and B reproduce the cases $\delta > r$ and $\delta = r$ respectively and C reproduces the case $\delta < r$. The graphs show the relevance of the sign of the cost of carry, $r - \delta$, for the probability of investing. If it is negative or zero we have a probability declining with σ, that is an increase in volatility lowers the probability of investing, confirming the prediction of the theory. On the contrary, if the difference is greater than zero, the $\Pr(\tau < T)$ rises for "moderate" increases of uncertainty, until a maximum is reached after which it begins to decline.

[Figure 1 about here.]

3.2 The risk-adjusted rate of return and Sarkar's results

So far, in our numerical exercise we have treated δ and σ as independent parameters. However, when the uncertainty on the project value changes the asset price equilibrium relationship, $\mu - \lambda\sigma = r - \delta$ must continue to hold. Therefore, with given μ, assuming that δ is constant means that the project price of risk λ should adjust in the opposite direction of σ to restore equilibrium. In other words, it is the project risk premium $\lambda\sigma$ that is constant, and figure 1 shows that this condition has a different impact on the probability of investing depending on the sign of the cost of carry. On the contrary, in Sarkar (2000) it is the "dividend" yield δ that plays the crucial role of the parameter which moves to restore the market equilibrium. In fact, to recover his result we firstly need to allow δ adjusting as σ changes by means of the asset price relationship $\delta = r - (\mu - \lambda\sigma)$ with λ constant. Secondly we need to explicit λ in terms of the market price of risk, λ_M, and of correlation of the asset with the market portfolio, ρ_M, i.e. $\lambda = \lambda_M \rho_M$, so that $\delta = r - (\mu - \lambda_M \rho_M \sigma)$. Finally, setting, as in Sarkar (2000)[3] $\lambda_M = 0.4$ and $\rho_M = 0.7$ we get figure 2 where the continuous curve exactly reproduces the graph reported in Sarkar, with $T \to \infty$, while the dashed curve is the probability of investing evaluated with a trigger value computed for an American call

[3]In particular, recalling the meaning of δ as the instantaneous cash payout (CP) from the project (i.e. like a dividend yield), Sarkar assumes $\delta = CP/V$, with $CP = 0.1$.

option with maturity ($T = 5$).

[Figure 2 about here.]

4 Concluding remarks

In this note we have closely examined the probability of investing in a capital project. The investment decision is modeled as an American call option and the equilibrium condition is simply based on the equality between the risk adjusted expected rate of return and the cost of carry of the investment. The analysis shows that the shape of the probability of investing in response to changes in the project uncertainty σ crucially depends on which market parameters are called to restore the asset price equilibrium condition. If we assume the dividend yield δ as the exogenously fundamental market parameter, than the effect of an increase in uncertainty on the probability of investing relies on the sign of the cost of carry, and in particular it is positive for moderate values of σ only when the cost of carry is positive. On the contrary, if we rely on the capital asset pricing model to evaluate the project price of risk λ, we get a moderate increase in the probability of investing only when λ is the fundamental market constant.

Which of these two viewpoints is more plausible is, in general, an empirical matter and depends on the type of project we are analyzing. However, if in Sarkar (2000) the validity of the CAPM can be read as a way of justifying a constant project price of risk, it appears to be controversial, to say the least, that the correlation between the project value and the market portfolio is constant when σ changes.

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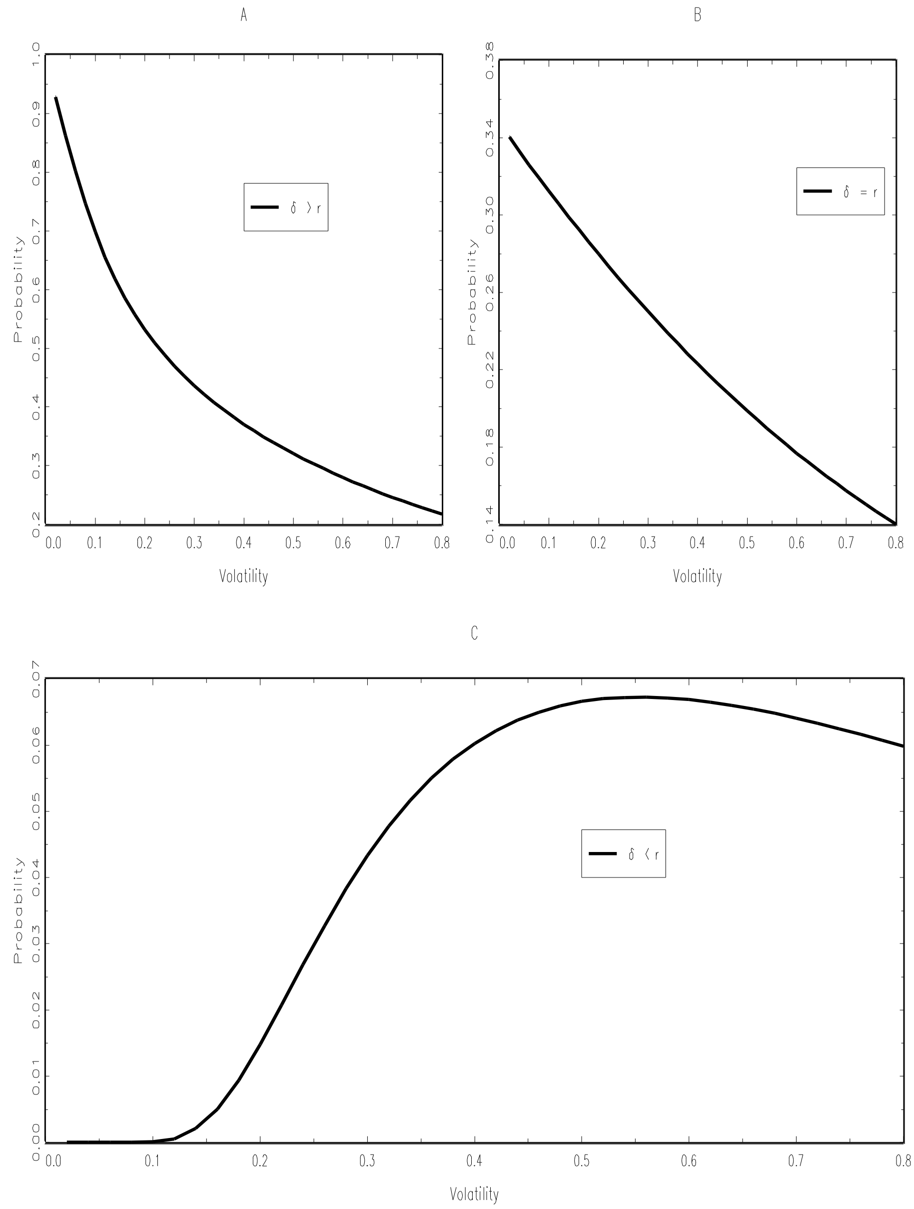

Figure 1: American call with $\delta > r$ (A), $\delta = r$ (B), $\delta < r$ (C).

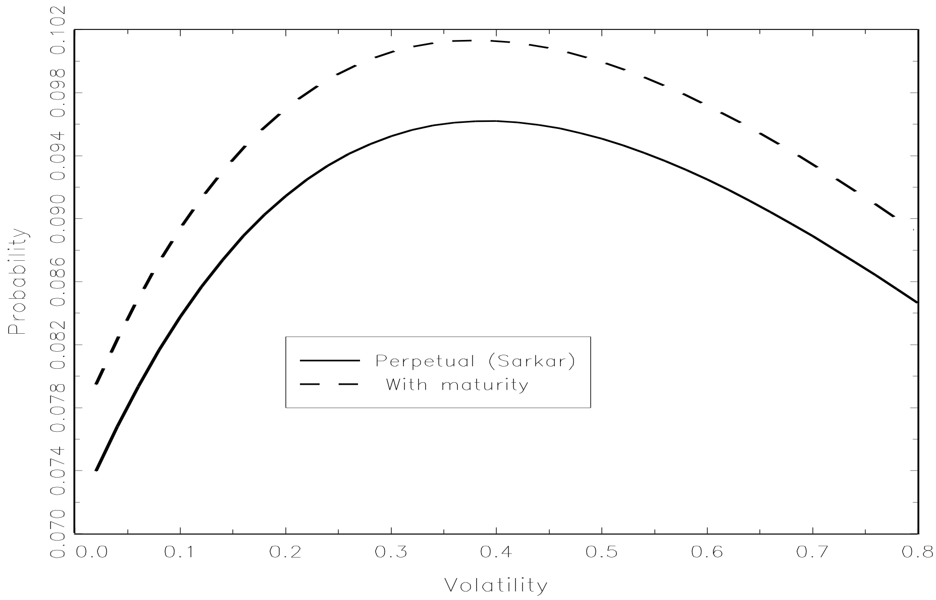

Figure 2: Sarkar's case.