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A Sequential Approach to the Characteristic Function and the Core in Games with Externalities*

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Abstract

This paper propose a formulation of coalitional payoff possibilities in games with externalities, based on the assumption that forming coalitions can exploit a "first mover advantage". We derive a characteristic function and show that when outside players play their best response noncooperatively, the core is always nonempty when the game has strategic complementarities. We apply this result to cartel formation in Bertrand oligopoly and in Shapley-Shubik (1977) strategic market games.

Keywords: Core, Cooperative Games, Externalities.

JEL Classification: C7

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1 Introduction

One major open problem in the theory of games is the determination of coalitional possibilities in games with externalities[1]. These games are characterized by the fact that while the set of available actions for a coalition is usually independent of the actions of the excluded players, its payoff is not. This is the case of most meaningful economic problems in which group formation is a relevant issue: cartel formation in oligopolies, international cooperation on trade or environmental issues, formation of joint ventures and R&D associations, and so on. In all these problems, a coalition of players cannot by itself determine its own payoff in the game; this is jointly determined by the choice of its members and of the outside players.

Cooperative game theory expresses coalitional possibilities in terms of the maximal payoff attainable by a coalition through the choice of a joint strategy in the underlying game. A real valued "characteristic" function associates with each coalition this maximal payoff.[2] The fact that outside players may affect the payoff of coalitional members was already accounted for in the seminal work by Von Neumann and Morgenstern (1944). There, the characteristic function was defined as the maximal aggregate payoff of a coalition under the assumption that outside players act in order to minimize its payoff (see also Aumann (1959)). In this sense, their α and β functions express the maximal payoff that coalitions can *guarantee* to their members. Although appealing because immune from any *ad hoc* assumption on the reaction of the outside players (indeed, their minimizing behavior is here not meant to represent the expectation of S but rather as a mathematical way to determine the lower bound of S's aggregate payoff), still this approach has important drawbacks: deviating coalitions are often too heavily penalized, while outside players often end up bearing an unreasonably high cost in their attempt to hurt deviators. These problems have motivated the introduction of rationality requirements in the reaction of outside players.

Most recent game theory has taken a different approach to the analysis of cooperation,

[1]In this way, for instance, the nobel laureate Kenneth Arrow has expressed his authoritative view during the final round table at the First International Meeting in Games Theory, Bilbao 24th-28th July 2000.

[2]In games without transferable utility, this functions associates with each coalition a utility frontier.

viewing the choice to form a coalition as a strategy in an explicitly defined game (see Hart and Kurz (1983), Yi (1997) and Bloch (1997) for a survey). In these games, players announce coalitions, and specific rules map strategy profiles into actual coalition structures. A fixed imputation rule, mapping each coalition structure into a payoff imputation for the players, yields a well defined game. Equilibrium coalition structures (and, consequently, payoff imputations) have been defined by applying concepts of equilibrium to these games. Two main coalition formation rules have been extensively studied in the literature: the *gamma* and the *delta* rules (see Hart and Kurz (1983)). In the first case, a coalition forms if and only if all its members have announced it; in the second, all players announcing the same coalition finally belong to the same coalition (not necessarily the one they announced). An interesting difference between these rules attains to the consequences of the formation of a coalition S in objection to the grand coalition: under the gamma assumption, S forms and outside players split up into singletons; under the delta assumption, S forms and outside players merge into a single coalition.

The structure of the coalition formation models can be used to impose minimal rationality requirements on the reaction of outside players in the derivation of a characteristic function. In particular, the coalition formation rules can be used to determine which coalition structure prevails among outside players at the formation of a coalition S. Interpreting the formation of S as an objection to some proposal coming from the grand coalition, the reaction of outside players can be identified as their *optimal* choices in the coalition structure induced by the formation of S. Following a common practice in the literature, first suggested by Ichiishi (1981), these choices are assumed to simultaneously maximize the payoff of each group in the induced partition of outside players.

One proposal of construction of a characteristic function based on the *gamma* rule comes from the theory of environmental agreements[3] and was put forward by Chander and Tulkens (1997). Chander and Tulkens define a characteristic function by associating with each coali-

[3]However, we find the same issue in industrial organization papers dealing with the problem of cartel formation. See, for instance, Rajan (1989).

tion its Nash equilibrium payoff in the game played against the outside players acting as singletons. The gamma rule can be here defended on institutional grounds: in some instances of international environmental agreements, treaties require the formation of at most one coalition (see, for instance, Murdoch and Sandler (1997) on the regulation of chlorofluorocarbon emissions). Similarly, the assumption of one coalition with fringe outside players is extensively used in the theory of industrial organization for the analysis of horizontal mergers (see Salant et al.,(1983) Deneckere and Davidson, (1985)).

The construction of a characteristic function based on the Nash equilibrium of the game between the forming coalition and the various elements of the induced coalition structure, implicitly assumes that coalitional payoffs originate in two stages: a coalition formation stage, in which the coalition forms and outside players get organized in some coalition structure; a normal form game, in which Nash strategies are played. This because the very use of the Nash equilibrium concept is justified only if all players know who the players are and what their payoff functions and strategy sets are in the game. If coalitional possibilities are considered as threats to the stability of cooperation, here coalitional deviations from a generally agreed joint strategy are assumed to be carried out by first publicly abandoning the negotiation process (as, for instance, a group of countries leaving the international negotiation table) and then playing the Nash equilibrium strategies of the induced strategic form game. We will refer to this way of determining the characteristic function a "simultaneous conversion" of a game.

However, in various economic situations, the coalitional deviations from an agreed joint strategy are carried out by directly choosing an alternative strategy in the underlying game. Firms defect from an industrial cartel by setting a lower price; countries not complying with internationally agreed pollution abatements simply set higher levels of production, and so on. In these cases, the formation of a deviating coalition occurs secretly, and is publicly monitored only when the new strategies have been played (indeed, the new strategies are the only element which is monitored). Forming coalitions can therefore exploit a positional advantage, much as the leader in a Stackelberg game, while outside players must react as Stackelberg followers.

In this paper we explore this idea and accordingly derive a characteristic function capturing the assumption that deviating coalitions move first. In particular, for an arbitrary coalition structure $\pi(S)$ on the set of outside player $N \backslash S$, we define the characteristic function as the aggregate payoff of S of the perfect equilibrium in the sequential game in which S chooses a strategy as a leader, and the elements of $\pi(S)$ simultaneously react in a noncooperative manner. We refer to this operation as "sequential conversion" of a game.

The idea that coalitions can behave as leaders is not new: many works on cartel formation assume that the (unique) cartel sets the price (or the quantity), while the fringe firms optimally react to it (Donsimoni et al., (1986)). Fringe firms have been treated both as competitive players (d'Aspremont et al. (1982)) and as strategic player (Martin, (1979) and Shaffer, (1995)). In this paper, however, we take a different approach: while these works look for stable coalitions when these can behave as leaders, we follow a core-theoretic approach and look for those payoff imputations which would not be objected by coalitions acting as leaders. Our approach can yield interesting results in those situations in which the core selection is too large under the simultaneous conversion. For instance, an interesting application, presented in the paper, shows that the sequential conversion selects a unique core-stable imputation in a cartel formation game under symmetric Cournot oligopoly. More importantly, although various economic problems turn out to have an empty core using the characteristic function we propose, we show that the class of games with strategic complementarities have a nonempty core under the gamma assumption (i.e., if players outside a forming coalition split up as singletons).

The paper is organized as follows: the next section presents the general setup and introduces both the simultaneous and the sequential conversions. Section 3 presents our existence result. Section 4 briefly illustrates some economic applications. Finally, section 5 concludes the paper.

2 The Model

2.1 Setup

We consider a set of players $N = \{1, .., i, .., n\}$, each endowed with a set X_i of feasible actions and a payoff function $u_i : X \to \mathbf{R}$, where $X \equiv \prod_{i \in N} X_i$. For each $S \subseteq N$ we denote by $u_S : X \to \mathbf{R}$ the function defined for all $x \in X$ by $u_S(x) \equiv \sum_{i \in S} u_i(x)$. We assume that utility is transferable, so that $u_S(x)$ is a well defined index of the aggregate utility of S. We will only consider continuous payoff functions. The strategic form game $\Gamma = (N, (X_i, u_i)_{i \in N})$ is obtained from the above elements. A Nash equilibrium $\overline{x}$ of the game Γ is defined in the usual way. Throughout the paper, we will only consider games with a unique Nash equilibrium.

We will associated to the game Γ various cooperative games (N, v) by specifying characteristic functions $v : 2^N \to R_+$, where $v(S)$ expresses the maximal aggregate payoff attainable by coalition S in Γ. An imputation for (N, v) is a vector $z \in \mathbf{R}^n_+$.

Definition 1 *The core of the cooperative game* (N, v)*, denoted* $C(N, v)$*, is the set of imputations* $z \in \mathbf{R}^n_+$ *such that* $\sum_{i \in N} z_i = v(N)$ *and, for all* $S \subset N$*,* $\sum_{i \in S} z_i \geq v(S)$ *.*

We will denote by $C(N, v)$ the core of the game (N, v).

2.2 Simultaneous Conversions

As argued in the introductory section, a first approach to the derivation of a characteristic function for the game Γ views the value $v(S)$ as resulting from an implicit two stage process. At the first stage players announce coalitions, and S forms. At the second stage, the coalitions that have formed choose their strategies according to some equilibrium concept. The rules of the game played at the first stage determine which coalitions form for any given profile of strategies (see Hart and Kurz (1983)). The gamma assumption states that only those coalitions declared by all their members will form. The delta assumption is less restrictive, in that all members announcing the same coalition end up together. As we said, these rules generate different coalition structure in the occurrence of a coalition S deviating from the joint strategy in which all players announce the grand coalition: under the gamma rule, outside players split up as singletons, while under the delta rule, outside players merge. Obviously, these rules describe different institutional and economic environment in which cooperation occurs; while the gamma game is appropriate in representing situations in which only one coalition can form, the delta game better represents situation in which cooperation is easy and not too regulated. These rules generate different characteristic functions.

Under the gamma rule, we associate with each coalition S a coalition structure $\pi_\gamma(S)$ whose elements are S and all players outside S as singletons. Letting $\Gamma(S, \pi_\gamma)$ denote the strategic form game played by the elements of $\pi_\gamma(S)$, the characteristic function $v_\gamma(S)$ is thus defined as the aggregate payoff of S in the (unique) Nash equilibrium $\bar{x}$ of the game $\Gamma(S, \pi_\gamma)$, *i.e.*,

$$v_\gamma(S) = \sum_{i \in S} u_i(\bar{x}). \tag{1}$$

Similarly, under the delta rule we can define a characteristic function $v_\delta(S)$ considering the Nash equilibrium payoff of S in the two player game $\Gamma(S, \pi_\delta(S))$ played by S and its complement $N \backslash S$. Denoting by $\hat{x}$ this equilibrium we have

$$v_\delta(S) = \sum_{i \in S} u_i(\hat{x}).$$

2.3 Sequential Conversions

In various economic situations it makes little sense to assume that deviating coalitions operate in two stages, first publicly announcing its formation and then playing their preferred strategy in a simultaneous game against outside players, however organized. It is often the case that deviations take place after a secret coalition formation process. We have argued that in some cases the formation of a coalition can only be deduced from the observation that some strategies have changed in the game. Therefore, outside players, at least for some transitional period, have to react to these changes very much as followers in a Stackelberg game. We will now propose a conversion capturing this sequential structure in the definition of the characteristic function. Let, as before, $\pi(S) = (S, T_1, ..., T_p)$ be a coalition structure associated with the formation of a coalition S. Let $\Psi(S,\pi)$ be the sequential game in which S moves first choosing an action $x_S \in X_S$ and, at the second stage, the other elements of $\pi(S)$ simultaneously choose an element out of their respective strategy sets. A perfect equilibrium of $\Psi(S,\pi)$ is a pair $\left(x_S^*, \left(x_{T_k}^*(x_S)\right)_{k=1,...,p}\right)$ satisfying the following conditions:

$$x_S^* \in \arg\max_{x_S \in X_S} \sum_{i \in S} u_i \left(x_S, \left(x_j^*(x_S)\right)_{j \notin S}\right) \tag{2}$$

and, $\forall k = 1, ..., p$,

$$x_{T_k}^*(x_S) \in \arg\max_{x_{T_k} \in X_{T_k}} u_j \left(x_S, x_{T_k}, \left(x_{T_j}^*(x_S)\right)_{j \neq k}\right). \tag{3}$$

We denote by $x^*(S)$ the strategy profile $\left(x_S^*, \left(x_{T_k}^*(x_S^*)\right)_{k=1,...,p}\right)$. The assumption of continuous payoffs and the closedness of the Nash correspondence (see, for instance, Fudenberg and Tirole, 1991) directly imply that S faces a continuous maximization problem in (2) so that, by application of Weiestrass' theorem, a perfect equilibrium $\Psi(S,\pi)$ always exists. A characteristic function is now defined by assigning to each coalition S its aggregate payoff at the relevant perfect equilibrium:

$$v_\phi(S) = \sum_{i \in S} u_i \left(x^*(S)\right). \tag{4}$$

3 A Class of Games with a Nonempty Core

In this section we identify a class of games allowing for a nonempty core under the sequential conversion. In particular, we show that under two additional symmetry conditions, all games with *strategic complementarities* generate cooperative games with core-stable imputations. We obtain this result for the gamma assumption, *i.e.,* for the case in which the formation of S induces outside players to split up as singletons. The issue of existence is of particular interest in dealing with the sequential conversion; since for all coalitions S it is trivially true that $v_\phi(S) \geq v_\gamma(S)$, all games in which the core is empty under the simultaneous conversion replicate the same property in this case. However, the analytical structure of the perfect equilibrium allows us to obtain an existence result which, if applied to the simultaneous conversion, can be itself regarded as a new contribution (we remind here that the only existence result for the core of (N, v_γ) was obtained by construction by Chander and Tulkens (1997) for a multilateral externalities game with quasilinear preferences). For simplicity, we limit the analysis to games in which strategy sets are subsets of R and payoff functions are twice continuously differentiable. For this case, strategic complementarity is equivalent to the following property (see Topkis (1998)):

$$\frac{\partial^2 u_i(x)}{\partial x_i \partial x_j} \geq (x)\, 0 \; \forall i, j \in N, i \neq j, \forall x \in X. \tag{5}$$

Let now $f_{N \backslash S} : X_S \rightarrow X_{N \backslash S}$ map the joint strategies of S into the Nash equilibrium of the reduced game $\Gamma(N \backslash S, x_S)$. A well known theorem by Milgrom and Roberts (1990) directly implies that if payoffs satisfy (5) then the function $f_{N \backslash S}$ is increasing in x_S (see Theorem 4.2.2 in Topkis (1998)).[4]

For our main result, stated in theorem 1, we need some additional assumptions.

Assumption 1 (symmetric players). *Let $x, x' \in X$ be such that $x_{N \backslash i \cup j} = x'_{N \backslash i \cup j}$ and $x_j = x'_i$ and $x_i = x'_j$ for $i, j \in N$. Then $u_i(x) = u_j(x')$.*

Assumption 2 (symmetric externality). *Either $\dfrac{\partial u_i(x)}{\partial x_j} \geq 0$ for all $i, j \in N$ and $x \in X$ or $\dfrac{\partial u_j(x)}{\partial x_i} \leq 0$ for all $i, j \in N$ and $x \in X$.*

Assumption 3 (strict concavity). *$u_i(x)$ strictly concave in x_i for all $i \in N$.*

Assumption 1 requires that all players are identical in the following sense: if two players play the same action, both facing the same action profile of the other $(n-1)$ players, they get the same payoff. The second assumption has been shown to play a crucial role in various cooperative game theory results (see, for instance, Milgrom and Roberts (1996), Yi (1999)), and requires that the sign of the effect of each player's action on the payoff of the rest of

[4] In order to apply this result, we here exploit the fact that R is a chain (which, together with condition (8), implies that the game is supermodular) and the assumption of a unique Nash equilibrium (that implies that $f_{N \backslash S}$ is singlevalued, so that the greatest and least elements coincide).

players is the same. We will denote the case of a positive sign as "positive externality" and the case of a negative sign as "negative externality".

The lemmas that follow are an extension of some well known results in the leader-follower duopoly literature (see, for instance, Gal-Or, 1985). To simplify notation, for a given action profile x_S we will denote by $\left(x_{S\backslash i}, y\right)$ the vector $(x_1, ...x_{i-1}, y, x_{i+1}, .., x_s)$.

Lemma 1 *Let $S \subset N$ and consider an interior equilibrium $x^*(S)$.*
(a) If externalities are positive, then $i \in S$ and $j \in N\backslash S$ imply $x_i^* \geq x_j^*$;
(b) If externalities are negative, then $i \in S$ and $j \in N\backslash S$ imply $x_i^* \leq x_j^*$.

Proof. **(a).** We proceed by contradiction. Suppose $x_i^* < x_j^*$ for some $i \in S$ and $j \in N\backslash S$. The next series of inequalities follows:

$$\frac{\partial u_i\left(x_S^*, x_{N\backslash S}^*\right)}{\partial x_i} > \frac{\partial u_i\left(x_{S\backslash i}^*, x_j^*, x_{N\backslash S}^*\right)}{\partial x_i} \geq \frac{\partial u_i\left(x_{S\backslash i}^*, x_j^*, x_{(N\backslash S)\backslash j}^*, x_i^*\right)}{\partial x_i} = \frac{\partial u_j\left(x_S^*, x_{N\backslash S}^*\right)}{\partial x_j} = 0 \tag{6}$$

The first inequality follows by the strict concavity (assumption 3); the second by condition (5); the third by assumption 1, and the fourth by the first order conditions of the problem (3) defining the equilibrium strategy profile $x^*(S)$. Note next that every $i \in S$ first order condition of problem (2) can be rewritten as

$$\sum_{h\in S}\left(\frac{\partial u_h\left(x^*\right)}{\partial x_i} + \sum_{j\in N\backslash S}\frac{\partial u_h\left(x^*\right)}{\partial x_j}\frac{\partial f_j\left(x_S^*\right)}{\partial x_i}\right) \equiv 0 \tag{7}$$

where $f_j\left(x_S^*\right)$ is the j-th element of $f_{N\backslash S}(x_S^*)$. Let us examine an arbitrary element h of the summation over S in (7): by the assumption of case (a) of this lemma, the first term is non-negative if $h \neq i$; moreover, by (6), this term is strictly positive for i. This facts, together with the fact that $f_{N\backslash S}$ is increasing, imply that condition (7) can be satisfied only if $\frac{\partial u_h(x^*)}{\partial x_j} < 0$ for some $h \in S$, which contradicts the assumption of the lemma.

(b). The same contradiction argument used for case (a) can be proved by inverting the inequality signs in (6) in the appropriate manner. ■

Lemma 2 *Let $S \subset N$ and consider an interior equilibrium $x^*(S)$. If $j \in N\backslash S$ and $i \in S$ then $u_j(x^*) \geq u_i(x^*)$.*

Proof. The following inequalities hold for all $j \in N \backslash S$ and $i \in S$:

$$u_j \left(x_S^*, x_{N \backslash S}^* \right) \geq u_j \left(x_S^*, x_{(N \backslash S) \backslash j}^*, x_i^* \right) \geq u_j \left(x_{S \backslash i}^*, x_j^*, x_{(N \backslash S) \backslash j}^*, x_i^* \right). \tag{8}$$

The first part is implied by condition (3), while the second follows from lemma 1 and assumption (2). By the assumption of identical players, we also have

$$u_j \left(x_{S \backslash i}^*, x_j^*, x_{(N \backslash S) \backslash j}^*, x_i^* \right) = u_i \left(x_S^*, x_{N \backslash S}^* \right). \tag{9}$$

Inequalities (8) and (9) imply

$$u_j \left(x^* \right) \geq u_i \left(x^* \right),$$

which proves the result. ■

Theorem 1 *The game (N, v_ϕ) has a nonempty core.*

Proof. We prove the theorem showing that the equal split allocation giving $\frac{v_\phi (N)}{n}$ to each player in N is in the core of the game (N, v_ϕ). Suppose not, so that $v_\phi (S) > v_\phi (N)$ for some $S \subset N$. By lemma 2, the maximal payoff of players in S in weakly lower than the minimal payoff of players in $N \backslash S$. This implies that

$$\frac{\sum_{j \in N \backslash S} u_j \left(x^* \right)}{n - s} \geq \frac{\sum_{i \in S} u_i \left(x^* \right)}{s} = \frac{v_\phi (S)}{s},$$

so that

$$\frac{v_\phi (S)}{s} > \frac{v_\phi (N)}{n} \Rightarrow \frac{\sum_{j \in N \backslash S} u_j \left(x^* \right)}{n - s} > \frac{v_\phi (N)}{n}.$$

This in turns implies that

$$s \frac{\sum_{i \in S} u_i \left(x^* \right)}{s} + (n - s) \frac{\sum_{j \in N \backslash S} u_j \left(x^* \right)}{n - s} > s \frac{v_\phi (N)}{n} + (n - s) \frac{v_\phi (N)}{n}$$

or

$$\sum_{i \in N} u_i \left(x^* \right) > v_\phi (N)$$

which contradicts efficiency of $v_\phi(N)$. ■

The following corollary directly follows from the fact that $v_\phi(S) \geq v_\gamma(S)$ for all S.

Corollary 2 *The game (N, v_γ) has a nonempty core.*

4 Examples

In this section, we first apply theorem 1 to two well known economic examples of games characterized by strategic complementarities: Bertrand oligopoly and strategic market games. The third example does not satisfy strategic complementarity and is introduced for a different purpose; the case of linear Cournot oligopoly illustrates how the core under the sequential conversion can serve as a useful refinement of the core under the simultaneous conversion.

4.1 Bertrand Oligopoly with Differentiated Goods

Let N be a set of oligopolistic firms competing in prices in a market for differentiated goods. The strategy space for every firm is the interval $[0, \bar{p}]$ in $\mathbf{R}$. Following the model of Shubik and Leviatan (1980), the payoff of every firm i is its profit, given that its market share is a function of the difference between its own price and the average price of the market. More precisely

$$u_i(p_1, p_2, ..., p_n) = \left(a - p_i - \beta \left(p_i - \frac{1}{n} \sum_{h \in N} p_h \right) \right) (p_i - c), \tag{10}$$

where $c > 0$ is every firm's marginal cost and $\beta > 0$ is a parameter expressing the degree of differentiation between goods.

We first note that each payoff function is strictly concave in one's own strategy, and that the term $\frac{\partial u_i(p_1,p_2,...,p_n)}{\partial p_j}$ is always strictly positive. Now, to establish non-emptiness of the core, we just need to check whether the condition (5) expressing the strategic complementarity property holds for every i. Using (10), it is straightforward to obtain:

$$\frac{\partial^2 u_i(p_1, p_2, ..., p_n)}{\partial p_i \partial p_j} = \frac{\beta}{n} > 0.$$

Consequently, from the above result we can conclude the grand coalition cartel is stable under both the sequential and simultaneous conversions, i.e., $C(N, v_\phi)$ and $C(N, v_\gamma)$ are both nonempty. It is relatively easy to see that this result can be generalized to every Bertrand game respecting the conditions required by the theorem 1 and, in general, to all symmetric games with multidimensional action spaces that are supermodular (Topkis (1998)). In the next section we show another classical game usually possessing the strategic complementarity property: the strategic market game *à la* Shapley and Shubik (1977).

4.2 Strategic Market Games

Consider N identical players and, for every $i \in N$, a transferable utility function represented by:

$$u_i(a_i, m_i) = \ln(a_i) + m_i$$

where a_i denotes player i's consumption of commodity a, and m_i player i's consumption of money. Let ω_i^m be player i's endowment of money and ω_i^a player i's endowment of commodity a. Let also, without loss of generality, $\omega_i^m = 1$ and no credit be available in the economy, so that the interval $[0, 1]$ constitutes every player's strategy set. Trade occurs as follows: each player i submits a bid b_i representing the amount of money he offers to buy commodity a. If $(b_1, ..., b_n)$ is the profile of bids, player i obtains a payoff equal to:

$$u_i\left(b_i, \sum_{j \neq i} b_j\right) = \ln\left(b_i \frac{\sum_{i \in N} \omega_i^a}{\sum_{i \in N} b_i}\right) + (\omega_i^m - b_i). \tag{11a}$$

It is easy to see that every player's payoff function (11a) is strictly concave in his own bid b_i, also showing negative externalities with respect with every other player's action. The Nash (non trivial) equilibrium is computed by solving the following n first order conditions for a maximum:

$$\frac{\partial u_i(b_1, b_2, ..., b_n)}{\partial b_i} = \frac{1}{b_i} - \frac{1}{\sum\limits_{i \in N} b_i} - 1 = 0, \ \forall i \in N \tag{12}$$

whose only interior positive solution gives $\overline{b}_1 = \overline{b}_2 = ... = \overline{b}_n = \frac{n-1}{n}$. Moreover, since $\frac{\partial^2 u_i(b_1, b_2, ..., b_n)}{\partial b_i \partial bj} = \left(\sum_{i \in N} b_i\right)^{-2} > 0$, the strategic complementarity property is respected and, theorem 1 can be applied to conclude that both $C(N, v_\phi) \neq 0$ and $C(N, v_\gamma)$ are nonempty. It is interesting to note that theorem 1 can also be applied both to specific classes of strategic market games as the "bilateral oligopoly" introduced by Gabszewicz and Michel (1997) that usually respects strategic complementarity when goods are substitutes and to more general version of the model with multidimensional players' action spaces (see, for instance, the market games with multiple trading posts (Koutsougeras (1999)) when the game is supermodular.

4.3 Cournot Oligopoly

Let N be a set of firms competing in quantities in a common market for a single homogeneous good. Each firm i produces the quantity y_i, facing the same linear inverse demand function $p(y) = a - by$, where $y \equiv \sum y_i$, $a > 0$ and $b > 0$. Each firm's payoff is given by its profit, so that $u_i(y_1, ..., y_n) = p(y)\, y_i - cy_i$, where cy_i is the cost function for firm i, with $a > c \geq 0$. The choice set of each firm is bounded above by some finite capacity constraint $\overline{y}_i$. Under continuity assumptions on $p(y)$ it can be easily proved that the Cournot game admits a unique Nash equilibrium. Also, very simple algebra yields the following two characteristic functions (see appendix for derivations):

$$v_\gamma(S) = \sum_{i \in S} u_i(\bar{y}) = \frac{(a-c)^2}{(n-s+2)^2\, b};\ v_\phi(S) = \sum_{i \in S} u_i(y^*) = \frac{(a-c)^2}{4(n-s+1)\, b}.$$

Using the above specifications, we can easily check that in this case, the core of the game (N, v_ϕ) consists of the (unique) efficient equal split allocation. To see this, without loss of generality set $\frac{(a-c)^2}{b} = 1$, so that the equal split allocation gives to each player in N a payoff of $\frac{v_\phi(N)}{n} = \frac{1}{4n}$ and $v_\phi(S) = \frac{1}{4(n-s+1)}$, where $s = |S|$ and $n = |N|$. We first show that the equal split allocation belongs to the core. Consider the value $\frac{v_\phi(S)}{s}$ for an arbitrary coalition S. We have that for all S such that $s \leq n$,

$$\frac{v_\phi(S)}{s} = \frac{1}{4s(n-s+1)} \leq \frac{1}{4n} = \frac{v_\phi(N)}{n}. \tag{13}$$

In fact, the above inequality reduces to

$$s(n-s+1) \geq n \tag{14}$$

which is satisfied for all $1 \leq s \leq n$. It follows that if coalition S forms, at least one player gets a payoff less than or equal to $\frac{v_\phi(S)}{s}$, and therefore less than or equal to $\frac{v_\phi(N)}{n}$. This implies that the equal split allocation belongs to $C(N, v_\phi)$. To see that the equal-split is the unique allocation contained in $C(N, v_\phi)$, note that (14) is satisfied with equality only for $s = n$ and for $s = 1$. This means that $v_\phi(i) = \frac{v_\phi(N)}{n}$ for all $i \in N$. Thus, consider the allocation z different from the equal split allocation; at z, some player j receives a payoff $z_j < \frac{v_\phi(N)}{n}$.

Player j can thus improve upon z by getting $v_\phi(i) = \frac{v_\phi(N)}{n}$, which implies that z is not in $C(N, v_\phi)$. Since both $C(N, v_\phi)$ and $C(N, v_\phi)$ are nonempty and closed subsets of $\mathbf{R}^n_+$, with $C(N, v_\phi) \subset C(N, v_\gamma)$, it follows that the set $C(N, v_\gamma) \backslash C(N, v_\phi)$ contains a continuum of points in $\mathbf{R}^n_+$.

The above result is obviously not generic. The introduction of non-linearities in costs is sufficient to generate an empty core in the associated game (N, v_ϕ). Similarly, it can be shown that the core is empty in the multilateral externalities game studied by Chander and Tulkens (1997, 1995) under a log-linear specification of preferences.

5 Concluding Remarks

In this paper we have presented two alternative approaches to the determination of the characteristic function in games with externalities. The appropriateness of either approach depends on the strategic context in which players choose their actions. The sequential conversion seems appropriate in settings in which players' actions are perfectly monitored and players can fully commit to their strategies. In these cases, the derived characteristic function simply formalizes the assumption that a forming coalition anticipates the (optimal) reaction of outside players who face its formation as a *fait accompli.* On the other hand, the simultaneous conversion seems appropriate when the formation of a deviating coalition can be monitored before its strategy choice, and all strategies are chosen only once the new coalition structure has formed. We have shown that the sequential structure of the game characterizing the payoff of forming coalitions can be exploited to find sufficient conditions for the non-emptiness of both core concepts. The crucial property for non-emptiness (strategic complementarity) is often encountered (and easily testable) in economic applications. Moreover, the property of the core under the sequential conversion to act as a *refinement* of the core under the simultaneous conversion yields an interesting result for the case of linear Cournot oligopoly, in which the former selects the equal split efficient allocation out of the continuum of allocations contained in the latter.

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Appendix: Derivation of v_ϕ for the Cournot game.

The best-reply for $j \in N \backslash S$ is obtained from (3) as

$$y_j(y_S) = g(y_S) = \frac{a - b\sum_{i \in S} y_i - c}{b(n-s+1)}.$$

Coalition S maximizes:

$$\sum_{i \in S} \pi_i(y_S, (n-s)y_j(y_S)) = \left(a - b\sum_{i \in S} y_i - b(n-s)\frac{a - b\sum_{i \in S} y_i - c}{b(n-s+1)}\right)\sum_{i \in S} y_i - c\sum_{i \in S} y_i.$$

The FOC for an internal solution of this problem is

$$a - 2b\sum_{i \in S} y_i - (n-s)\frac{a - c - 2b\sum_{i \in S} y_i}{(n-s+1)} - c = 0$$

from which

$$\sum_{i \in S} y_i^* = \frac{a-c}{2b}$$

and for all $j \in N \backslash S$

$$y_j^*(y_S^*) = \frac{a - b\left[\frac{a-c}{2b}\right] - c}{b(n-s+1)} = \frac{a-c}{2b(n-s+1)}.$$

To obtain $v_\phi(S)$, we first compute the equilibrium price:

$$p(y^*) = a - b\sum_{i \in S} y_i^* - b(n-s)y_j^*(y_S^*),$$

which, after substitutions, becomes

$$p(y^*) = a - b\left(\tfrac{a-c}{2b}\right) - b(n-s)\tfrac{a-c}{2b(n-s+1)} = \tfrac{a+2(n-s)c+c}{2(n-s+1)}.$$

Finally,

$$v_\phi(S) = \sum_{i \in S} \pi_i\left(y_S^*, (n-s)y_j^*(y_S^*)\right) = p(y^*)\sum_{i \in S} y_i^* - c\sum_{i \in S} y_i^*$$

that is,

$$v_\phi(S) = \tfrac{a+2(n-s)c+c}{2(n-s+1)}\tfrac{a-c}{2b} - c\tfrac{a-c}{2b} = \tfrac{(a-c)^2}{4b(n-s+1)}.$$

By applying the above formula, the worth of the grand coalition $(n = s)$ can be written as:

$$v_\phi(N) = \tfrac{(a-c)^2}{4b}.$$