Liquidity: What can a "Hausbank" do that other lenders cannot do?

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FIRST DRAFT

Abstract

Our interest here concerns liquidity supply as a distinctive feature of the bank-borrower relationship. Any agent facing an opportunity or a commitment may find him/herself unexpectedly illiquid, and hence he/she may find it profitable to borrow "on call" if this costs less than missing the opportunity or defaulting on the commitment, or costs less than using non-money goods as means of payment. This is the essence of what Hicks called "the overdraft economy". Accordingly, we call a debt contract inclusive of the liquidity service an "overdraft debt contract", and we investigate its efficiency properties in a continuous time stochastic model of a repeated bank-borrower relationship where the key problem is the credibility of their mutual commitment between the two parties. Our main finding is that efficient, i.e. cost-minimizing, overdraft debt contracts emerge in the absence of perfect commitment and enforceability as the borrower and the bank can extert *mutual threat of termination.*

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1 Introduction

The first principle of the theory of financial intermediation is that any private borrower-lender relationship may face severe efficiency limits in dealing with transaction costs, risk management, agency problems and liquidity requirements. In this paper we focus on liquidity as a key financial problem and a distinctive feature of bank intermediation.

Liquidity is a difficult economic concept. As a first approximation, it means availability of means of payment. at (almost) zero transaction-costs. In principle all goods might be used as means of payment against other goods, but the existence of *different* transaction costs may make some goods more liquid than others. Transactions in modern market economies rely upon the existence of an artificial good called "money" created to be the *perfectly liquid means of payment.* Lack of money in the presence of a profitable transaction or of a payment commitment entails a loss because the transaction cannot be done or the commitment cannot be honoured, or because of the costs of using non-money goods as means of payment. The costs of illiquidity are a well-known explanation of the reason why rational agents are willing to hold money as an apparently worthless asset, on the one hand, and of the existence of specialized agents who are able to manage liquidity efficiently on the other. Efficiency in liquidity management may increase for the economy as a whole in two main ways: 1) by offering deposit services to a large number of money holders with uncorrelated liquidity needs so that total "idle" deposits can be less than the sum of individual money holdings, 2) by offering liquidity "on call" so that agents are allowed to economize their individual holdings of the worthless asset. These two services have long been viewed as a key explanation, in terms of social efficiency gain, of the emergence of bank intermediation (see e.g. Hicks, 1967, 1989; Goodhart, 1989). Though it is now clear that the social efficiency gains that can be associated with each of these services do not provide a self-contained theory of the bank (for instance one should still explain why the two services are, or should be, performed by a single subject: see e.g. Fama, 1985; Mayer, 1994), we shall assume that anyone who offers deposit and liquidity services is a bank

Our interest here concerns the liquidity service as a distinctive feature of the bank-borrower relationship. The problem has been outlined above: any agent facing an opportunity or a commitment may find him/herself unexpectedly illiquid, and hence he/she may find it profitable to borrow "on call" if this costs less than missing the opportunity or defaulting on the commitment, or costs less than using non-money goods as means of payment. A crucial preliminary aspect of this problem is that no rational agent plans to be illiquid. Take as a benchmark a standard intertemporal optimal programme: at each point in time the agent should be able to realize the relevant time step of the programme, which implies that he/she should dispose of the money value of the programme at each point in time. Now introduce uncertainty: at time 0 the agent *expects* he/she will be able to realize his/her planned optimal step at each future time, which implies that the agent *expects* he/she will dispose of the relevant money value. What if this latter expectation is violated *in any particular point in time* (suppose that the agent's cash flow is a stochastic variable following any probability law: by Tchebycheev inequality the probability that any single realization differs from the expected value by any arbitrary number is strictly positive)? This part of the problem is not made explicit in standard intertemporal optimization; an auxiliary assumption is present: the agent can freely lend or borrow at a single market interest rate *vis-à-vis* any unexpected positive or negative excess of liquidity (under the usual end-of-time condition of zero present value of lending and borrowing); as a particular case, the market interest rate for liquidity can be zero, i.e. there exists infinite liquidity supply. Alternatively, one can find the opposite extreme assumption known as "cash-in-advance constraint", which means that the agent can borrow liquidity in no circumstance, i.e. there exists zero liquidity supply (consequently the end-of-time condition of zero present value of wealth is supplemented with a time-by-time cash constraint which forces money holdings into the agent's programme). It is often recognized that both cases should, to say the least, only be taken as a first approximation, but the liquidity-supply part of the problem is generally not developed.

To address this problem we use a highly stylized model of an agent (a firm) with an intertemporal programme (an investment project) characterized by a payment commitment (interests) towards another agent (a bank) at each point in time, a stochastic cash flow (profit) following a trendless geometric Brownian motion, and a non-negative net present value of the programme (the agent may well be a household with cash flow given by income, or a sovereign state with cash flow given by its trade balance, etc.). Hence this agent faces a liquidity problem as soon as the cash flow falls short of the payment commitment. Note that we work with a stochastic process such that, as is typically the case, the liquidity problem is perceived as *a temporary* *shortage of cash*: in fact, though the observed level of cash flow is the best expectation of the future levels, a positive probability exists that the cash flow recovers (the initial net present value of the programme is indeed non-negative). To focus on the problem at hand we assume that the agent has non-money goods that can be liquidated at a cost but no worthless money holdings, which can be replaced by liquidity supply "on call". All agents are risk neutral.

In the first place, in section 2 we shall see that the standard measure of financial value (the excess of the *expected* cash-flow present value over the principal) is a correct measure only if infinite liquidity supply exists, i.e. only if each time the cash flow is less than the payment commitment the agent can receive liquidity at no cost. Liquidity can be provided by the same subject with whom the agent is committed, typically a bank which allows the agent's account to be temporarily in the red. This is the essence of what Hicks called "the overdraft economy" (1967), and that he regarded as the ideal type of modern credit economies; accordingly, we shall call a debt contract inclusive of the liquidity service an "overdraft debt contract" (ODC).

In the second place, in section 3 we shall give a precise content to the idea that the ODC may increase efficiency by comparing it with the so-called "standard debt contract" (SDC). According to the usual definition (e.g. Freixas and Rochet, 1998, sec. 4.2), a SDC establishes that as soon as a debtor reports to be illiquid, 1) he/she is audited, and 2) in case of insolvency is declared bankrupt. In our framework, the SDC corresponds to the zero liquidity-supply case. However, truncation of the agent's programme at the first illiquidity state entails a loss of value measured by the expected liquidation cost of non money-goods (recall that the initial net present value of the programme is non negative). From this point of view, if on the one hand the SDC can be considered as the optimal financial contract under costly state verification, on the other it rules the bank's liquidity service out of analysis. The point also relates to the issue of the specificity of the bank as a financial intermediary. Consider for instance the case of a firm that can finance an investment either with a bank SDC or on the stock market. If it chooses the bank, the firm should pay a fixed coupon whose discounted value equals the investment value. If it chooses the stock market, the firm is committed to pay shareholders a dividend; arbitrage opportunities imply that the discounted value of the dividend at each point in time cannot be lower than the invested capital. Given a single market interest rate, by Modigliani-Miller theorem the firm should be indifferent between the two choices. Indifference holds in good as well as in bad times. Suppose the firm chooses the stock market: if at any date the firm's profit falls below the no-arbitrage dividend, i.e. the firm is illiquid, shareholders should disinvest from that firm (if liquidating the firm is costly, disinvestment will only be delayed: see e.g. Dixit-Pindyk, 1994). At that point there is no difference for the firm between being liquidated by the bank or by the shareholders. However, as we said previously, a positive probability exists that the profit can return above the no-arbitrage dividend (coupon), and hence it may be profitable for the firm to receive a liquidity injection instead of being shut down. In essence, we meet here the so-called "short-termism" that may affect the stock market (Keynes, 1936, ch.12; Mayer, 1988; Hoshi, 1989), and, alternatively, the idea that an explanation of the comparative advantage of the bank should be sought for in its ability to grant ODCs.

In the third place, in section 4 we shall remove both the infinite and zero liquidity-supply cases and shall investigate the case of costly ODCs. i.e. debt contracts in which overdrafts are allowed at a specific extra-cost. From previous considerations, it is clear that ODCs should be framed into long-term contracts analysis and more generally within the literature that relates the specific advantages of the bank-borrower relationship in the long-term, personal, non-marketable nature of this relationship (von Thadden, 1990; Hellwig, 1991; Mayer, 1994). For this reason, and in hommage to the German tradition of relationship banking, we have named the bank in our paper "Hausbank". A significant part of this literature aims at showing how long-term bank-borrower relationships resolve asymmetric information problems and agency problems. In the same framework it has been argued that long-term ties can also induce inefficiency since the bank can have the opportunity to charge extra-costs to "informationally captured" firms (Sharpe, 1990). Here we wish to show how the repeated relationship between bank and borrower may solve the problem of designing efficient ODCs in a setup where the key problem is not asymmetric information but the credibility of the mutual commitment between the two parties.

As a first step we assume perfect commitment, and we find a whole set of feasible values of the marginal cost of liquidity that can be charged by the bank up to an upper bound ("liquidity premium"). Liquidity injections allow the borrower to carry on his/her initial programme, but the "liquidity premium" redistributes part of the programme's value from the borrower to the bank; hence this result can be viewed as an extension of the idea that long-term relationships can solve specific borrower-lender problems only at a cost. Then we remove perfect commitment and address three core issues in long-term ODCs analysis. 1) *Non private enforceability* or the borrower's willingness to pay problem (Bolton-Scharfstein, 1990; Hart-Moore, 1996, sec.4): after the ODC has been signed, and even though the borrower's state is freely observable, the bank may be unable to force the borrower to pay unless payment is enforced by an external legal authority; sure payments are only those based on the borrower's willingness to pay. 2) *Renegotiation*: a long-term contract may be "non renegotiation-proof" (Salanie', 1997), i.e. the borrower, once he/she finds him/herself in specific states, may, in alternative to paying the contractual sum and in order to avoid bankruptcy, induce the bank to accept a renegotiation of the contractual terms. 3) *Threat of termination*: as a consequence of the previous two points, it is generally argued that the bank's "threat of termination" is an essential part of the contractual equilibrium (Allen, 1983; Stiglitz-Weiss, 1983; Bolton-Scharfstein, 1990). The analytical framework that we have chosen allows us to deal with all three issues in a coherent and general way (for instance, most models in the field are limited to two or three-period contracts).Our main finding is that efficient, i.e. cost-minimizing and "renegotiation-proof", ODCs do emerge in the absence of perfect commitment as the borrower and the bank can exert *mutual threat of termination.*

2 Infinite liquidity supply as benchmark

We consider a risk neutral agent such as a firm which owns an infinitely-lived investment project of initial amount I, illiquid non-money goods $K \leq I$, $K = kI, \in (0,1]$, and no worthless money balances. The investment project generates instantaneous profits π_t which are uncertain and driven by a trendless geometric Brownian motion.

$$d\pi_t = \sigma \pi_t dW_t, \text{ with } \pi_0 = \pi > 0, \sigma > 0 \tag{1}$$

where dW_t is the standard increment of a Wiener process (or Brownian motion), uncorrelated over time and satisfying the conditions that $E(dW_t) = 0$ and $E(dW_t^2) = dt$. Therefore $E(d\pi_t) = 0$ and $E(d\pi_t^2) = (\sigma\pi_t)^2 dt$.

Assumption 1. A constant market interest rate r is given. By Modigliani-Miller theorem, the minimum no-arbitrage constant payment b due to investors is

$$\int_0^\infty e^{-rt} b dt \equiv \frac{b}{r} = I \tag{2}$$

The project can be undertaken.

By the standard net present value (NPV) technique the project's value for the firm, $V^l(\pi; b)$ can be expressed by (e.g. Harrison, 1985, pag.44):

$$V^l(\pi; b) = E_0 \left\{ \int_0^\infty e^{-rt}(\pi_t - b)dt \mid \pi_0 = \pi \right\} = \frac{\pi - b}{r} \equiv \frac{\pi}{r} - I \geq 0 \tag{3}$$

where $E_0(.)$ denotes expectation given available information at time 0.

However, the stochastic process driving the firm's profits implies that there always exists a positive probability that at a point in time t, $\pi_t < b$. At that point the firm is unable to pay the no-arbitrage sum b because it is illiquid, and the project's value is out of the market. Therefore, the project cannot be consistently valued on the pure NPV basis unless an auxiliary assumption is added concerning the existence of infinite liquidity supply at any illiquidity state. For reasons that we do not formalize here, but that can be related to standard arguments of comparative efficiency in liquidity management, no private investor can supply liquidity "on call" except banks (Hicks, 1967, 1989; Goodhart, 1989). We therefore add the following:

Assumption 1bis. The firm signs an "overdraft debt contract" (ODC) with a bank, with interest rate r and a fixed coupon payment b. In all states in which $\pi_t < b$, the firm obtains an instantaneous costless injection of liquidity equal to $b - \pi_t$.

The project's NPV for the bank is simply given by:

$$W^l(b) = \frac{b}{r} - I = 0. \tag{4}$$

where the superscript l indicates the infinite liquidity case

3 Case with zero liquidity supply

In this section we wish to compare the benchmark case of infinite liquidity supply, provided by costless overdraft facilities in a bank ODC, with the opposite case of zero liquidity supply. To this end, instead of assumption 1bis let us consider the following:

Assumption 2. The firm signs a "standard debt contract" (SDC) with a bank. 1) The firm pays $\min(\pi_t, b)$ at each time t, 2) bankruptcy is declared when the firm is marginally insolvent, i.e. the first time that $\pi_t < b$, 3) the bank liquidates the firm and obtains the value of the collateralized non-money goods K.

For the time being we assume the SDC is optimal. We shall investigate this property further in the subsequent parts of the paper. Letting $T = \inf(t \geq 0 : \pi_t \leq b)$ be the stochastic bankruptcy time, the project's NPV for the firm as of time 0 in this case becomes:

$$V^i(\pi; b) = E_0 \left\{ \int_0^T e^{-rt}(\pi_t - b)dt - e^{-rT}K \mid \pi_0 = \pi \right\} \tag{5}$$

where the superscript i indicates the illiquidity case.

By the usual dynamic programming decomposition we may split the above conditional expectation into the contribution over the infinitesimal time interval 0 to dt and the integral from dt to T. Because the investment yields a cash flow up to the time T that the project is shut down, the return from holding it, over the small time interval dt, is given by $(\pi - b)dt$ plus the capital gain $E(dV^i(\pi; b))$. Hence, in the continuation region (i.e. the values of π for which is worth for the bank to keep the firm alive) we get the following Bellman equation:

$$rV^i(\pi; b)dt = (\pi - b)dt + E(dV^i(\pi; b))$$

Since π_t is driven by (1), applying *Ito's Lemma* to dV^i the asset equilibrium condition yields the following differential equation (Dixit and Pindyck, 1994, pag. 147-152):

$$\frac{1}{2}\sigma^2\pi^2 V^{i\prime\prime} - rV^i = -(\pi - b) \qquad \text{for } \pi \in [b, \infty), \tag{6}$$

with boundary conditions:

$$\lim_{\pi \to \infty} [V^i(\pi; b) - \frac{\pi - b}{r}] = 0 \tag{7}$$

$$V^i(b; b) = -K \tag{8}$$

As usual, equation (7) states that, when the profits go to infinity the value of the firm must be bounded. In fact, the second term in (7) represents the discounted present value of shareholders' profit flows over an infinite horizon starting from price level π as in (3). Boundary condition (8) stems from assumption 2, it says that when the profits reach the level b, the bank closes the firm and its value must be equal to its liabilities (*matching value condition*).

By the linearity of differential equation (6) and making use of (7), the general solution takes the form:

$$V^i(\pi; b) = \frac{\pi - b}{r} + A\pi^{\beta_2}, \qquad \text{for } \pi \in [b, \infty). \tag{9}$$

where $A < 0$ is a constant to be determined and β_2 is the negative root of the characteristic equation $\Phi(\beta) = \frac{1}{2}\sigma^2\beta(\beta - 1) - r = 0$. The matching value condition, that is the project's NPV for the firm at the bankruptcy threshold $\pi = b$, yields the value of the constant A:

$$A = -Kb^{-\beta_2} \tag{10}$$

The difference between the project's NPV in (3) and in (9) highlights the role of the assumption concerning liquidity supply:

$$V^l(\pi; b) - V^i(\pi; b) = -A\pi^{\beta_2} > 0$$

The difference is positive, i.e. there is a loss of value due to expected bankruptcy at the first illiquidity state. This loss of value arises from the truncation of the firm's life at T and is measured by:

$$\begin{aligned} A\pi^{\beta_2} &= -(\frac{\pi - b}{r} + K)(\frac{\pi}{b})^{\beta_2} \\ &= ((1 - k)I - \frac{\pi}{r})(\frac{\pi}{b})^{\beta_2} \end{aligned}$$

Finally, for assumption 2, to be intertemporally consistent the bank should have no incentive to force the firm into bankruptcy as long as $\pi > b$, i.e. the project's NPV for the bank, $W^i(\pi; b)$, should satisfy:

$$W^i(\pi; b) = E_0 \left\{ \int_0^T e^{-rt} b dt + e^{-rT} K \mid \pi_0 = \pi \right\} - I \leq 0. \tag{11}$$

As T is the random first time the process π_t reaches the bound b starting from the general initial position $\pi > b,$ the bank's NPV (11) reduces to:[1]

$$\begin{aligned} W^i(\pi; b) &= \frac{b}{r}\left[1 - \left(\frac{\pi}{b}\right)^{\beta_2}\right] + K\left(\frac{\pi}{b}\right)^{\beta_2} - I \leq 0 \\ &= \frac{b}{r}(k-1)\left(\frac{\pi}{b}\right)^{\beta_2} \leq 0 \end{aligned} \tag{12}$$

Since $k \in (0, 1]$, the above condition is always satisfied for any k prior to bankruptcy. By contradiction suppose it is not: then it would be more valuable to the bank to quit the SDC than to receive the coupon b for the rest of project's life. But this condition would hold at time 0 as well, so that the contract would never be signed. In particular, note that with maximum collateral, $k = 1$, $W^i(\pi; b) = 0$, so that the bank is indifferent towards bankruptcy throughout the project's life. By contrast, for any $k < 1$, bankruptcy entails a loss for the bank too, due to the truncation of firms' payments.

4 Case with costly liquidity supply

In the previous section we have seen that the SDC is equivalent to a zero liquidity-supply case: the first time the firm cannot pay b it should be liquidated. Under the SDC, this happens regardless of the bank or non-bank nature of investors[2]. However, the expectation of bankruptcy at the first illiquidity state reduces the project's value. Since there always exists a positive probability that the firm's profits recover, the firm's liquidation may not be efficient. Hence the SDC rules the role of the bank as liquidity supplier out of analysis. Therefore, in this section we return to ODCs as distinctive of bank intermediation and we wish to analyze their efficiency properties in detail.

First of all, we wish to examine a more general case in which liquidity supply in neither zero nor free. To this end, we modify assumption 1 as follows:

[1]Solution of (11) can be obtained through the usual dynamic programming decomposition. However, for a more general approach to calculate this expression see Harrison (1985, p.42).

[2]If the firm's liquidation is costly, there will be a delay in the liquidation timing. This is a standard result in irreversibility theory that at present we do not model here (see Dixit-Pindyk, 1994).

Assumption 3. In any state $\pi_t < b$, the bank is committed to supply a compensative liquidity injection with an additional coupon payment.

Note that for the sake of comparison with previous cases, the ODC can be thought of as composed of two parts, the debt contract part paying the fixed coupon b, and the overdraft facility. In practice, whenever $\pi_t < b$, the bank does not execute the bankruptcy procedure and adds the overdue amount to the overall debt charging an appropriate fee.

In order to examine this new problem, we now need model the cost of liquidity supply. Here, we treat liquidity supply as a regulation mechanism, in the sense of Harrison (1985), of the firm's profits. The bank regulates the process π_t by means of instantaneous, infinitesimal "liquidity injections" U_t never allowing π_t to go below b. More formally, the process $\pi_t \in [0, \infty)$ is free to move as dictated by (1) as long as $\pi_t > b$, but the instant π_t crosses b from above, it is reflected at the lower barrier b (see Appendix).

As to the cost of liquidity, for the sake of simplicity, we assume they are linearly increasing with the liquidity injections by the amount:

$$dC_t = c \times [\text{injections of liquidity}].$$

The introduction of a cost for liquidity changes the picture with respect to the two previous basic cases. The bank and the firm have now different possible options during the project's life, which are represented in figure 1.

[Figure 1]

We start with a verbal description of the choices tree as an introduction to the more formal treatment we shall give below. Node A corresponds to the first insolvency state of the firm (time T as defined in sec.2). There are two options:

- AB: no liquidity injection and firm's closure
- AC: liquidity injection

If AB is chosen, the ODC is actually broken. If AC is chosen the ODC is performed: the bank refinances the firm and charges an appropriate fee, the firm stays in business and is expected to pay the original coupon b and the additional fee from that point onwards. After refinancing, say at time $T' > T$ and node C, the firm may find itself in such conditions that a new alternative opens up:

- CD: go on complying with the ODC
- CE: breach the ODC

If the firm chooses CE, the bank should find out the most appropriate reaction; this can be one of the two following alternatives:

- EG: exert the bankruptcy procedure and close the firm
- EF: do not close the firm

Which alternative is chosen at each node, and hence the properties of the ODC, essentially depend on two factors:

a) the cost of liquidity supply

b) the commitment technology behind the ODC

4.1 Full commitment.

In the first place, we study a reference case where we assume that the ODC is supported by *full commitment* by both sides. This means that the ACD path in figure 1 is always chosen unconditionally.[3]

Assumption 4. By full commitment we mean that the contract's conditions are perfectly verifiable at zero cost, i.e. a) each realization π_t is observable, b) the bank obtains the due amount with certainty.

[3] In other words, the ODC signed by the bank and the firm covers the whole duration of the relationship and it cannot be breached or renegotiated.

Coming to the choices at node A, AB is obviously equivalent to zero liquidity supply in section 3, and we already know that the NPV of the project is $V^i(\pi; b)$ for the firm and $W^i(\pi; b)$ for the bank. As to the alternative choice, if it is taken, by assumption 4 both the bank and the firm will always fulfill the ODC provisions so that the project will last throughout its life; hence studying the choices at node A under full commitment is equivalent to studying the conditions of existence of the ODC. The project's NPV for the firm under the ODC becomes:

$$\begin{aligned} V^c(\pi; b) &= E_0\left\{\int_0^\infty e^{-rt}[(\pi_t - b + U_t)dt - dC_t] \mid \pi_0 = \pi\right\} \quad (13) \\ &= \frac{\pi - b}{r} + E_0\left\{\int_0^\infty e^{-rt}(U_t dt - dC_t) \mid \pi_0 = \pi\right\} \end{aligned}$$

To compute the discounted expectation in (13) we repeat the arbitrage calculation, but now with a different condition at the (reflecting) barrier b. That is, $V^c(\pi; b)$ is obtained by solving the following Bellman equation (see Appendix):

$$\frac{1}{2}\sigma^2\pi^2 V^{c\prime\prime} - rV^c = -(\pi - b) \qquad \text{for } \pi \in [b, \infty), \quad (14)$$

with boundary conditions:

$$\lim_{\pi \to \infty}[V^c(\pi; b) - \frac{\pi - b}{r}] = 0 \quad (15)$$

$$V^{c\prime}(b; b) = c \quad (16)$$

While (15) is equal to (7) in the previous case, and has the same meaning, condition (16) replaces the matching value condition (8). In fact, since liquidity is now costly, it is necessary that at each liquidity injection the marginal value of continuing the project's life does not fall below the marginal cost of liquidity (*smooth pasting condition*).[4] Again, by the linearity of the differential equation (14) and making use of (15) and (16), the general solution of (13) takes the form:

[4]The smooth pasting condition (16) is the first order derivative of the expected present value of a function of a Brownian motion. It does not involve any optimizing role of the barrier and requires only the continuity of the first derivative of V^c in b (Dixit, 1993, p. 27).

$$V^c(\pi; b) = \frac{\pi - b}{r} + B\pi^{\beta_2} \tag{17}$$

with:

$$B = \frac{(c - 1/r)b^{1-\beta_2}}{\beta_2}$$

The second term on the r.h.s. of (17) measures the NPV of total liquidity supply for the firm, i.e. the difference between the NPV of the liquidity injections $U_t dt$ and that of the additional fees dC_t:

$$B\pi^{\beta_2} = E_0 \left\{ \int_0^\infty e^{-rt}(U_t dt - dC_t) \mid \pi_0 = \pi \right\} \tag{18}$$

Taking into account the initial loan I and the liquidity injections U_t, the expected discounted value of the firm's overdraft payments matches the total value of overdrafts (see Appendix):

$$E_0 \left\{ \int_0^\infty e^{-rt} dC_t \mid \pi_0 = \pi \right\} = crE_0 \left\{ \int_0^\infty e^{-rt} U_t dt \mid \pi_0 = \pi \right\} \tag{19}$$

We are now in a position to check whether the ODC is feasible for the firm, (i.e. which of the two choices AB, ACD is more valuable to it). For the ODC to be feasible for the firm, it is simply necessary and sufficient that the project's NPV under the ODC is not less than under the zero liquidity case, which implies the following overall a participation constrain (or *value matching condition):*

$$V^c(\pi; b) - V^i(\pi; b) = B\pi^{\beta_2} - A\pi^{\beta_2} \geq 0. \tag{20}$$

Knowing the constants A e B, the firm's participation condition becomes:

$$\frac{(c - 1/r)b^{1-\beta_2}}{\beta_2} + Kb^{-\beta_2} \geq 0.$$

Note that such a condition implies a constraint upon the determination of the marginal liquidity cost c by the bank:

$$c \leq \frac{1}{r} - \frac{K\beta_2}{b}.$$

Since $\beta_2 < 0$, $c > 1/r$ is a feasible value, hence c can *exceed the perpetual discount rate* $1/r$ by an amount proportional to the cost K and the parameter

β_2. To grasp the meaning of this result, recall that K represents an exit cost to the firm which may induce it to stay in business instead of quitting immediately. Given our assumption about the collateralization degree k, i.e. $K = kI$, and given that $b = Ir$, we obtain the feasible interval for c:

$$c \leq \frac{(1 - k\beta_2)}{r} \tag{21}$$

In words, c can exceed $1/r$ up to the factor $-k\beta_2/r > 0$, which we may call the "liquidity premium".

We now turn to the bank. The project's NPV for the bank under the ODC is:

$$\begin{aligned} W^c(\pi; b) &= E_0 \left\{ \int_0^\infty e^{-rt}[(b - U_t)dt + dC_t] \mid \pi_0 = \pi \right\} - I \qquad (22) \\ &= \frac{b}{r} - I - B\pi^{\beta_2} \\ &= -B\pi^{\beta_2} \end{aligned}$$

This is an obvious result since, as we know, $B\pi^{\beta_2}$ is the NPV of total liquidity supply for the firm, so that $-B\pi^{\beta_2}$ is the same variable valued by the bank. Note that therefore if the ODC is implemented, it implies a zero-sum *redistribution of value* between bank and firm. To establish whether the ODC is valuable to the bank we should compare its NPV with the NPV of the alternative choice of exerting the bankruptcy procedure at node A. As we know from section 3, bankruptcy entails a loss for the bank measured by $W^i(\pi; b) \leq 0$, hence the ODC is valuable to the bank provided that:

$$W^c(\pi; b) - W^i(\pi; b) = -B\pi^{\beta_2} + \frac{b}{r}(k - 1)\left(\frac{\pi}{b}\right)^{\beta_2} \geq 0$$

After substituting the value of B, like in the case of the firm, we obtain a participation condition of the bank which implies a restriction on the marginal refinancing cost c:

$$c \geq \frac{[1 + (1 - k)\beta_2]}{r} \equiv \frac{(1 - k\beta_2)}{r} + \frac{\beta_2}{r} \tag{23}$$

Therefore, c can be set *below* $1/r$ up to the factor $(1 - k)\beta_2/r < 0$ that we may call the "liquidity discount". Note that if $c \leq 1/r$, $B \geq 0$, i.e. the NPV of total liquidity supply is negative for the bank. Nonetheless, as a noteworthy consequence of the fact that the alternative to implementing the ODC is the firm's bankruptcy, and this may be a loss for the bank depending on $k < 1$, the ODC remains valuable to the bank even for values $c < 1/r$ up to the point where the loss in the NPV of total liquidity supply does not exceed the NPV of the bankruptcy loss. In other words, if owing to incomplete collateralization the firm's bankruptcy is costly to the bank, the latter is ready to pay a liquidity discount that the firm may in turn exploit by claiming for a lower c.

As a result of our inspection of the ODC participation conditions of the firm and the bank at node A (i.e. from (21) and (23)), we have obtained:

Remark 1 *The feasible set of values of the marginal refinancing cost c is :*

$$\frac{1 + (1-k)\beta_2}{r} < c \leq \frac{1 - k\beta_2}{r} \tag{24}$$

At the upper bound of the set, $c = \frac{(1-k\beta_2)}{r}$, the bank exploits the firm's willingness to pay for liquidity and charges the liquidity premium $-k\beta_2/r$; at the lower bound of the set, $c = \frac{1+(1-k)\beta_2}{r}$, the firm exploits the bank's loss of value in exerting the bankruptcy procedure and claims for the liquidity discount $\frac{(1-k)\beta_2}{r}$. The width of the set of values of c depends on the collateralization degree k and on the absolute value of β_2 which is in turn increasing in the variance of the π_t process (i.e. $|\frac{\partial\beta_2}{\partial\sigma}| > 0$). A large variance process induces the bank to charge a high liquidity premium and the firm to claim for a high liquidity discount. A high collateral increases the firm's willingness to pay in order to stay in business, and hence it jeopardizes the firm's ability to extract a liquidity discount and strenghtens the bank's ability to charge a liquidity premium. For $k = 1$, there will be no way for the firm to induce the bank to accept a liquidity discount and c will necessarily be set above $1/r$.

For any c in the feasible set, the move ACD dominates the move AB for both the firm and the bank, hence there may be bargaining over c. We do not formalize this problem because we instead wish to stress the crucial role of the full commitment assumption. Suppose for the time being that the bank has all the bargaining power because it is on the long side of market (the bank can refuse the whole ODC to the firm whereas the firm cannot). Therefore:

Proposition 1 *Under full commitment, the bank can maximize the project's NPV by setting* $c^* = \frac{(1-k\beta_2)}{r}$*, and extracts a rent from the firm's value by the amount* $-B(c^*) > 0$*.*

We shall see in the next section that it is sufficient to remove the full commitment assumption to obtain an efficient, endogenous solution.

4.2 Observable profits but non enforceable payments.

So far we have examined situations in which once the parties have signed the ODC, both comply with the contract terms forever. Yet any borrower-lender relationship raises the fundamental question: why should the borrower pay his/her debt to the lender? The popular answer is that the borrower will pay the lender as long as the latter can monitor and audit the latter or to the extent that insolvency, leading to bankruptcy, is more costly than the debt payment. For instance, the basic model of SDC arises out of the premise that the debt payment problem exists owing to asymmetric information and costly state verification (Townsend, 1979; Diamond, 1984; Gale-Hellwig, 1985). Then it is shown that the optimal debt contract is precisely tailored to solve the problem by *a*) minimizing the lender's costs of auditing the borrower's true state, and *b*) forcing the borrower into bankruptcy when he/she is truly insolvent.

Implicit in this result are two assumptions: 1) auditing, albeit costly, is always effective (i.e. auditing is always sufficient for the lender to obtain the amount due), 2) bankruptcy *vis-à-vis* insolvency is always viable to the lender. More recent studies have re-examined the issue after relaxing assumption 1). The more general setup is one of "*non-enforceable contracts*". The idea is that, whatever the borrower's true state, and even when he/she is technically solvent, the lender's ability to bear auditing costs, or even his/her free access to information, may be of little help in the absence of the borrower's *willingness to pay.* The most typical case is when the borrower is a sovereign State (Eaton-Gersowitz, 1981; Eaton et al., 1986). However, even in the case of private relationships, there may be several reasons other than asymmetric information that may prevent debt contract provisions contingent on the borrower's state from being enforced (Bolton-Scharfstein, 1990; Hart-Moore, 1996, sec.4). One basic reason is that a lender (in civilized countries) has no private means to force the borrower to pay the amount due. Another complementary reason may be "non-verifiability" (which is indeed tantamount to infinite state verification costs), that is to say the borrower's state can be "technically" known to the two private parties, but it is not possible to have it verified by an independent legal authority (e.g. the courts). In these cases, unless the borrower's willingness to pay is assumed a priori, the bankruptcy option must be effective at any point in time.[5] This rationale of viable borrower-lender relationships is often referred to as "threat of termination", which is generally viewed as an efficient incentive (or better disincentive) device (Stiglitz-Weiss, 1983; Allen, 1983; Bolton-Scharfstein, 1990; Haubrich, 1989).

This leads us to the second assumption underlying standard debt contracts -that bankruptcy *vis-à-vis* insolvency is always viable to the lender. It is well-known to bankers that "a 1000 dollars debt is a debtor's problem, a 1,000,000 dollars debt is a creditor's problem". In early standard debt contract models, bankruptcy is indeed a forced choice because the debt contract has a fixed deadline. However, in a borrower-lender long-run relationship, it is often the case that bankruptcy is not the most profitable (least costly) lender's choice. In all cases whereby termination entails a loss for the lender, the lender's "threat of termination" is weakened while the borrower's "threat of insolvency" is strengthened. Under such *mutual threat* some form of renegotiation may be more profitable. To put it differently, under the conditions under discussion, the standard debt contract may no longer be "renegotiation-proof" (Salanié, 1997), which means that under specific circumstances the borrower may be able to induce the lender to renegotiate over the contract terms. Not by chance, in the field of sovereign State debt, where there is no enforceability of payments and the lenders' termination losses are high, renegotiation is the prevailing solution when insolvency is declared.

Thus, an interesting question arises: can we devise efficient, "renegotiation-proof" debt contracts which only rely upon the bankruptcy menace? And in particular, is our ODC efficient in this sense? To address this problem we first relax the assumption of perfect commitment that we replace with one of non verifiability. Hence we modify assumption 4 as follows:

Assumption 4bis. The ODC is non privately enforceable. If at any point in time the firm refuses to pay its obligation by declaring insolvency, the bank can only exert the bankruptcy procedure against the firm.

[5] Bankruptcy is generally a legal procedure enforced by the courts. This does not contradict the non verifiability assumption because the courts are not requested to ascertain the borrower's ability to pay, but to act upon the borrower's unwillingness to pay.

The situation depicted by assumption 4bis is relevant because the ODC is indeed "non renegotiation-proof", as we shall see promptly.

In the first place we show that, as argued above, when the firm's bankruptcy entails a loss for the bank, its "threat of termination" is not credible. More precisely, upon the firm's declaration of insolvency at any point in time, a fixed coupon $z < b$ exists such that the bank is at least indifferent between renegotiating the contract with the new coupon z and exerting the bankruptcy procedure.

At any point in time the NPV of termination for the bank, $W^t(b)$, i.e. the NPV of exerting the bankruptcy procedure at that time, is simply:

$$W^t(b) = K - I = \frac{b}{r}(k-1) \tag{25}$$

If we define $W^z(z;b)$ the project's NPV for the bank after renegotiation of the coupon z, this is given by:

$$\begin{aligned} W^z(z;b) &= E_t\left\{\int_t^{\infty} e^{-r(s-t)} z ds\right\} - I \\ &= \frac{z-b}{r} \end{aligned} \tag{26}$$

Therefore, the bank will be indifferent between the two options provided that:

$$W^z(z;b) - W^i(b) = \frac{z-kb}{r} \geq 0$$

or, simplifying:

$$z \geq kb \tag{27}$$

This result shows that, as long as $k < 1$, at any point in time the bank can be induced to renegotiate a new coupon $z < b$. In other words, the no-arbitrage coupon b cannot be enforced by "threat of termination". The reason, as already anticipated at the beginning of this section, is that $k < 1$, i.e. less than full collateral, entails that termination is costly to the bank; hence it can find it profitable to obtain at least kb (forever) instead of bearing the termination loss.

In the second place, we examine the firm's willingness to pay. After signing the ODC at time 0, it is trivial that the firm has always an incentive to retain the bank loan and minimize the debt payment no matter whether profit is high, $\pi > b$ (see equation (3)) or low, $\pi < b$ (see equation (17)). However, it is worth giving a formal proof.

Let us suppose that at a point in time $t < T$ the firm has a profit $\pi_t = x > b$ and can choose to pay the bank a fixed coupon z such that the consequent project's NPV is:

$$\begin{aligned} V^z(x;b,z) &= E_t\left\{\int_t^\infty e^{-r(s-t)}(\pi_s - z)ds \mid \pi_t = x > b\right\}, \quad \text{for } t < T \quad (28) \\ &= \frac{x-z}{r} \end{aligned}$$

Since at the same point in time the project's NPV under the ODC is $V^c(x;b) = \frac{x-b}{r}$, the firm has an incentive to renegotiate if:

$$V^z(x;b,z) - V^c(x;b) = \frac{b-z}{r} \geq 0$$

or:

$$z < b \qquad (29)$$

Now let us consider the same problem at a point in time $t > T$ with profit $\pi_t = x < b$, i.e. after the refinancing part of the ODC has been activated. Since the bank is committed to inject liquidity to fill any gap $x - b$, the firm will find it profitable to renegotiate if:

$$V^z(x;b,z) - V^c(b;b) = \frac{x-z}{r} - B(c)b^{\beta_2} \geq 0$$

or:

$$x - z - b\left(\frac{cr-1}{\beta_2}\right) \geq 0 \qquad (30)$$

To obtain the sum z that the firm is willing to pay, recall that:

- $B(c)b^{\beta_2}$ measures the NPV of total liquidity supply for the firm, which is decreasing in the marginal cost of liquidity c;
- For any value of $c < 1/r$, the term $B(c)b^{\beta_2} > 0$, i.e. refinancing redistributes value from the bank to the firm, and for any $c \geq 1/r$, $B(c)b^{\beta_2} \leq 0$, i.e. refinancing redistributes value from the firm to the bank;

- For any value of c within the feasible set (24), the firm prefers to stay in business under the ODC than to close. A *fortiori*, the firm prefers to pay $z < b$ than to close. On the other hand, the firm cannot pay a coupon that exceeds its expected profit, hence it must be $z \leq x$.

Therefore, 1) for any value of $c < 1/r$ in the feasible set, the firm is willing to pay any coupon $z < x - b(\frac{cr-1}{\beta_2}) < x < b$; 2) for any $c > 1/r$, the firm is willing to pay $z \leq x < b$. The difference between the two cases relates to the fact that the firm under the ODC gains value in the former and looses value in the latter; hence the firm is willing to pay less in the former than in the latter. Consequently, depending on the value of c, we can write:

$$z < \min\left[x,\ \ x - b(\frac{cr-1}{\beta_2})\right] \tag{31}$$

Note that $c > 1/r$ has an important consequence on the timing of renegotiation: this should take place as soon as the firm's profit crosses the threshold value b. Consider as an example the case that the bank is maximizing the refinancing premium by setting $c^* = \frac{1-k\beta_2}{r} > \frac{1}{r}$; then the solution to condition (30) is $z < x + kb$, which is however constrained to the subset $z \leq x$. As soon as $x = b$ the firm finds it profitable to renegotiate.

Our last step is to compare the renegotiation sets of z for the firm (z^f) and for the bank (z^b). Let us examine the case when the ODC includes a refinancing premium $c > 1/r$; hence, z^f is unconditionally the same in all states (see (30) and (31)), so that:

$$z^f < b, \quad \text{and} \quad z^b \geq kb, \tag{32}$$

which implies the following proposition:

Proposition 2 *Under non-enforceability, with less than full collateral, $k < 1$, at any point in time the ODC is non "renegotiation-proof", and can be renegotiated for any fixed coupon $z \in [kb, b)$.*

This proposition has a major consequence: unless the loan is fully collateralized, the bank will never sign the ODC, because it can only be sure to obtain a fraction k of the non-arbitrage coupon b even when the firm is technically solvent. By contrast, with full collateral, $k = 1$, the renegotiation set of z collapses to $z = b$, so that the ODC is indeed enforced by "threat of termination". This also implies that the bank is in a position to charge the maximum liquidity premium on the marginal cost of liquidity, $c^* = \frac{1-\beta_2}{r}$.

Yet suppose that the collateral market value is not independent of the firm's state. A reasonable case is that $k = 1$ whenever the firm is technically solvent, $\pi_t > b$, and $k < 1$ when the firm is technically insolvent, $\pi_t < b$. In this situation the ODC is renegotiation-proof as long as $\pi_t > b$, but it may no longer be as soon as $\pi_t < b$ and the overdraft facility of the contract should be activated. Our key argument is that the bank-firm "mutual threat" leads to an endogenous efficient solution for the value of c. In other words, the efficient, "renegotiation-proof" ODC emerges from a continuous rate of hearing between the firm and the bank. Such repetition of the relationship may substitute explicit long-term contracts and provides the firm with appropriate "refinancing options"

As we have seen above, the firm should start paying for liquidity as its profit falls below b, but it has the highest incentive to break the ODC contract and offer $z^f < b$ to the bank soon after the refinancing part of the contract has started. Clearly, the break-even sum $z^b = kb$ for the bank lies in the renegotiation set for the firm. Therefore, the ODC can only be made renegotiation-proof if the bank chooses c in such a way that it drives the firm's incentive to renegotiate to zero, given its willingness to pay $z \leq x$; hence, according to (30):

$$b(\frac{cr-1}{\beta_2}) = 0, \tag{33}$$

or:

$$c = \frac{1}{r}$$

We have found the following "efficiency propositions".

Proposition 3 *A) In order for a costly overdraft facility to be "renegotiation-proof" with less than full collateral, $k < 1$, the marginal cost of liquidity must be $c = 1/r$, which implies that the NPV of total liquidity supply is zero.*

B) Bank liquidity supply can be efficient, i.e. the NPV of total liquidity supply is zero, only with less than full collateral and "mutual threat"

To summarize, the rationale of propositions 2 and 3 is that with full collateral the bank's "threat of termination" is sufficient to enforce the ODC with the market coupon b, and with the maximum refinancing premium above the efficient marginal cost of liquidity. With less than full collateral, the bank and the firm operate under “mutual threat”. The firm’s “threat of insolvency” countervails the bank’s “threat of termination”, so that the firm is in a position to declare insolvency and induce the bank to renegotiate. To avoid this, the bank can only refrain from charging any refinancing premium. Hence the bank-firm “mutual threat” is an important mechanism in the emergence of efficient firm-bank long-term relationships.

5 Conclusions

Liquidity supply ”on call” by means of overdraft facilities granted to the borrower is a key specificity of the bank-borrower relationship. We have examined the efficiency properties of ODCs; in particular cost-minimization and ”renegotiation proof-ness” in a context of a long-term relationship, with symmetric information but non enforceability of payments. Our main finding is that efficient ODCs do emerge in the absence of perfect commitment as the borrower and the bank can exert *mutual threat of termination*. This condition arises whenever the bank’s threat of termination (exert the bankruptcy procedure if the borrower is unwilling to pay) is countervailed by the borrower’s threat of renegotiation (declare insolvency and offer a renegotiation of the contract). We find that mutual threat is effective under a simple and general condition: the loan is less than fully collateralized. In fact, if this is the case, bankruptcy entails a loss for the bank too, and hence it is willing to accept renegotiation. A ”renegotiation-proof” ODC exists however, provided that the marginal cost of liquidity is set efficiently, i.e. the bank charges no ”liquidity premium”. An important feature of these results is that they hinge on the long-term, repeated relationship between the borrower and his/her ”Hausbank”.

A Appendix

We define the regulation as the positive increment $d\pi_t$ to let π_t stay at b.[6] That is:

$$\tilde{\pi}_t \equiv \pi_t L_t, \quad \text{for } \tilde{\pi}_t \in [b, \infty), \tag{34}$$

or in term of the regulated process $\tilde{\pi}_t$, we get:

$$U_t = \tilde{\pi}_t - \pi_t \equiv (L_t - 1)\pi_t, \tag{35}$$

where:

- $i)$ π_t is a geometric Brownian motion, with stochastic differential as in (1);
- $ii)$ L_t is an increasing and continuous process, with $L_0 = 1$if $\pi_0 \geq b$, and $L_0 = b/\pi_0$ if $\pi_0 < b$, so that $\tilde{\pi}_0 = b$;
- $iii)$ L_t increases only when $\tilde{\pi}_t = b$.

In particular, although the process L_t may have a jump at $t = 0$ it is continuous and maintains π_t above the barrier b using the minimum amount of control, in that control takes places only when π_t would cross b from above with probability one in the absence of regulation. Applying Ito's lemma to (34), we get:

$$d\,\tilde{\pi}_t = \sigma\,\tilde{\pi}_t dW_t + d\tilde{L}_t, \qquad \tilde{\pi}_0 \in [b, \infty)$$

where $d\tilde{L}_t = \tilde{\pi}_t \frac{dL_t}{L_t} \equiv \pi_t dL_t$ is the infinitesimally small level of liquidity injection from the bank to the firm. By (34), if $\tilde{\pi}_t = b$, we get $d\,\tilde{\pi}_t = 0$ and the rate of variation of L_t is equal to that of π_t to keep $\tilde{\pi}_t$ constant. Therefore, referring to $d\tilde{L}_t$, the cost of liquidity can be expressed as:

$$dC_t = cd\tilde{L}_t \equiv c\pi_t dL_t \tag{36}$$

Making use of (35) and (36) we are able to rewrite (13) as:

$$V^c(\,\tilde{\pi}_0; b) = E_0 \left\{ \int_0^\infty e^{-rt}[(\,\tilde{\pi}_t - b)dt - cd\tilde{L}_t] \mid \tilde{\pi}_0 \in [b, \infty) \right\} \tag{37}$$

[6] We use the theory of regulated stochastic process (Harrison-Taksar, 1983, Harrison, 1985). Applications of this metodology to economic problems can be found in Bentolila-Bertola (1990), Moretto-Rossini (1999) and Moretto-Valbonesi (2000).

Since $\tilde{\pi}_t$ is a Markov process in levels (Harrison, 1985, proposition 7, pp.80-81), we know that the above conditional expectation is in fact a function solely of the starting state.[7] Keeping active the dependence of V^c on $\tilde{\pi}_t$, and assuming that it is twice continuously differentiable, by Ito's lemma we get:

$$\begin{aligned} dV^c &= V^{c\prime} d\,\tilde{\pi}_t + \frac{1}{2} V^{c\prime\prime} (d\,\tilde{\pi}_t)^2 \\ &= V^{c\prime}(L_t d\pi_t + \pi_t dL_t) + \frac{1}{2} V^{c\prime\prime} L_t^2 (d\pi_t)^2 \\ &= V^{c\prime}(\sigma\,\tilde{\pi}_t dW_t + d\tilde{L}_t) + \frac{1}{2} V^{c\prime\prime} \tilde{\pi}_t^2 \sigma^2 dt \\ &= \frac{1}{2} V^{c\prime\prime} \tilde{\pi}_t^2 \sigma^2 dt + V^{c\prime} \sigma\,\tilde{\pi}_t dW_t + V^{c\prime} d\tilde{L}_t \end{aligned} \tag{38}$$

where we have used the property that for a finite variation process as L_t, $(dL_t)^2 = 0$. Yet, as $dL_t = 0$ except when $\tilde{\pi}_t = b$ we are able to rewrite (38) as:

$$dV^c(\tilde{\pi}_t; b) = \frac{1}{2}\sigma^2 \tilde{\pi}_t^2 V^{c\prime\prime}(\tilde{\pi}_t; b) dt + \sigma \tilde{\pi}_t V^{c\prime}(\tilde{\pi}_t; b)\, dW_t + V^{c\prime}(b; b) d\tilde{L}_t \tag{39}$$

Equation (39) is a stochastic differential equation in V^c. Now integrating by part the process $V^c e^{-rt}$ we obtain (Harrison, 1985, pag.73):

$$\begin{aligned} e^{-rt} V^c(\tilde{\pi}_t; b) &= V^c(\tilde{\pi}_0; b) + \\ &+ \int_0^t e^{-rs} [\frac{1}{2}\sigma^2 \tilde{\pi}_s^2 V^{c\prime\prime}(\tilde{\pi}_s; b) - rV^c(\tilde{\pi}_s; b)] ds \\ &+ \sigma \int_0^t e^{-rs} \tilde{\pi}_s V^{c\prime}(\tilde{\pi}_s; b)\, dW_s + V^{c\prime}(b; b) \int_0^t e^{-rs} d\tilde{L}_s \end{aligned} \tag{40}$$

Taking the expected value of (40) and letting $t \to \infty$, if the following conditions hold:

[7] For $\pi_0 = \pi < b$ optimal control would require that L has a jump at zero so as to ensure $\tilde{\pi}_0 = b$. In this case the integral on the right of (37) is defined to include the control cost $c\tilde{L}_0$ incurred at $t = 0$, that is (see Harrison 1985, p.102-103):

$$\int_0^\infty e^{-rt} d\tilde{L}_t \equiv \tilde{L}_0 + \int_{(0,\infty)} e^{-rt} d\tilde{L}_t$$

(a) $\lim_{u\to\infty} \Pr[T(u) > T(b) \mid \tilde{\pi}_0 \in [b,\infty)] = 0$ per $b \leq \tilde{\pi}_t < u < \infty$, where $T(u) = \inf(t \geq 0 \mid \tilde{\pi}_t = u)$ and $T(b) = \inf(t \geq 0 \mid \tilde{\pi}_t = b)$;

(b) $V^c(\tilde{\pi}_0; b)$ is bounded in $[b,\infty)$;

(c) $e^{-rt}\tilde{\pi}_t V^{c\prime}(\tilde{\pi}_t; b)$ is bounded in $[b,\infty)$;

(d) $V^{c\prime}(b; b) = c$;

(e) $\frac{1}{2}\sigma^2\tilde{\pi}_t^2 V^{c\prime\prime}(\tilde{\pi}_t; b) - rV^c(\tilde{\pi}_t; b) = -(\tilde{\pi}_t - b)$,

we obtain the expression for $V^c(\tilde{\pi}_t; b)$ indicated in (37). Condition (a) says that the probability that the regulated process $\tilde{\pi}_t$ reaches infinity before reaching some other value within $[b,\infty)$ is nul. As $\tilde{\pi}_t$ is a geometric type of process this condition is , in general, always satisfied (Karlin-Taylor, 1981, pag. 228-230). Furthermore, if condition (a) holds and $V^c(\pi; b)$ is bounded in $[b,\infty)$,then also conditions (b) and (c) hold too. Finally, its is worth noting that for $\pi_0 \geq b$, $L_0 = 1$ and then $\tilde{\pi}_0 = \pi_0 = \pi$ so that $V^c(\tilde{\pi}_0; b) = V^c(\pi; b)$. On the other hand, if $\pi_0 < b$, we get $L_0 = b/\pi_0$, so that $\tilde{\pi}_0 = b$ and $V^c(\tilde{\pi}_0; b) = V^c(b; b)$.

Finally, by the conditions $(i) - (iii)$, the policy L_t say that profits are augmented in the minimum amounts consistent with the restriction $\tilde{\pi}_t \geq b$. Further, the same conditions $(i) - (iii)$ uniquely determine L_t with the form (Harrison,1985; proposition 3, pag. 19-20):[8]

$$L_t \equiv \begin{cases} \max(1, b/\pi_0) & \text{for } t = 0 \\ \sup\limits_{0\leq v\leq t} (b/\pi_v) & \text{for } t \geq 0 \end{cases},$$

Now, we may verify that:

$$\begin{aligned} B\tilde{\pi}_0{}^{\beta_2} &= E_0\left\{\int_0^\infty e^{-rt}[(\tilde{\pi}_t - \pi_t)dt - cd\tilde{L}_t] \mid \tilde{\pi}_0 \in [b,\infty)\right\} \\ &= E_0\left\{\int_0^\infty e^{-rt}[(L_t - 1)\pi_t dt - c\pi_t dL_t] \mid \tilde{\pi}_0 \in [b,\infty)\right\} \end{aligned} \tag{41}$$

[8] This is an applicantion of a well known result by Levy (1948) for which the process:

$$\ln\tilde{\pi}_t \equiv \ln\pi_t + \ln L_t \equiv \ln\pi_t + \sup_{0\leq v\leq t}(\ln b - \ln\pi_v)$$

has the same distribution of the "reflected Brownian process" $\mid \ln b - \ln\pi_t \mid$.

First, for each $T > 0$, the integration by parts gives:

$$\int_0^T e^{-rt}\pi_t dL_t = e^{-rT}\pi_T L_T - \pi_0 L_0 + r\int_0^T e^{-rt}\pi_t L_t dt - \int_0^T e^{-rt} L_t d\pi_t \quad (42)$$

Taking the expectation of both side and using the zero expectation property of the Brownian motion (Harrison, 1985, pag.62-63), then gives:

$$E_0\int_0^T e^{-rt}\pi_t dL_t = \pi_T L_T E_0[e^{-rT}] + rE_0\int_0^T e^{-rt}\pi_t L_t dt - \pi_0 L_0 \quad (43)$$

By the Strong Markov property[9] of $\tilde{\pi}_t$ it follows that $\pi_T L_T E_0[e^{-rT}] = \pi_T L_T \left(\frac{\pi}{\pi_T}\right)^{\beta_1}$ so that $\pi_T L_T \left(\frac{\pi}{\pi_T}\right)^{\beta_1} \to 0$ almost surely as $T \to \infty$. Substituting in (43) and rearranging we obtain:

$$E_0\int_0^\infty e^{-rt}\tilde{\pi}_t dt = \frac{\pi_0 L_0}{r} + \frac{1}{r}E_0\int_0^\infty e^{-rt}\pi_t dL_t \quad (44)$$

Therefore, rearranging we get that the expected discounted value of the firm's overdraft payments matches the total value of overdrafts:

$$\frac{1}{r}E_0\int_0^\infty e^{-rt}dC_t = cE_0\int_0^\infty e^{-rt}U_t dt \quad (45)$$

Finally, recalling that $\pi_0 L_0 = \pi$ if $\pi_0 \geq b$ and $\pi_0 L_0 = b$ if $\pi_0 < b$, equation (44) and (45) allow, by verification, to conclude that:

$$(cr - 1)\left[E_0\int_0^\infty e^{-rt}U_t dt\right] \equiv B\pi^{\beta_2} \text{ or } Bb^{\beta_2}$$

[9] The Strong Markov Property of regulated Brownian motion processes stresses the fact the stochastic stopping time T and the stocastic process $\tilde{\pi}_t$ are independent (Harrison, 1985, proposition 7, pp.80-81). That is, as L_t depends only on the primitive exogenous process π_t, the Markov property extends to the endogenous regulated process $\tilde{\pi}_t$.

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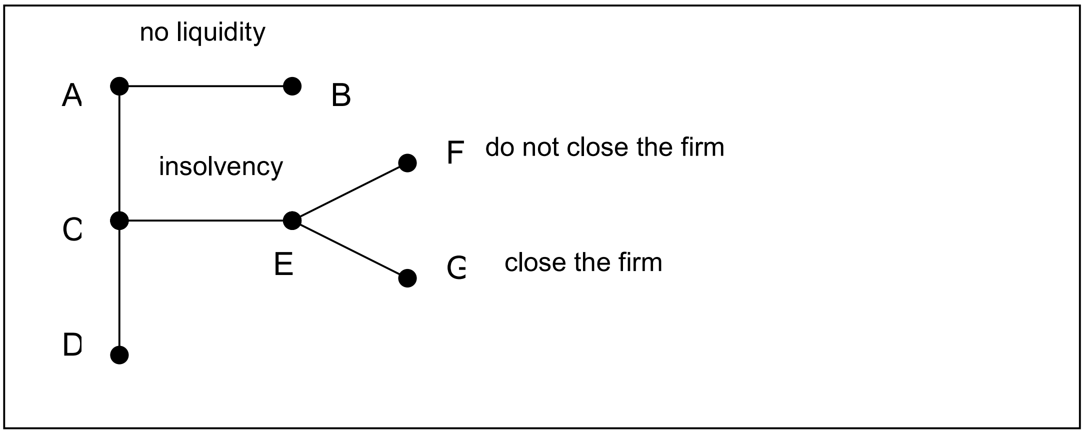
Figure 1:

Figure 1


no liquidity
A
B
insolvency
F
do not close the firm
C
E
G
close the firm
D