Time horizon and the discount rate

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Abstract

We discuss the selection of the socially efficient discount rate for public investment projects that entail costs and benefits in the far distant future. We show that the discount rate should be a decreasing function of time horizon under some specific restrictions on the distribution of uncertain growth and on preferences. We consider a logarithmic random walk for consumption. The benchmark result is that, in the absence of any risk of recession, the yield curve is decreasing if relative risk aversion is decreasing. Relaxing the assumption on the absence of recession requires more restrictions on preferences, as increasing relative prudence.

Keywords: Discounting, uncertain growth, prudence, long term.

JEL Classification: D81, D91, Q25, Q28.

Non technical Summary

Many public decision makers are reluctant to use discounting for very far distant cash flows because of the exponential effect of a constant discount rate. Various authors suggested to use a discount rate that is decreasing with time horizon in order to solve this problem. In this paper, we examine what the classical theory of discounting can do to justify the use of a decreasing discount rate. There is an intuitive argument for this. The uncertainty about the future provides a strong incentive to make more efforts for this uncertain future. This is illustrated for example by the observation that households increase their precautionary saving when their future incomes are more uncertain.
Observe now that the uncertainty about the future is increasing with the time horizon. Risks accumulate over time. Combining these two observations implies that more efforts should be done for periods more distant in the future. This is done by reducing the discount rate with respect to time horizon. Notice however that this precautionary effect is potentially counterbalanced by a wealth effect: future generations are expected to be wealthier, at least in expectation. Therefore, we should core less about them. This is done by increasing the discount rate with respect to time horizon. The paper provides some necessary and sufficient conditions on preferences that guarantee that the precautionary effect dominates the wealth effect. The simplest case arises when there is no risk of recession in the economy, neither in the short run, nor in the long run. In that case, it is socially efficient to reduce the discount rate for longer horizon if and only if relative risk aversion is decreasing with wealth. This hypothesis is sustained by the observation that wealthier people invest a larger share of their wealth in risky assets.
A simple calibration of the model using a real expected growth rate of GDP per capita of 2% per year and a standard deviation of 2.5% suggests the use of a real discount rate around 5% for the short run, going down to around 2% in the very long run.

1 Introduction

Under the pressure of environmentalists, public decision-makers have been asked to include the long term effect of their decision in the standard cost-benefit analysis. The carbon dioxide that one emits today will not be recycled for a couple of centuries, yielding long term costs like global warming. Some nuclear wastes like Plutonium have half-life in the tens of thousands years.

Following Weitzman (1998), there must be something wrong with discounting when the far distant future is at stake. Discounting these far distant costs and benefits of our current actions at the same rate than for the shorter terms is equivalent to forget these long term effects. For example, one should spend no more than twenty cents today to eliminate a one-million damage happening 200 years from now if one uses a discount rate of 8%. Notice that 8% is considered as an acceptable discount rate. Several countries like the USA, France and Germany recommend the use of a discount rate somewhere between 5% and 8% for their public investment policy.

A first question is about the theoretical foundations and the level of the discount rate. With a sure positive growth of the economy, we don't want to do too much for future generations which will enjoy a larger GDP per capita anyway. Under decreasing marginal utility of consumption, one more unit of consumption in the future is less valuable than one more unit of consumption today. This is the standard argument for using a positive discount rate. The smaller the rate at which marginal utility decreases, the larger our willingness to transfer consumption in the future, the smaller the optimal discount rate. Of course, the growth of the economy is affected by random shocks which should be taken into account in the selection of the discount rate. The effect of the risk affecting the willingness to improve the future is well-known since Leland (1968), Drèze and Modigliani (1972) and Kimball (1990): if people are prudent, the existence of an uncertain growth rate of the economy should induce us to reduce the discount rate.

The selection of the discount rate for long term investments is both crucial and controversial, because of this exponential nature of discounting. It is controversial for the same reason, since even the smallest positive discount rate leads ultimately to a disenfranchising of future generations. So, the question is why wouldn't we consider a smaller discount rate for fast distant cash-flows? In fact, what is wrong is not the concept of discounting, but rather the idea that the discount rate should be the same for all time horizons.

The objective of this paper is to examine the relationship that exists between the socially efficient discount rate and time horizon. In other words, we look at the design of the term structure of interest rate. The theoretical foundations of a decreasing yield curve in the long term are not obvious however. They are two conflicting effects of the longer horizon on our attitude towards the intertemporal substitution of consumption . The first effect is the larger expected GDP per capita enjoyed by the more distant generation. It should induce us to reduce our efforts for them. The second effect is the accumulation of risks faced by that generation, which should induce us to select a more conservative action for the future, under prudence. Usually, the first effect is larger than the second effect. This means that the minimal gross payoff on an investment of one dollar today is larger when the benefit is expected to mature later. But this does not tell us whether the minimal rate of return *per period* will be smaller or larger for longer maturities. For it to be decreasing, we need that the effect of the accumulation of uncertainty over time be strong enough to induce more prudence for the long term. In other words, there would be something bad to the accumulation of risks. If this is true, then the socially efficient discount rate would be decreasing with maturity.

We see that the difficulty is to determine the representative agent's attitude towards adding more risk to an initially risky situation. To illustrate, suppose that the growth rate of future incomes is subject to an i.i.d. shock at each period. The agent has the opportunity to save money today to increase her consumption t periods from now. Suppose that the yearly return on saving is independent of maturity. Would she save more if savings is devoted for increasing consumption two periods from now than for increasing consumption next period? If the answer to this question is positive, this would provide a theoretical foundation for a decreasing deiscount rate.

The attitude towards the accumulation of risks is not a new question. For example, Pratt and Zeckhauser (1987) examined how the attitude toward one risk is affected by the presence of a second independent risk. Together with Kimball (1993) and Gollier and Pratt (1997), they argue that the impact of the sum of independent risks on welfare is larger than the sum of the welfare effects of each of these risk, i.e. there is something bad in the accumulation of risks.[1] Our problem in this paper is to examine the effect of accumulated

[1] Ross (1998) argues that independent risks could be complements rather than substi-

future risk not on welfare, but on savings and on the equilibrium risk free rate. Our intuition is again that the accumulation of risks over time is bad, inducing people to save more. As in Pratt and Zeckhauser (1987), we will show that this hypothesis is correct only if some restrictions are made on higher derivatives of the utility function.

This work is also related to a recent paper by Martin Weitzman (1998)[2] who also proves that the discount rate should be decreasing with time horizon. Weitzman's conclusion is obtained in a much different framework, with risk neutral agents together with a simple early revelation of future uncertain productivity of capital. His conclusion relies on the fact that the NPV is a convex function of the discount rate, whereas our result relies on more complex assumptions on preferences. The two approaches are complement.

Using discount rates that are decreasing with time horizon seems to be in contradiction with the observation that the yield curve most often has a positive slope. However a possible explanation of this phenomenon is the presence of friction on credit markets. For example, in the presence of a liquidity constraint, the return of long term credit contracts contains an illiquidity premium. More generally, we are suspicious about using financial market prices for public cost-benefit analyses because of our difficulty to fit observed market prices with any realistic economic modelling. The equity premium puzzle and the risk free rate puzzle are still puzzles (see Kocherlakota (1996)).

2 Description of the economy

There is a fixed set of identical infinitely living consumers.[3] Let z_t denote the level of consumption of each agent at date t. It can be seen as the GDP per capita. The grow rate of z is uncertain. We assume that it follows a Markov process characterized by

$$\widetilde{z}_t = f(z_{t-1}, \widetilde{x}_t), \tag{1}$$

tutes, contrary to what is suggested by this literature.

[2]See also Gollier (1999) for a discussion and extensions of Weitzman (1998).

[3]The only reason why we consider infinitely living agents is to escape any ethical considerations. Ethical considerations in this model are linked to the selection of the discount factor β, which is left arbitrary in this paper. It has an effect on the level of the yield curve, not on its slope.

where $\widetilde{x}_t$ is a random shock. These shocks are exogenous.[4] In this paper, we consider shocks that are independent over time. Agents are assumed to be expected utility maximizers with a time separable utility function u which is increasing and concave. Parameter β denotes the rate of pure preference for the present.

Let us consider an investment that costs one unit of consumption per capita at date 0, and whose yields a unique sure cash flows C_n at date n. Its gross rate of return per period equals $y_n = \sqrt[n]{C_n}$. The socially efficient discount rate corresponding to maturity n is the critical rate of return y_n that let the expected discounted utility of the representative agent unchanged. It is given by

$$(y_n(z))^n \beta^n E\left[u'\left(\widetilde{z}_n\right) \mid \widetilde{z}_0 = z\right] = u'(z), \tag{2}$$

where $z = z_0$ is the GDP per capita at date 0. The right-hand side of this expression is the marginal utility cost of reducing consumption by one unit at time $t = 0$. The left-hand side is the increase in the discounted expected utility generated by consuming $C_n = y_n^n$ at date n. If the rate of return of the investment is larger than the solution y_n of equation (2), it would raise the lifetime welfare of the representative agent. This equation is rewritten as

$$(y_n(z))^n = \frac{u'(z)}{\beta^n E\left[u'\left(\widetilde{z}_n\right) \mid \widetilde{z}_0 = z\right]}. \tag{3}$$

Notice that y_n would be the equilibrium interest rate for maturity n in competitive and frictionless markets for credit. This is why we will indifferently use the terms "discount rate" and "interest rate" in the remaining of this paper.

The yield curve is a plot of the term structure, that is, a plot of y_n against n. We are interested in determining the properties of this curve. There is a wide literature on the equilibrium form of the yield curve. The most cited references on this topic are Vasicek (1977) and Cox, Ingersoll, and Ross (1985a,b). The form of the yield curve is a complex function of the attitude towards risk and time, and of the statistical relationships that may exist in

[4] We do not examine the origin of the long-term shocks to growth, which are mostly due to innovations in our real world. In fact, our model can be made compatible to any model of growth as long as the representative agent is an expected utility maximizer.

the temporal growth rates of the economy. The properties of the yield curve are not yet completely understood, even in the case of i.i.d. shocks.

We will mostly be interested in the case of the classical case of multiplicative shocks to the economy. This corresponds to the case where $f(z,x) = zx$. This means that the growth rate of the economy is a stationary process. We also consider the case of additive shocks to consumption with transition function $f(z,x) = z+x$. In fact, it is easy to go from one model to the other one by defining the following items: function U is such that $U'(Z) = u'(\exp Z)$, $Z = \ln z$ and $X_t = \ln x_t$. The pricing formula that we obtain in the multiplicative case

$$(y_n(z))^n = \frac{u'(z)}{\beta^n E u' \left[z \prod_{i=1}^n \widetilde{x}_i \right]} \tag{4}$$

can be rewritten as

$$(y_n(z))^n = \frac{U'(Z)}{\beta^n E U' \left[Z + \sum_{i=1}^n \widetilde{X}_i \right]}, \tag{5}$$

which corresponds to the case of additive shocks. Any property in the multiplicative model has is equivalence in the additive model. Notice that the assumption of decreasing relative risk aversion in the multiplicative model corresponds to the assumption of decreasing absolute risk aversion in the additive model. Indeed, we have that

$$\frac{-U''(Z)}{U'(Z)} = \frac{-zu''(z)}{u'(z)}. \tag{6}$$

3 The yield curve under certainty

When there is no uncertainty on the future changes in GDP per capita, the equilibrium risk free rate reflects the combination of impatience and the aversion to fluctuations of consumption over time. In the absence of any growth over time ($z_t = z_0$ for all t), the interest rate would just counterbalance impatience, and $y_n = \beta^{-1}$. If the growth of GDP is positive, agents will accept to invest to improve further the future only if the return of the investment is sufficiently large to compensate for the more unequal distribition of consumption over time/generations. The larger the growth rate, the smaller the equilibrium interest rate.

If the growth is small, one can use a first-order approximation of $u'(z_1)$ around z_0 to get the following approximation:

$$y_1(z_0) \simeq \beta^{-1} + \frac{z_1 - z_0}{z_0} R(z_0), \tag{7}$$

where $R(z) = -zu''(z)/u'(z)$ is the relative degree of aversion to fluctuations of consumption over time. The socially efficient discount rate is approximately equal to the sum of the rate of pure preference for the presence and of the product of the growth rate of the economy by R.

We now examine two related questions. First, how does a change in wealth affect the short term interest rate? Second, under which condition is the interest rate decreasing with respect to maturity? We start by answering to these questions in the multiplicative case, where $(z_1 - z_0)/z_0$ is independent of z_0. Approximation (7) suggests that the short-term interest rate y_1 is decreasing in the current GDP per capita z_t if R is decreasing. This is confirmed by the following condition that holds for the multiplicative model:

$$z_0 y_1'(z_0) = y_1(z_0)\left[R(z_0 x) - R(z_0)\right]. \tag{8}$$

We conclude that, with positive growth, the short term interest rate is decreasing in the GDP per capita if the relative aversion to consumption fluctuation is decreasing (DRA). The opposite result holds in case of a recession. DRA means that people care less about the unequal distribution of consumption over time when they are wealthier.

This result can be used to solve the problem of the term structure of interest rates under certainty. The following condition holds under certainty:

$$y_n(z_0)^n = \prod_{\tau=0}^{n-1} y_1(z_\tau). \tag{9}$$

One can improve the far distant future by a sequence of short term investments whose benefits are reinvested at each period. This roll-over s-tartegy implies that our attitude towards long term transfers must be link to our future attitude towards short term transfers. Equation (9), which is a classical arbitrage condition in finance, makes this link explicit. If growth is positive and if relative aversion is decreasing, then we know that $y_1(z_0) \geq y_1(z_1) \geq ... \geq y_1(z_{n-1})$: the short-term interest rate is decreasing in time. This is because we will be less concerned by consumption smoothing in the future. It implies that the right-hand side of equation (9) is smaller than $(y_1(z_0))^n$. It implies that $y_n(z_0)$ is smaller than $y_1(z_0)$. More generally, we can show that if the growth rate of the economy is certain, constant over time and positive, DRA implies that the slope of the yield curve is decreasing (y_n decreasing in n). Similarly, when the growth rate is negative, we know from equation (8) that DRA implies that the short term interest rate is increasing with GDP under DRA. It yields again short term interest rates that are decreasing over time, which implies in turn a decreasing yield curve is decreasing. The opposite results hold under increasing relative risk aversion.

Proposition 1 *Suppose that the growth rate $(z_{t+1}-z_t)/z_t$ is certain and constant over time. Suppose also that relative risk aversion $R(z) = -zu''(z)/u'(z)$ is uniformly decreasing (resp. increasing).Then the yield curve is decreasing (resp. increasing).*

A simple extension of this result is obtained when considering an economy whose economy is increasing at a decreasing rate: $x_1 \geq x_2 \geq ....$ Because the interest rate is an increasing function of the growth rate of the economy, the decreasing growth rate adds to the fact that the future short term interest rates will be smaller than today. This reinforces the negative slope of the yield curve under DRA. Another immediate corollary of this result is that, in the additive model $z_{t+1} = z_t + x$, $x > 0$, the short term interest rate is decreasing in the GDP per capita if the absolute aversion is decreasing (DARA). Under the same conditions, the yield curve is decreasing.

4 The yield curve under uncertainty

4.1 The precautionary effect of uncertain growth

Again, let us focus on the standard multiplicative model. If the growth rate $\widetilde{x}_t = \widetilde{z}_t/z_{t-1}$ is random, a third determinant to the short term interest rate must be added to impatience and consumption smoothing. This is really this additional effect which will be difficult to control when we will turn to the problem of the term structure under uncertainty. In short, the presence of uncertainty on the growth of incomes induces the representative agent to save more for the precautionary saving motive. The equilibrium interest rate will be smaller than the one prevailing in the risk free economy.

One way to quantify the effect of the uncertain growth on the interest rate is to define the "precautionary equivalent"[5] growth rate, the certain growth rate that yields the same interest rate. The precautionary equivalent growth rate $\widehat{x}_1(z)$ is thus characterized by the following condition:

$$Eu'(z\widetilde{x}_1) = u'(z\widehat{x}_1). \tag{10}$$

Obviously $\widehat{x}_t$ is smaller than $E\widetilde{x}_t$ if u' is convex, a condition that we call "prudence". Under prudence, the presence of uncertainty on the growth rate has the same effect on the interest rate as a sure reduction of this rate, i.e., it reduce the equilibrium interest rate. As suggested by Kimball (1990), the precautionary equivalent growth rate can be approximated by

$$\widehat{x}_1(z) \cong E\widetilde{x}_1 - \frac{1}{2}\sigma^2_{\widetilde{x}_1}P(z), \tag{11}$$

where $P(z) = -zu'''(z)/u''(z)$ is the degree of relative prudence. Condition (11) indicates that the effect of the uncertainty on growth on the interest rate is the same as a sure reduction of the growth rate by the product of half its variance by relative prudence. The more prudent people, the smaller the precautionary equivalent growth rate, and the smaller the equilibrium interest rate. If the precautionary effect is larger than the wealth effect combined with the effect of impatience, the equilibrium interest rate will be negative.

To sum up, two different characteristics of the utility function affect the level of the equilibrium risk free rate. A first question is to determine the effect of uncertainty. Which certain growth rate should we consider as e-quivalent to the uncertain growth rate we face? We showed that the degree of relative prudence determines the impact of the riskiness of growth on the precautionary equivalent growth rate. The more prudent we are, the smaller should the equivalent certain growth rate be. Prudence is measured by the degree of convexity of u'. The second question is to determine by how much we should substitute consumption today by consumption tomorrow in the

[5]This should not be confused with the certainty equivalent growth rate, the certain growth rate that generates the same expected utility of the representative agent in the future.

face of this precautionary equivalent growth rate. This depends upon the degree of fluctuation aversion. This is measured by the degree of concavity of u. The more resistant to intertemporal substitution we are, the larger the discount rate, for a given certainty equivalent growth rate. This can be seen more explicitly by combining conditions (7) and (11). It yields

$$y_1(z) \cong \frac{1}{\beta} + \left[(E\widetilde{x}_1 - 1) - 0.5Var(\widetilde{x}_1)\frac{-zu'''(z)}{u''(z)} \right] \frac{-zu''(z)}{u'(z)}. \tag{12}$$

Hansen and Singleton (1983) obtained a similar formula with power utility functions and lognormal growth under continuous time. An advantage of our formula is to exhibit the three determinants of the equilibrium risk free rate, with the three terms in the right-hand side of the above equality being respectively the pure preference for the present, the consumption smoothing effect and the precautionary effect.

As is standard, we can calibrate this model by using CRRA utility functions for which $u'(z) = z^{-\gamma}$, $-zu''(z)/u'(z) = \gamma$ and $-zu'''(z)/u''(z) = \gamma + 1$. We use data for the U.S. growth of GDP per capita over the period 1963-1992, for which we have $E\widetilde{x} - 1 = 1.86\%$ and $Var(\widetilde{x}) = (2.41\%)^2$. With $\beta = 1$, condition (12) is rewritten as

$$y_1 - 1 = 0.0183\gamma - 0.00029\gamma^2. \tag{13}$$

Thus, the equilibrium risk free rate is increasing in the degree of risk aversion as long as γ is less than 31. Then, the risk free rate is decreasing in γ to eventually become negative when relative risk aversion is larger than 63. For $\gamma = 1$, the Hansen-Singleton formula yields an equilibrium risk free rate approximately equal to 1.80%, whereas, for $\gamma = 4$, it yields a risk free rate around 6.86%.[6] This should be compared to the historical risk free rate that fluctuated around an average value of 1% over the last century. Because interval $[1, 4]$ is widely accepted for realistic values of R[7], we conclude following Weil (1989) that the theoretical model overpredicts the equilibrium risk free rate. The growth of consumption has been so large and its volatility has been so low than the model is unable to explain why households saved so much during this period. A much larger risk free rate than the actual one would have been necessary to explain the actual consumption growth. Notice

[6]The exact values are 1.80% and 7.04% respectively for $\gamma = 1$ and $\gamma = 4$.

[7]See for example Drèze (1981).

that even for a large $\gamma = 4$, the precautionary effect reduces the equilibrium interest rate by less than half a percent. This is the risk free rate puzzle.

4.2 The yield curve when there is no risk of recession

We now turn to the objective of this paper, which is to examine the slope of the yield curve under uncertainty. As under certainty, this question is linked to the sensitivity of the short term interest rate to changes in GDP per capita. Let $\rho(z)$ denote the short term interest rate that will prevail at date $t = 1$ if the GDP per capita is z. We have

$$\rho(z) = \frac{u'(z)}{\beta E u'(z\widetilde{x}_2)}. \tag{14}$$

Parallel to equation (9), the log of the long term interest rate $y_2(z_0)$ is an harmonic mean of the log of future short term rates $y_1(z_0)$ and $\rho(z_0\widetilde{x}_1)$, as indicated by the following condition:

$$(y_2(z_0))^{-2} = (y_1(z_0))^{-1} E\left[\frac{u'(z_0\widetilde{x}_1)}{Eu'(z_0\widetilde{x}_1)}(\rho(z_0\widetilde{x}_1))^{-1}\right]. \tag{15}$$

This condition extends equation (9) to the case of uncertainty. Our attitude towards a transfer of consumption in the long term depends upon our future attitude towards transfers in the short term. From this equation, we see that the yield curve is decreasing if and only if

$$E\left[\frac{u'(z_0\widetilde{x}_1)}{Eu'(z_0\widetilde{x}_1)}(\rho(z_0\widetilde{x}_1))^{-1}\right] \geq (y_1(z_0))^{-1}. \tag{16}$$

There are a lot of reasons for why this inequality would hold. For example, if one anticipates a recession in period 2 or a large increase in the risk on growth during that period, ρ will be smaller than y_1,thereby implying condition (16). We thus need to make a stationarity assumption. From now on, we make the following assumption:

$$\rho(z_0) = y_1(z_0) \tag{17}$$

This means that the future interest rate would be the same as today if it would happened that the GDP per capita would not change. Technically, this condition could be rewritten as $Eu'(z_0\widetilde{x}_2) = Eu'(z_0\widetilde{x}_1)$. This is the case for example when $\widetilde{x}_1$ and $\widetilde{x}_2$ are i.i.d., an extreme form of stationarity.

Suppose that there is no risk of recession in the short run, i.e., that $\widetilde{x}_1$ is larger than unity almost surely. Then, we would be done if the future short term rate $\rho(z_0 x_1)$ be decreasing in x_1. Indeed, it would imply that $\rho(z_0\widetilde{x}_1)$ would be smaller than $\rho(z_0)$ almost surely. Because we assumed that $\rho(z_0) = y_1(z_0)$, this would yield condition (16).

Under which condition can we guarantee that $\rho(z)$ is decreasing? Differentiating ρ defined by equation (14), we obtain that

$$\begin{aligned} z\rho'(z) &= \rho(z)\left[\frac{-Ez\widetilde{x}_2 u''(z\widetilde{x}_2)}{Eu'(z\widetilde{x}_2)} - \frac{-zu''(z)}{u'(z)}\right] \\ &= \rho(z)\left[E\left[\frac{u'(z\widetilde{x}_2)}{Eu'(z\widetilde{x}_2)}R(z\widetilde{x}_2)\right] - R(z)\right], \end{aligned} \tag{18}$$

a condition which extends equation (8) to the case of uncertainty. It implies that

$$\rho'(z) \leq 0 \quad \Longleftrightarrow \quad E\left[\frac{u'(z\widetilde{x}_2)}{Eu'(z\widetilde{x}_2)}R(z\widetilde{x}_2)\right] \leq R(z). \tag{19}$$

The short term discount rate is decreasing in GDP if the risk-neutral expectation of the future relative aversion $R(zx_2)$ is smaller than the relative aversion evaluated at the current GDP z. When this is the case, an increase in z reduces the gap between the current and the future (expected) marginal utility of consumption, thereby reducing the socially efficient discount rate.

Let us assume that $\widetilde{x}_2$ is also larger than unity almost surely, i.e., that there is no risk of recession in the long term. If we combine this assumption with DRA, we have that $R(z\widetilde{x}_2)$ is smaller than $R(z)$ almost surely, for all z. From condition (18), it implies that $\rho(z)$ would be decreasing. Thus, if $\widetilde{x}_1$ and $\widetilde{x}_2$ are larger than unity almost surely, DRA implies that the yield curve is decreasing. We can also prove that the same property holds if $\widetilde{x}_1$ and $\widetilde{x}_2$ are smaller than unity almost surely. In that case, ρ would be increasing, implying that $\rho(z_0\widetilde{x}_1)$ is smaller than $\rho(z_0) = y_1(z_0)$ with probability one. This concludes the proof of the following result.[8]

[8]Gollier (1998) proved this result by observing that DRRA is equivalent to the log supermodularity of $u'(z_0x_1x_2)$. If x_1 and x_2 are larger than unity, this implies that $u'(z_0x_1x_2)u'(z_0)$ is larger than $u'(z_0x_1)u'(z_0x_2)$. Taking the expectation with respect to $\widetilde{x}_1$ and $\widetilde{x}_2$ immediately yields that $y_2(z_0)$ is smaller than $y_1(z_0)$.

Proposition 2 *Suppose that the growth rate of GDP per capita never changes sign. Then, the yield curve is decreasing (resp. increasing) if relative risk aversion is decreasing (resp. increasing).*

Observe also that when relative risk aversion is constant, ρ is independent of z. Under the stationarity assumption (17), this implies that the yield curve is flat. This result holds independent of any restrictions to the distributions of $\widetilde{x}_1$ and $\widetilde{x}_2$, except the stationarity condition. However, one may question the assumption of constant relative risk aversion. For example, Ogaki and Zhang (1999) tested whether relative risk aversion is decreasing or increasing from various consumption data in developing countries. They obtained strong evidences that relative risk aversion is decreasing. Another argument in favor of DRA is based on the observation that the share of wealth invested in risky assets is increasing with wealth in most developed countries.[9] This is possible only under DRA. Thus, we have here an argument for using smaller discount rates for longer time horizons. Notice that if growth would be an additive phenomenon, i.e., if $f(z,x) = z + x$, then condition DRA in Proposition2 would be replaced by DARA which is a well-established feature of preferences.

The problem with Proposition 2 is that DRA is sufficient for a decreasing yield curve only if the growth is either positive with probability 1 or negative with probability 1. If $\widetilde{x}_1$ or $\widetilde{x}_2$ has a support containing 1, DRA is not sufficient anymore for a decreasing yield curve. Let us illustrate this point by the following example, using One-Switch utility functions introduced by Bell (1988). Take $u'(z) = a + z^{-b}$ with $a > 0$ and $b > 0$. It yields $-zu''(z)/u'(z) = b\left[az^{-b} + 1\right]^{-1}$, which is decreasing in z. In addition, take $a = b = 1$ together with a pair $(\widetilde{x}_1, \widetilde{x}_2)$ of i.i.d. variables that are distributed as $\widetilde{x}$, with $\widetilde{x} - 1 \sim (-50\%, 1/3; +100\%, 2/3)$. In such a situation, straightforward computations generate $y_1 = y_2 = 0$: the yield curve is flat in spite of DRA! Thus, extending the analysis to economies with a risk of a recession requires restricting the set of DRA utility functions to guarantee that the interest rate be decreasing with maturity. In the next two sections, we look for such restrictions.

[9] See Kessler and Wolf (1991) and Guiso et al. (1996) for the case of U.S. portfolios.

4.3 The yield curve with a risk of recession in the long run

In this section, we maintain the assumption that there is no risk of recession in the short run: $\widetilde{x}_1$ is larger than unity with probability 1. What we do here is to relax the condition that $\widetilde{x}_2$ is larger than unity almost surely. Notice that this condition implies that $\beta\rho(z)$ is larger than unity, or $Eu'(z\widetilde{x}_2) \leq u'(z)$, for all z. In this section, we replace condition $\widetilde{x}_2 \geq_{a.s.} 1$ by the condition that $\beta\rho(z)$ is larger than 1 for all z in the support of $z_0\widetilde{x}_1$. This means that the consumption smoothing effect of growth on the interest rate dominates the precautionary effect, i.e. that the equilibrium interest rate is larger than the rate of pure preference for the present. This condition is obviously much weaker than the absence of risk of recession. As shown in section 3, this is a reasonably weak assumption. Using Hansen-Singleton approximation formula (13) for CRRA functions, this assumption holds whenever relative risk aversion is less than 63!

Because we continue to assume that $\widetilde{x}_1 \geq_{a.s.} 1$, it is still true that the yield curve will be decreasing if the future short term interest rate is decreasing with GDP, i.e., if

$$\frac{-Ez\widetilde{x}_2 u''(z\widetilde{x}_2)}{Eu'(z\widetilde{x}_2)} \leq \frac{-zu''(z)}{u'(z)}, \tag{20}$$

for all z in the support of $z_0\widetilde{x}_1$. We want that condition (20) holds whenever $Eu'(z\widetilde{x}_2) \leq u'(z)$, or, to sum up, that

$$Eu'(z\widetilde{x}_2) \leq u'(z) \quad \Longrightarrow \quad \frac{-Ez\widetilde{x}_2 u''(z\widetilde{x}_2)}{Eu'(z\widetilde{x}_2)} \leq \frac{-zu''(z)}{u'(z)}. \tag{21}$$

We can rewrite the above condition in its additive form, with $U'(Z) = u'(\exp Z)$, $Z = \ln z_0$ and $X = \ln x_2$:

$$EU'(Z+\widetilde{X}) \leq U'(Z) \Longrightarrow \quad \frac{-EU''(Z+\widetilde{X})}{EU'(Z+\widetilde{X})} \leq \frac{-U''(Z)}{U'(Z)}. \tag{22a}$$

In words, this condition means that *any expected-marginal-utility decreasing risk reduces the degree of absolute risk aversion towards any other independent small risk*. This links this question to the literature on the interaction

of independent risks whose main papers are Pratt and Zeckhauser (1987), Kimball (1993) and Gollier and Pratt (1996). The closest work to this problem is the one by Kimball (1993) who examines the condition on preferences that guarantees that any expected-marginal-utility increasing risk increases the demand for any independent risky asset. Kimball (1993) showed that this condition holds if and only if both absolute risk aversion and absolute prudence are decreasing. We obtain a similar result here. Absolute measures of concavity are replaced by the corresponding relative measures, because we focus on multiplicative risks.

Proposition 3 *Suppose that there is no risk of recession in the short run and that the future short term interest rate is larger than the rate of pure preference for the present, almost surely. Then, the yield curve is decreasing if and only if relative risk aversion $-zu''(z)/u'(z)$ is decreasing and relative prudence $-zu'''(z)/u''(z)$ is increasing.*

Proof: By definition, we have

$$\frac{-zu''(z)}{u'(z)} = \frac{-U''(Z)}{U'(Z)} \quad \text{and} \quad \frac{-zu'''(z)}{u''(z)} = \frac{-U'''(Z)}{U''(Z)} + 1. \tag{23}$$

The decreasing relative risk aversion of u is equivalent to the decreasing absolute risk aversion of U. The increasing relative prudence of u is equivalent to the increasing absolute prudence of U. Thus, we have to prove that property (22a) holds for all Z and $\widetilde{X}$ if and only if U exhibits decreasing absolute risk aversion and increasing relative prudence. Define the precautionary equivalent $\psi(Z)$ of risk $\widetilde{X}$ at wealth Z by $EU'(Z+\widetilde{X}) = U'(Z+\psi(Z))$. Because we assume that $EU'(Z+\widetilde{X})$ is smaller than $U'(Z)$, $\psi(Z)$ is positive. As shown by Kimball (1990), ψ is decreasing in Z if the absolute prudence of U is increasing. We then easily obtain that

$$\frac{-EU''(Z+\widetilde{X})}{EU'(Z+\widetilde{X})} = (1+\psi'(Z))R(Z+\psi(Z)) \leq R(Z+\psi(Z)) \leq R(Z). \tag{24}$$

The first inequality comes from increasing prudence, whereas the second inequality comes from decreasing risk aversion. This proves the sufficiency part of the Proposition. The necessity is proven in a similar way by contradiction. The necessity of DRA is obtained by taking degenerated random variables $\widetilde{x}_1$ and $\widetilde{x}_2$. Proposition 1 generates the result. The necessity of increasing prudence is derived from taking a random variable $\widetilde{X}$ such that $\psi(Z) = 0$. Under increasing prudence, the inequalities in (24) would be reversed, yielding a contradiction.■

Under the same conditions on $\widetilde{x}_1$ and $\widetilde{x}_2$, the yield curve would be increasing under increasing relative risk aversion and decreasing relative prudence. The proof of this result is left to the reader. The bottom line is that relaxing the assumption of the absence of any risk of recession in the long run requires more restrictions on the utility function. Whereas we just needed conditions on the third derivative of u (DRA) in the previous section, we are here forced to go up to the fourth derivative, with increasing relative prudence.

Up to our knowledge, it is the first time that increasing absolute prudence arises as a useful condition in the economics of uncertainty. The only available reference is Kimball (1990,1993) who justifies the assumption of decreasing absolute prudence. But decreasing absolute prudence is compatible with either increasing or decreasing relative prudence. The only other direct application of increasing relative prudence is derived from conditions (10) and (11). Remember that the precautionary equivalent growth rate of a random growth is the certain growth rate that would generate exactly the same equilibrium interest rate. Under increasing relative prudence, the precautionary equivalent growth rate $\widehat{x}_1(z)$ of any given random growth rate $\widetilde{x}_1$ is a decreasing function of the current GDP per capita. Whether actual preferences exhibit such a property remains an open question.

4.4 The yield curve when there is a risk of a recession at each period

When there is a chance of a recession during the first period, we cannot anymore rely our analysis on whether the future short term interest rate is decreasing or increasing with GDP, as we did before. We will not be able to separate what will happen in period 2 from the risk in period 1. This contrasts with condition (21) for example, where we were able to focus on the analysis to period 2 only. Our objective in this section is to show that the set of constraints on preferences that would guarantee a decreasing yield curve when there is a risk of recession in the short run becomes quite sophisticated. To do this, let us look at the limit case where the equilibrium interest rate in period 1 is just equal to the rate of pure preference for the present, i.e., where $\beta y_1(z_0)$ equals unity. This is the case either if the growth is zero with certainty, or if the growth is uncertain with a positive probability of a recession. The yield curve would be decreasing if $\beta y_2(z_0)$ is smaller than unity. This property is summarized as follows:

$$\left.\begin{array}{rcl} Eu'(z_0\widetilde{x}_1) & = & u'(z_0) \\ Eu'(z_0\widetilde{x}_2) & = & u'(z_0) \end{array}\right\} \quad \Longrightarrow \quad Eu'(z_0\widetilde{x}_1\widetilde{x}_2) \geq u'(z_0). \tag{25}$$

The first condition to the left corresponds to our assumption that $\beta y_1(z_0) = 1$, whereas the second condition is the stationarity assumption that we made all along this paper. The condition to the right means that $\beta y_2(z_0)$ is less than unity. The interpretation of condition (25) in terms of saving behaviour is simple. Suppose that the agent does not want to save when the uncertain growth per period of her income is either $\widetilde{x}_1$ or $\widetilde{x}_2$. Does it imply that she would save in the presence of growth risk $\widetilde{x}_1\widetilde{x}_2$? Condition (25) states that she wants to save more. That would reduce the equilibrium interest rate.

Define function v in such a way that $v(z) = -u'(z)$ for all z. Under risk aversion ($u'' \leq 0$) and prudence ($u''' \geq 0$), function v is increasing and concave, i.e., it looks like a utility function. Next, we define function V as $V(Z) = v(\exp Z)$. We can then rewrite property (25) as follows:

$$\left.\begin{array}{rcl} EV(Z+\widetilde{X}_1) & = & V(z_0) \\ EV(Z+\widetilde{X}_2) & = & V(z_0) \end{array}\right\} \quad \Longrightarrow \quad EV(Z+\widetilde{X}_1+\widetilde{X}_2) \leq V(Z). \tag{26}$$

In words, condition (26) means that two lotteries on which the agent with utility function V is indifferent when taken in isolation are jointly undesirable. In some sense, this is equivalent to say that independent risks are not complements. Pratt and Zeckhauser (1987) called this condition "properness". A proper utility function V is a function which satisfies condition (26) for any Z and any pair of random variables ($\widetilde{X}_1, \widetilde{X}_2$). They showed that this condition is satisfied for all function V that are $HARA$, a set of functions that contains all power, logarithmic and exponential functions. Now, remember that $V(Z) = -u'(\exp Z)$. It implies for example that One-switch utility functions satisfy condition (25). Indeed, with $u'(z) = a + z^{-\gamma}$, we have $V(Z) = -a + b\exp Z$, which belongs to the class of HARA functions.[10]

[10] The counterexample at the end of section 4.2 is based on this function, with $\widetilde{x}_1$ and $\widetilde{x}_2$ being such that condition 25 is satisfied as an *equality*. This is in fact a limit case.

More interestingly, Pratt and Zeckhauser (1987) also showed that a necessary condition for V to be proper is that

$$\left[\frac{-V''(Z)}{V'(Z)}\right]'' \geq \left[\frac{-V''(Z)}{V'(Z)}\right]' \left[\frac{-V''(Z)}{V'(Z)}\right]. \tag{27}$$

This condition is necessary in the sense that its violation would imply the existence of a pair (X_1, X_2) that would violate condition (26). Pratt and Zeckhauser also showed that this condition is necessary and sufficient for condition (26) to hold when $\widetilde{X}_1$ and $\widetilde{X}_2$ are small risks. With $V(Z) = -u'(\exp Z)$, we have that

$$\frac{-V''(Z)}{V'(Z)} = P(\exp Z) - 1 \tag{28}$$

$$\left[\frac{-V''(Z)}{V'(Z)}\right]' = (\exp Z)P'(\exp Z) \tag{29}$$

$$\left[\frac{-V''(Z)}{V'(Z)}\right]'' = (\exp Z)P'(\exp Z) + (\exp Z)^2 P''(\exp Z). \tag{30}$$

This yields the following result.

Corollary 1 *A necessary condition for the yield curve to be nonincreasing under the stationary condition (17) is that*

$$zP''(z) \geq P'(z)(P(z) - 2) \tag{31}$$

for all z, where $P(z) = -zu'''(z)/u''(z)$ is relative prudence. This condition is necessary and sufficient when the risk on growth is small and the short term interest rate equals the rate of pure preference for the present.

Necessary condition (31) is sophisticated, as it requires conditions on the fifth derivative of the utility function. This means that introducing the risk of recession in the long term and in the short term makes it really a hard task to guarantee that long term discount rates are smaller than short term ones.

5 Conclusion

Economists who have already been confronted to the cost-benefit analysis of actions having an impact to the welfare of future generations felt the difficulty to discount future benefits at a constant rate of return. In this paper, we showed that it is potentially compatible with efficiency to discount far distant cash flows at a decreasing rate. This is the case for example when there is no risk of recession if relative risk aversion is decreasing. This assumption on preferences is plausible, as it corresponds to the well-documented observation that wealthier people invest a larger share of their wealth in risky assets. We also showed that the problem becomes more complex when a risk of recession is introduced. More conditions on preferences must be added to guarantee that the yield curve is decreasing. One such necessary condition is that relative prudence be increasing, a condition that is difficult to test.

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