Market Formation in Bilateral Oligopolies

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Abstract

This paper investigates the structure of bilateral oligopolies – a simple version of Shapley Shubik games with two types of traders and two commodities. It shows that interior equilibria exist, studies the example of CES utility functions to uncover the relation between the complementarity of products in the utility functions and the shape of the reaction functions of the traders, and proves that the number of trading posts is irrelevant. Even if traders can split their offers on different markets, they never choose to specialize and all equilibria are equivalent to an equilibrium where all agents trade on a single market.

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Nontechnical abstract: Many markets for intermediate goods or primary commodities are characterized by the small number of traders active on the market. On those thin markets, traders enjoy high market power and exploit this market power by reducing the amount they demand or supply to the market. The strategic market game introduced by Shapley and Shubik in 1975 provide a simple framework to analyze market power on markets where both buyers and sellers behave strategically. In this paper, we investigate the structure of simple strategic market games, the bilateral oligopolies characterized by two sets of traders with corner endowments in two commodities.

We provide conditions for the existence of a nontrivial equilibrium, and analyze whether the model exhibits strategic complements or strategic substitutes. This last analysis enables us to show that, as goods become more and more complementary, the incentives to exploit market power vanish and, in the limit, the equilibrium converges to a competitive equilibrium. Hence, when traders trade in complementary goods, the loss in efficiency due to market power disappears and there is no reason for public intervention.

In the second part of the paper, we raise the following question. Suppose that traders could choose on which market to trade, do they have an incentive to form separate markets or to coordinate their trades on a single market? We exhibit an endogenous *agglomeration* force by which traders prefer to group on a single market. This agglomeration force stems from the very existence of market power on the two sides of the market: each buyer (or seller) prefers to see more traders on the market, as an increase in the number of traders dilutes market power and increases each trader's utility on the market.

Our analysis thus provides a justification for the existence of single market places on which all trades are conducted. This centralization of trade can easily be observed ion markets for raw materials, where trades occur on a centralized commodity exchange. Our model also suggests that as traders become larger, trade on financial markets should converge on a single market place. Hence, if traders have increasing power on stock exchanges, we should observe an increasing concentration of trade on a few stock exchanges.

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1 Introduction

The strategic market games introduced by Shapley and Shubik (Shapley (1976), Shubik (1973) and Shapley and Shubik (1977)) provide an easy framework to study trade on thin markets. Recently, Gabszewicz and Michel (1997) have proposed a simple version of Shapley-Shubik games, called bilateral oligopolies, with two types of agents who have corner endowments in two commodities. Bilateral oligopolies are interesting to study for two main reasons. First, due to its very simplicity, the model of bilateral oligopoly avoids difficulties which are present in general Shapley-Shubik games. In a bilateral oligopoly, since there are only two commodities and traders can only be on one side of the market, there is no need to distinguish between the different versions of Shapley-Shubik games. Second, as noted by Gabszewicz and Michel (1997), bilateral oligopolies can be viewed as generalizations of the traditional Cournot oligopoly model where both sides of the market (buyers and sellers) behave strategically. In fact, as the number of traders on one side of the market grows large, the outcome of the game converges to the outcome of the Cournot oligopoly.

In recent work, Bloch and Ghosal (1997) have used bilateral oligopolies to study the emergence of markets and the stability of trading groups. Under very restrictive assumptions on utilities, they have shown the existence and uniqueness of equilibrium, and proved that the only stable trading structure is the grand coalition, where all traders buy and sell on the same market. They also assume that the formation of trading groups is exclusive: traders are forced to trade on a single market. In this paper, we investigate further the structure of bilateral oligopolies and generalize the analysis in Bloch and Ghosal (1997) to a model with general utility functions where traders can split their offers on different markets.

We first prove the existence of an interior equilibrium in bilateral oligopolies. The issue of existence of interior equilibria in Shapley-Shubik games has been raised since the late seventies (see Dubey and Shubik (1978) and Amir et al. (1990) for contributions in this direction.) In bilateral oligopolies, Cordella and Gabszewicz (1998) provide an example where interior equilibria fail to exist. We provide an easy sufficient condition on utilities (that the example in Cordella and Gabszewicz (1998) does not satisfy) to guarantee existence of equilibrium.

In the second part of the paper, we turn to the structure of reaction functions in bilateral oligo-polies. In traditional oligopoly models, the relation between the substitutability of the products and the shape of reaction functions is well known (see Bulow, Geanakoplos and Klemperer (1985)). In bilateral oligopolies, we show that this relation is not easily derived. Using the example of a CES utility function, we prove that offers of traders on the two sides of the markets are strategic complements (substitutes) if and only if the products are substitute (complements). On the other hand, offers of traders on the same side of the market are neither strategic complements nor substitutes.

In the third part of the paper, we tackle the problem of market formation. In our model, each trader submits a vector of bids on different trading posts. In equilibrium, we show that all traders are present on the same trading posts. Furthermore, prices are identical across trading posts so that all equilibria are equivalent to an equilibrium where all traders trade on a single market. Our results thus provide an additional justification to the assumption that there exists a single market on which all agents trade. In imperfectly competitive markets, the traders' ability to influence prices creates an "agglomeration" effect which leads all agents to trade on the same markets. When all agents are active on the same markets, arbitrage opportunities preclude the emergence of different prices on different markets, so that the equilibrium allocations are independent of the number of active markets.[1].

The rest of the paper is organized as follows. We introduce the model of bilateral oligopolies in Section 2. We prove existence of equilibrium in Section 3, and study the example of CES production functions in Section 4. Section 5 contains our results on market formation, and Section 6 some concluding comments.

2 The Model

In a bilateral oligopoly, there are two commodities labeled x and y and two types of agents. Agents of type I, indexed by $i = 1, 2, ..., I$ are endowed

[1] However, it should be noted that this last result may depend on the particular version of strategic market games that we are using. Using a more general version of strategic market games, where traders can both buy and sell on the same market, Koutsougeras (1998a) and (1998b) provides examples where different prices emerge on different trading posts. The robustness of these examples to variations on the structure of the strategic market games remains an open question.

with the first commodity. Agents of type II, indexed by $j = 1, 2, ..., J$ are endowed with the second commodity. We let α^i denote the initial endowment of a trader i of type I, and $u^i : \Re^2 \to \Re$ her utility function. Similarly, we denote by β^j and v^j the initial endowment and utility function of a trader j of type II. For any variable t, and any function f, we let f_t denote the partial derivative of f with respect to t. We make the following assumptions on the utility functions.

1. The utility functions are strictly increasing and strictly concave.
2. The two goods are strict complements: $u^i_{xy} > 0$ and $v^j_{xy} > 0$.
3. The utility functions satisfy the following boundary conditions $\lim_{x\to 0} u^i_x = \lim_{y\to 0} u^i_y = \infty$ and $\lim_{x\to 0} v^j_x = \lim_{y\to 0} v^j_y = \infty$.

We suppose that there are M markets on which the agents can trade. We denote by b^i_m the bid of trader i on market m and by g^j_m the bid of trader j on market m. Each trader of type I selects a vector of bids on the M markets, $b^i = (b^i_1, b^i_2, ..., b^i_M)$ and each trader of type II chooses a vector of bids $g^j = (g^j_1, g^j_2, ..., g^j_M)$. The strategy sets of the traders of type I and II are thus given by

$$S^i = \{b^i \in \Re^M | b^i_m \geq 0, \textstyle\sum_m b^i_m \leq \alpha^i\}$$
$$S^j = \{g^j \in \Re^M | g^j_m \geq 0, \textstyle\sum_m g^j_m \leq \beta^j\}$$

On any market m, the total amount of bids of traders of type I is given by $B_m = \sum_i b^i_m$ and the bids of traders II by $G_m = \sum_j g^j_m$. The relative price of good x on market m is obtained as $p_m = \frac{G_m}{B_m}$ if $B_m > 0$ and $p_m = 0$ if $B_m = 0$. Hence, the final allocations of the two types of traders are

$$(x^i, y^i) = \left(\alpha^i - \textstyle\sum_m b^i_m, \sum_m b^i_m \frac{G_m}{B_m}\right),$$
$$(x^j, y^j) = \left(\textstyle\sum_m g^j_m \frac{B_m}{G_m}, \beta^j - \sum_m g^j_m\right).$$

Definition 1 *A market equilibrium is a strategy profile* (b^{i*}, g^{j*}) *such that*
$$u^i\left(\alpha^i - \textstyle\sum_m b^{i*}_m, \sum_m b^{i*}_m \frac{\sum_j g^{j*}_m}{\sum_i b^{i*}_m}\right) \geq$$
$$u^i\left(\alpha^i - \textstyle\sum_m b^i_m, \sum_m b^i_m \frac{\sum_j g^{j*}_m}{\sum_{k\neq i} b^{k*}_m + b^i_m}\right) \forall b^i \in S^i,$$

$$v^j\left(\sum_m g_m^{j*}\frac{\sum_i b_m^{i*}}{\sum_j g_m^{j*}}, \beta^j - \sum_m g_m^{j*}\right) \geq$$

$$v^j\left(\sum_m g_m^j \frac{\sum_i b_m^{i*}}{\sum_{k\neq j} g_m^{k*}+g_m^j}, \beta^j - \sum_m g_m^j\right) \forall g^j \in S^j.$$

3 Existence of Interior Market Equilibrium

It is well known that the strategic market game we analyze always admits a trivial equilibrium, where all traders put zero bids on all the markets. As a first step in the analysis, we show that there exists an equilibrium where all traders make positive bids on a single trading post. To this end, we first define an *equilibrium point*, as in Dubey and Shubik (1978) and Amir et al. (1990), as the limit of a sequence of equilibria of perturbed games.

A perturbed game Γ^ε is a game where an outside agency puts a fixed quantity $\varepsilon > 0$ of the two goods on trading post $m = 1$

. Hence, the price of good x on trading post 1 in the perturbed game Γ^ε is $p_1^\varepsilon = \frac{G_1+\varepsilon}{B_1+\varepsilon}$. We let $(b^{i*,\varepsilon}, g^{j*,\varepsilon})$ denote a market equilibrium of the perturbed game Γ^ε.

Definition 2 *An equilibrium point is a market equilibrium* (b^{i*}, g^{j*}) *such that there exists a sequence* ε_n*, with* $\lim_{n\to\infty}\varepsilon_n = 0$ *and market equilibria of the perturbed games* Γ^{ε_n} *with* $\lim_{n\to\infty}(b^{i*,\varepsilon_n}, g^{j*,\varepsilon_n}) = (b^{i*}, g^{j*})$.

In words, equilibrium points are market equilibria which can be approached by a sequence of equilibria of perturbed games. We recall the following result from Dubey and Shubik (1978) which will prove helpful in the proof of existence of an interior equilibrium.

Lemma 3 *(Dubey and Shubik, 1978). In a game with more than two traders of the two types, there exist positive constants* C *and* D *such that* $C < p_1^{*,\varepsilon} = \frac{G_1^{*,\varepsilon}+\varepsilon}{B_1^{*,\varepsilon}+\varepsilon} < D$ *for all* $\varepsilon > 0$.

Notice that this Lemma shows that there exist *uniform bounds* on the relative prices of the two commodities. However, it does not guarantee that equilibrium bids are bounded away from zero. In fact, examples can be constructed to show that all bids can be equal to zero at an equilibrium point. (See Cordella and Gabszewicz (1998) for a simple example with linear utilities.) In our model, the boundary assumptions on traders' utilities guarantee the existence of an equilibrium point where all traders put positive bids on market 1.

Theorem 4 *There exists an equilibrium point where all traders put positive bids on market* 1*.*

Proof: Consider the perturbed game Γ^{ε}. We consider an equilibrium where all traders put zero bids on all markets $m = 2, .., M$. For simplicity, we drop the market subscript on all variables. Consider the behavior of a trader of type I. She solves the following problem.

$$\max_{b^i \in [0,\, \alpha^i]} u^i \left(\alpha^i - b^i, b^i \frac{G + \varepsilon}{b^i + \sum_{k \neq i} b^k + \varepsilon} \right)$$

Taking derivatives we obtain

$$\frac{\partial u^i}{\partial b^i} = -u^i_x + u^i_y \frac{(G + \varepsilon)(\sum_{k \neq i} b^k + \varepsilon)}{\left(b^i + \sum_{k \neq i} b^k + \varepsilon\right)^2}.$$

The second derivative is given by

$$\begin{aligned} \frac{\partial^2 u^i}{\partial b^{i2}} &= u^i_{xx} - 2u^i_{xy} \frac{(G + \varepsilon)(\sum_{k \neq i} b^k + \varepsilon)}{\left(b^i + \sum_{k \neq i} b^k + \varepsilon\right)^2} + u^i_{yy} \left(\frac{(G + \varepsilon)(\sum_{k \neq i} b^k + \varepsilon)}{\left(b^i + \sum_{k \neq i} b^k + \varepsilon\right)^2} \right)^2 \\ &\quad -2u^i_y \frac{(G + \varepsilon)(\sum_{k \neq i} b^k + \varepsilon)}{\left(b^i + \sum_{k \neq i} b^k + \varepsilon\right)^3} \end{aligned}$$

Since the utility function is strictly increasing, strictly concave, and the two goods are strict complements, $\frac{\partial^2 u^i}{\partial b^{i2}} < 0$. The boundary conditions on the utility function guarantee that $\lim_{b^i \to 0} \frac{\partial u^i}{\partial b^i} > 0$ and $\lim_{b^i \to \alpha^i} \frac{\partial u^i}{\partial b^i} < 0$. Hence there exists a unique interior solution to trader i's maximization problem. Define this solution as $\phi^i \left(\underset{k \neq i}{b^k}, g^j \right)$ and construct the function $\phi : \times_i [0, \alpha^i] \times_j$

$[0, \beta^j]$, $\phi = \times_i \phi^i \times_j \phi^j$. The function ϕ is a continuous function over a compact, convex subset of a Euclidean space and admits a fixed point by Brouwer's fixed point theorem. We denote this fixed point by $(b^{i*,\varepsilon}, g^{j*,\varepsilon})$.

In the second step of the proof, we show that for any trader i there exist constants D^i, E^i such that

$$0 < D^i \leq b^{i*,\varepsilon} \leq E^i < \alpha^i.$$

A similar argument can be used to show that the bids of traders of type II are uniformly bounded as well. We first show $b^{i*,\varepsilon} < \alpha^i$. Fix the strategy choices of the other players in equilibrium, and consider trader i's marginal utility,

$$\frac{\partial u^i}{\partial b^i} = -u_x^i + u_y^i \frac{(G^{*,\varepsilon} + \varepsilon)(\sum_{k \neq i} b^{k*,\varepsilon} + \varepsilon)}{\left(b^i + \sum_{k \neq i} b^{k*,\varepsilon} + \varepsilon\right)^2}$$

Note that

$$\frac{(G^{*,\varepsilon} + \varepsilon)(\sum_{k \neq i} b^{k*,\varepsilon} + \varepsilon)}{\left(b^i + \sum_{k \neq i} b^{k*,\varepsilon} + \varepsilon\right)^2} < \frac{(G^{*,\varepsilon} + \varepsilon)}{\left(b^i + \sum_{k \neq i} b^{k*,\varepsilon} + \varepsilon\right)}$$

Now consider a fixed lower bound z^i on trader i's bid, with $z^i > 0$ independent of ε. Then

$$\begin{aligned}
\frac{(G^{*,\varepsilon} + \varepsilon)}{\left(b^i + \sum_{k \neq i} b^{k*,\varepsilon} + \varepsilon\right)} &\leq \frac{(G^{*,\varepsilon} + \varepsilon)}{\left(z^i + \sum_{k \neq i} b^{k*,\varepsilon} + \varepsilon\right)} \leq \frac{(G^{*,\varepsilon} + \varepsilon)}{(z^i + \varepsilon)} \\
&< \frac{G^{*,\varepsilon}}{z^i} + 1 < \frac{\sum_j \beta^j}{z^i} + 1
\end{aligned}$$

Hence, for all $b^i > z^i$ and for all ε,

$$\frac{\partial u^i}{\partial b^i} < -u_x^i + u_y^i \left(\frac{\sum_j \beta^j}{z^i} + 1 \right).$$

Next observe that since $C < p_1^{*,\varepsilon} < D$, $b^i p_1^{*,\varepsilon}$ and u_y^i are uniformly bounded. Furthermore, $\lim_{b^i \to \alpha^i} u_x^i = \infty$, so that $\lim_{b^i \to \alpha^i} \frac{\partial u^i}{\partial b^i} < 0$ for all $\varepsilon > 0$, showing that there exists a uniform bound E^i such that $b^{i*,\varepsilon} \leq E^i < \alpha^i$.

Now since $b^{i*,\varepsilon} \leq E^i < \alpha^i$, in equilibrium

$$\frac{\partial u^i}{\partial b^i} = -u_x^i + u_y^i \frac{(G^{*,\varepsilon} + \varepsilon)(\sum_{k \neq i} b^{k*,\varepsilon} + \varepsilon)}{(B^{*,\varepsilon} + \varepsilon)^2} \leq 0$$

Since $C < p_1^{*,\varepsilon} < D$, $b^i p_1^{*,\varepsilon}$ and u_x^i are uniformly bounded. Furthermore, $\frac{B^{*,\varepsilon}+\varepsilon}{G^{*,\varepsilon}+\varepsilon} < \frac{1}{C}$. Hence there exists a uniform bound F such that

$$u_y^i \frac{(\sum_{k \neq i} b^{k*,\varepsilon} + \varepsilon)}{(B^{*,\varepsilon} + \varepsilon)} \leq F$$

Now consider

$$\tilde{U}_y = \max_i u_y^i$$

Summing over i, we obtain:

$$\tilde{U}_y \frac{(I-1)\, B^{*,\varepsilon} + I\varepsilon}{(B^{*,\varepsilon} + \varepsilon)} \leq IF$$

Implying

$$u_y^i \leq \tilde{U}_y \leq (I/(I-1))F.$$

But since $\lim_{b^i \to 0} u_y^i = \infty$, and $C < p_1^{*,\varepsilon} < D$, there must exist a uniform lower bound D^i on trader i's bid, such that $0 < D^i \leq b^i$ for all $\varepsilon > 0$.

To complete the proof, consider a sequence ε_n converging to 0. We have just shown that $\forall i$, $b^{i*,\varepsilon_n} \in [D^i, E^i]$. Similarly, $\forall j$, $g^{j*,\varepsilon_n} \in [D^j, E^j]$. The sequence $(b^{i*,\varepsilon_n}, g^{j*,\varepsilon_n})$ is thus defined over a compact set. Taking subsequences if necessary, it converges to a limit point (b^{i*}, g^{j*}) where $b^{i*} \in [D^i, E^i]$ and $g^{j*} \in [D^j, E^j]$. By continuity of the utility functions, it is clear that (b^{i*}, g^{j*}) is a Nash equilibrium of the game Γ.■

4 Strategic Substitutes and Complements in Bilateral Oligopolies : An Example

We now investigate the structure of market equilibria. In particular, we study whether reaction functions are increasing or decreasing, i.e. whether offers of traders are strategic substitutes or complements. We consider a simple example, with one trading post, and where all agents have an initial endowment of 1 and a common CES utility function,

$$U(x,y) = (x^\rho + y^\rho)^{\frac{1}{\rho}}, \text{ with } \rho \leq 1.$$

This utility function encompasses both situations where the goods are substitutes ($\rho \geq 0$) and complements ($\rho \leq 0$). When $\rho = 1$, the goods are perfect substitutes; when $\rho = 0$, the utility function is Cobb-Douglas, and when $\rho \to -\infty$, the goods are perfect complements. This exchange economy has a unique competitive equilibrium given by $p^\star = (1,1)$ and $(x^\star, y^\star) = \left(\frac{1}{2}, \frac{1}{2}\right)$, for traders of types I and II.

To compute the reaction functions of the two types of traders, we solve the maximization problem faced by agents of type I

$$\underset{0 \leq q_i \leq 1}{\text{Max}} \; U_i(q_i, b_j).$$

We obtain

$$\frac{\partial U_i}{\partial q_i} = A(q_i) \cdot B(q_i), \text{ where}$$

$$A(q_i) = -(1-q_i)^{\rho-1} + \left(\sum b_j\right)^\rho \frac{\sum\limits_{k \neq i} q_k}{\left(\sum q_i\right)^{\rho+1}} q_i^{\rho-1}$$

$$B(q_i) = \left[(1-q_i)^\rho + \left(q_i \frac{\sum b_j}{\sum q_i}\right)^\rho\right]^{\frac{1}{\rho}-1} > 0.$$

Next note that $\lim\limits_{q_i \to 0} A(q_i) = +\infty$ and $\lim\limits_{q_i \to 1} A(q_i) = -\infty$, and

$$\begin{aligned}\frac{\partial A}{\partial q_i} &= (\rho-1)(1-q_i)^{\rho-2} + \left(\sum b_j\right)^\rho \sum_{k \neq i} q_k \left[\frac{q_i^{\rho-2}\left((\rho-1)\sum q_i - (\rho+1) q_i\right)}{\left(\sum q_i\right)^{\rho+2}}\right] \\ &< 0.\end{aligned}$$

We conclude that there exists $\bar{q}_i$ such that $\forall q_i \leq \bar{q}_i$, $\dfrac{\partial U_i}{\partial q_i} \geq 0$ and $\forall q_i > \bar{q}_i$, $\dfrac{\partial U_i}{\partial q_i} < 0$.

Hence, the maximization problem has a unique interior solution given by

$$A(q_i) = \left[-(1-q_i)^{\rho-1} + \left(\sum b_j\right)^\rho \frac{\sum\limits_{k \neq i} q_k}{\left(\sum q_i\right)^{\rho+1}} q_i^{\rho-1}\right] = 0.$$

We use this reaction function to study whether offers of traders on the two sides of the market and on the same side of the market are strategic substitutes or complements.

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Lemma 5 *The offers of traders on the two sides of the market are strategic complements (substitutes) if and only if the goods are substitutes (complements). If the goods are complements, offers of traders on the same side of the market are strategic complements. If the goods are substitutes, offers of traders on the same side of the market are neither strategic complements nor strategic substitutes.*

Proof. : By implicit differentiation, we get

$$\frac{\partial q_i}{\partial b_j} = -\frac{\rho\left(\sum b_j\right)^{\rho-1}\dfrac{\sum\limits_{k\neq i} q_k}{\left(\sum q_i\right)^{\rho+1}} q_i^{\rho-1}}{\dfrac{\partial A}{\partial q_i}}.$$

Hence, $\frac{\partial q_i}{\partial b_j} \geq 0$ iff $\rho \geq 0$. Similarly, we compute

$$\frac{\partial q_i}{\partial q_j} = -\frac{\left(\sum b_j\right)^{\rho}\dfrac{\left(q_i - \rho\sum\limits_{k\neq i} q_k\right)}{\left(\sum q_i\right)^{\rho+2}} q_i^{\rho-1}}{\dfrac{\partial A}{\partial q_i}}$$

The sign of $\frac{\partial q_i}{\partial q_j}$ thus depends on the sign of $(q_i - \rho\sum\limits_{k\neq i} q_k)$.

We now turn to the computation of the equilibrium. We first show that all traders on the same side of the market adopt the same strategy. Suppose by contradiction that for two traders i and k of type I, $q_i \neq q_k$. Without loss of generality, let $q_k > q_i$. The following two equations must hold

$$\begin{aligned}
-\left(1-q_i\right)^{\rho-1} + \left(\sum b_j\right)^{\rho}\frac{\sum\limits_{t\neq i} q_t}{\left(\sum q_i\right)^{\rho+1}} q_i^{\rho-1} &= 0 \\
-\left(1-q_k\right)^{\rho-1} + \left(\sum b_j\right)^{\rho}\frac{\sum\limits_{t\neq k} q_t}{\left(\sum q_i\right)^{\rho+1}} q_k^{\rho-1} &= 0.
\end{aligned}$$

Thus, we have

$$\frac{(1-q_i)^{\rho-1}}{\sum\limits_{t\neq i} q_t q_i^{\rho-1}} = \frac{(1-q_k)^{\rho-1}}{\sum\limits_{t\neq k} q_t q_k^{\rho-1}}.$$

Since $q_k > q_i$, we have $\sum\limits_{t\neq i} q_t > \sum\limits_{t\neq k} q_t$, which implies that

$$\left(\frac{q_i}{1-q_i}\right)^{1-\rho} > \left(\frac{q_k}{1-q_k}\right)^{1-\rho}$$

yielding

$$q_i > q_k,$$

contradicting the assumption.

Since the equilibrium is symmetric among traders, on the same side of the market, we may denote by q and b the offers of traders of type I and type II on the market. The reaction functions are given by

$$\begin{aligned} \frac{\partial U_i}{\partial q_i} &= -q\,(1-q)^{\rho-1} + \frac{(n-1)}{n} b^{\rho} = 0 \\ \frac{\partial U_i}{\partial b_j} &= -b\,(1-b)^{\rho-1} + \frac{(n-1)}{n} q^{\rho} = 0. \end{aligned} \tag{1}$$

In the Appendix, we show that the system of equations (1) characterizes a unique symmetric Nash equilibrium, where all traders adopt the same strategy.

Proposition 6 *: The strategic market game has a unique interior Nash equilibrium. All traders adopt the same strategy:*

$$q^{\star} = b^{\star} = \frac{1}{\left(\frac{n}{n-1}\right)^{\frac{1}{1-\rho}} + 1}.$$

Lemma 7 *: As the degree of substitution of the two goods increases the equilibrium offers $q^{\star}$ and $b^{\star}$ decrease.*

Proof. : We compute $\dfrac{\partial b^\star}{\partial \rho} = -\dfrac{1}{(1-\rho)^2} \dfrac{\log\left(\frac{n}{n-1}\right)\left(\frac{n}{n-1}\right)^{\frac{1}{1-\rho}}}{\left[1+\left(\frac{n}{n-1}\right)^{\frac{1}{1-\rho}}\right]^2} < 0.$

We obtain for a linear utility function ($\rho \to 1$) $q^\star = b^\star = 0$, for a Cobb-Douglas utility function ($\rho = 0$), $q^\star = b^\star = \frac{n-1}{2n-1}$. As the goods become perfect complements ($\rho \to -\infty$), $q^\star = b^\star = \frac{1}{2}$, the equilibrium outcome converges to the competitive equilibrium.

The study of the CES example shows that the offers of traders on the two sides of the market are strategic substitutes (or complements) if and only if the goods are complements (substitutes). To understand this result, consider the behavior of an agent i of type I. If the offer b_j of agents of type II increases, the amount of good y in agent i's allocation, y_i, increases. If the two goods are substitutes, this decreases the marginal utility of x_i and induces trader i to increase her offer q_i. If, on the other hand, the two goods are complements, this increases the marginal utility of x_i and induces trader i to reduce her offer. Note that this effect is not related to the traditional analysis of strategic substitutes and complements in oligopoly.

The analysis of this example also shows that, as the complementary between the two goods increases, the equilibrium offers on the two sides of the market increase. The intuition underlying this result is easily grasped. When the complementarity increases, the marginal utility of good y to traders of type I increases. Hence, for any fixed offer b_j of traders of type II, the offer q_i increases. In equilibrium, both offers b_j and q_i are increasing with the degree of complementarity between goods.

5 Market Formation in Bilateral Oligopolies

Finally, we turn to the following problem: In the general market game described in Section 2, do all traders trade on a single market, or do traders specialize on different trading posts? We prove the following Theorem, characterizing all market equilibria of the bilateral oligopoly.

Theorem 8 *In any market equilibrium, all agents are active on the same markets. Prices are identical across markets, so that any market equilibrium is equivalent to an equilibrium with a single trading post.*

Proof: We first show that all agents are active on the same markets. As a first step, we prove that the ranking of bids on different markets has to be identical. In other words, if $b^i_\mu > b^{i'}_\mu$ for some market μ and some traders i,i' of type I, then $b^i_m \geq b^{i'}_m$ for all markets $m = 1, 2, .., M$.

To prove this statement, consider market μ. We immediately obtain that

$$\sum_{k \neq i} b^k_\mu < \sum_{k \neq i'} b^k_\mu.$$

Hence

$$\frac{u^{i'}_x}{u^{i'}_y} \geq \frac{G_\mu}{B_\mu} \frac{\sum_{k \neq i'} b^k_\mu}{B_\mu} > \frac{G_\mu}{B_\mu} \frac{\sum_{k \neq i} b^k_\mu}{B_\mu} = \frac{u^i_x}{u^i_y}.$$

If there exists a market μ' on which $b^{i'}_{\mu'} > b^i_{\mu'} \geq 0$, the preceding inequality is reversed, yielding a contradiction. The above argument shows that the sets of agents participating to different markets are monotone: if $I(m)$ denotes the set of agents active at trading post m, then there exists an ordering of trading posts for which $I(1) = I$ and $I(m) \subseteq I(m-1)$ for all m. We will show that $I(m) = I(m-1)$ for all m such that $I(m) \neq \emptyset$.

Suppose to the contrary that there exist two markets m and m' such that $I(m') \supset I(m) \neq \emptyset$. Let i be a trader in $I(m)$ and i' a trader in $I(m') \backslash I(m)$.

Since $\dfrac{\partial u^{i'}}{\partial b^{i'}_{m'}} = 0$,

$$\frac{u^{i'}_x}{u^{i'}_y} = \frac{G_{m'}}{B_{m'}} \frac{\sum_{k \neq i'} b^k_{m'}}{B_{m'}}$$

Since $\dfrac{\partial u^{i'}}{\partial b^{i'}_m} < 0$

$$\frac{u^{i'}_x}{u^{i'}_y} > \frac{G_m}{B_m} \frac{\sum_{k \neq i'} b^k_m}{B_m} = \frac{G_m}{B_m}$$

Hence, we have

$$\frac{G_{m'}}{B_{m'}} > \frac{G_m}{B_m}$$

$>$From the equations characterizing the optimal behavior of trader i, we obtain

$$\frac{u^i_x}{u^i_y} = \frac{G_m}{B_m} \frac{\sum_{k \neq i} b^k_m}{B_m}$$
$$\frac{u^i_x}{u^i_y} = \frac{G_{m'}}{B_{m'}} \frac{\sum_{k \neq i} b^k_{m'}}{B_{m'}}$$

Since $\dfrac{G_{m'}}{B_{m'}} > \dfrac{G_m}{B_m}$,

$$\frac{\sum_{k\neq i} b_m^k}{B_m} > \frac{\sum_{k\neq i} b_{m'}^k}{B_{m'}}$$

Summing up over all agents in $I(m')\backslash I(m)$ and letting $\iota = \#(I(m')\backslash I(m))$,

$$\sum_{i'\in I(m')\backslash I(m)} \frac{\sum_{k\neq i} b_{m'}^k}{B_{m'}} < \sum_{i'\in I(m')\backslash I(m)} \frac{\sum_{k\neq i} b_m^k}{B_m} = \iota - 1.$$

Letting $B_{m'}^1 = \sum_{k\in I(m')\backslash I(m)} b_{m'}^k$ and $B_{m'}^2 = \sum_{k\in I(m)} b_{m'}^k$,

$$\sum_{i'\in I(m')\backslash I(m)} \frac{\sum_{k\neq i} b_{m'}^k}{B_{m'}} = \frac{(\iota-1)B_{m'}^1 + \iota B_{m'}^2}{B_{m'}^1 + B_{m'}^2} > \iota - 1,$$

yielding a contradiction.

We now prove that on any two active trading posts m and $m^{'}$, prices are identical. In equilibrium, we have

$$\frac{G_m}{B_m}\frac{\sum_{k\neq i} b_m^k}{B_m} = \frac{G_{m'}}{B_{m'}}\frac{\sum_{k\neq i} b_{m'}^k}{B_{m'}}$$

for all agents i. Summing up over the agents, we find

$$\frac{G_m}{B_m}(I-1) = \frac{G_{m'}}{B_{m'}}(I-1)$$

so that the prices are equal on the two markets.

Finally, observe that this last statement shows that any equilibrium with multiple markets is equivalent to an equilibrium with a single market. If (b^i, g^j) is an equilibrium with a single trading post, any fragmentation of the trade, where all agents put the same fractions $\lambda^m b^i$ and $\lambda^m g^j$ on market m is an equilibrium of the game with multiple trading posts. Conversely, starting from an equilibrium with multiple trading posts, strategies where all agents put on a single trading post the bids $\sum_m b_m^i, \sum_m g_m^j$ form an equilibrium of the game.■

6 Conclusion

This paper investigates the structure of bilateral oligopolies – a simple version of Shapley Shubik games with two types of traders and two commodities. It shows that interior equilibria exist, studies the example of CES utility functions to uncover the relation between the complementarity of products in the utility functions and the shape of the reaction functions of the traders, and proves that the number of trading posts is irrelevant. Even if traders can split their offers on different markets, they never choose to specialize and all equilibria are equivalent to an equilibrium where all agents trade on a single market.

This last result is reminiscent of the analysis of Bloch and Ghosal (1997) who showed, under more restrictive assumptions, that the only stable market structure is the grand coalition. In fact, the presence of market power on the two sides of the market induces an agglomeration effect whereby all traders find it in their interest to trade on the same market.

The model of bilateral oligopolies provides an easy framework to study trading on thin markets. It could be used to address new issues on those markets. For example, what are the traders' incentives to collude? How does collusion on one side of the market affect collusion on the other side? Another set of interesting problems are linked to imperfect information. What is the equilibrium of a bilateral oligopoly where traders are uncertain about the preferences of traders on the other side of the market? Is information revealed in equilibrium? What happens when the traders are replicated? All these questions deserve attention and form important topics for further research.

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A Appendix

A.1 Proof of Proposition 6.

The symmetric offers given in the Proposition clearly satisfy the system of equations (1). We show that this equilibrium is unique and globally stable.

Following Dixit (1986), a sufficient condition for uniqueness of equilibrium is

$$\left|\frac{\partial q}{\partial b}\right|\left|\frac{\partial b}{\partial q}\right| \leq 1.$$

By implicit differentiation,

$$\frac{\partial q}{\partial b} = \frac{\rho \frac{n-1}{n} b^{\rho-1}}{(1-q)^{\rho-2}(1-\rho q)}.$$

At equilibrium, $(1-q)^{\rho-2} = \dfrac{(1-q)^{\rho-1}}{1-q} = \dfrac{\frac{n-1}{n} b^{\rho}}{q(1-q)}$. Hence, we get

$$\frac{\partial q}{\partial b} = \frac{\rho q(1-q)}{b(1-\rho q)}.$$

A symmetric computation gives

$$\frac{\partial b}{\partial q} = \frac{\rho b(1-b)}{q(1-\rho b)}.$$

Hence,

$$\left|\frac{\partial q}{\partial b}\right|\left|\frac{\partial b}{\partial q}\right| = \frac{\rho^2(1-b)(1-q)}{(1-\rho q)(1-\rho b)}$$

and

$$\left|\frac{\partial q}{\partial b}\right|\left|\frac{\partial b}{\partial q}\right| \leq 1 \Longleftrightarrow \rho + 1 \geq \rho(b+q). \tag{2}$$

Next, note that the system of equations (1) gives

$$\left(\frac{1-q}{1-b}\right)^{\rho-1} = \left(\frac{b}{q}\right)^{\rho+1}. \tag{3}$$

If $\rho \geq -1$, equation (3) has a unique solution: $b = q$. So, the only case to consider is $\rho \leq -1$. Inequality (2) then becomes

$$b + q \geq 1 + \frac{1}{\rho}. \tag{4}$$

At the first step, we show that $b + q \leq 1$. Equation (3) gives

$$\frac{b}{q} = \left(\frac{1-q}{1-b}\right)^{\frac{\rho-1}{1+\rho}}.$$

Without loss of generality, suppose $b \geq q$. Then $1-q \geq 1-b$, and since $\frac{\rho-1}{1+\rho} \geq 1$,

$$\frac{b}{q} = \left(\frac{1-q}{1-b}\right)^{\frac{\rho-1}{1+\rho}} \geq \frac{1-q}{1-b},$$

implying $b+q \leq 1$.

Next suppose without loss of generality that $b \geq q$. Then,

$$(1-q)^{\rho-1} = \frac{n-1}{n}\frac{b^{\rho}}{q} \geq \frac{n-1}{n}\, b^{\rho-1}$$

yielding

$$\left(\frac{b}{1-q}\right)^{1-\rho} \geq \frac{n-1}{n} \geq \frac{1}{2}$$

or,

$$\frac{b}{1-q} \geq \left(\frac{1}{2}\right)^{\frac{1}{1-\rho}}$$

or finally,

$$2^{\frac{1}{1-\rho}}\, b + q \geq 1.$$

Now,

$$b+q-\left(1+\frac{1}{\rho}\right) = \left(2^{\frac{1}{1-\rho}} b + q - 1\right) + \left[b\left(1-2^{\frac{1}{1-\rho}}\right) - \frac{1}{\rho}\right].$$

To show inequality (4), it thus suffices to show that $b\left(1-2^{\frac{1}{1-\rho}}\right) - \dfrac{1}{\rho} \geq 0$. Since this expression is decreasing in b, a sufficient condition for inequality (4) is

$$1 - \frac{1}{\rho} \geq 2^{\frac{1}{1-\rho}}.$$

It is easy to see that this inequality is satisfied for all $\rho \leq -1$, since $\Phi(\rho) = 1 - \dfrac{1}{\rho} - 2^{\frac{1}{1-\rho}}$ is a strictly increasing function and $\lim\limits_{\rho \to -\infty} \Phi(\rho) = 0$. This concludes the proof.