Nonpoint Source Pollution Control Under Incomplete and Costly Information

by
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Abstract

This paper analyzes the efficient management of nonpoint source pollution (NPS) under limited pollution control budget and incomplete information inherent in NPS pollution. By incorporating information acquisition into a pollution control model, it focuses on the tradeoff between data collection and treatment efforts and derives conditions under which (i) a favorable change in the state of treatment cost at one site may lead to an *increase* in treatment level at another site, (ii) a higher data collection cost induces *more* data collection, and (iii) an increase in information productivity leads to an *increase* in the level of data collection. A numerical simulation of the model illustrates how in managing NPS pollution the value of information acquisition depends on the degree of heterogeneity of polluting sites.

JEL Classification: D61, D81, D83, Q28

Key Words: Nonpoint Source Pollution, uncertainty, costly information, constrained pollution control budget

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We thank Richard Howitt for invaluable suggestions, plus seminar participants at UC Berkeley and UC Davis. Financial support for Kaplan was provide in part by a grant (Project No. W-887) from the University of California Centers for Water and Wildland Resources, and the United States Environmental Protection Agency, "Science to Achieve Results," Graduate Fellowship Program.

I. Introduction

In this paper, we take a constrained management approach to nonpoint source (NPS) pollution control where the focus is no longer the decision of the social planner who maximizes social welfare, but rather, more realistically, the decision of a manager who maximizes pollution control given a limited budget. The management of sediment loading from abandoned logging roads in Redwood National Park, Orick, CA. motivated this research. During high storm events, run-off is diverted from stream channels, at each road crossing, causing sediment to enter tributaries as the run-off returns to the channel downstream from the crossing (Spreiter et al. [13]). The sediment loading threatens salmon spawning habitat and the Tall Trees Grove, home of the world's tallest trees.

The manager's problem is further constrained by the incomplete information inherent in NPS pollution. By the definition of NPS pollution, information on the linkage between polluting sites and ambient load is incomplete. More specifically, while the pollution manager can observe the total ambient load or the consequent damage, she is unable or it is prohibitively costly, to detect with certainty the pollution from each individual site. This incomplete information creates uncertainty about the efficient treatment frontier. The manager can reduce the treatment uncertainty by extracting information from collected data, which in turn increases treatment effectiveness by allowing the manager to reallocate treatment to relatively larger polluting sites. However, the manager faces an explicit tradeoff between treatment effort and effectiveness because data collection is costly and the manager is fiscally constrained.

The purpose of this paper is to examine the role of incomplete information in NPS pollution control. By incorporating information acquisition into a pollution control model, the analysis allows us to focus on the tradeoff between treatment effort and effectiveness, which has yet to be fully explored in the literature. Furthermore, we explicitly consider the role of heterogeneity in the manager's decision, where the manager exploits the heterogeneity among the polluting sites to lower the sediment related damage for a given expenditure on treatment activity. In addition, we analyze the influence data collection and treatment costs have on the optimal budget allocations.

Previous research on the constrained public manager appears in Barrett and Segerson [1][1]. Their work implies that misspecification of the behavioral objective, between the alternatives of social welfare maximization and damage minimization subject to a manager's budget constraint, matters and can lead to a wrong policy prescription. This research extends the work of Polinsky and Shavell [9] by considering the alternative assumption of the fiscally constrained manager and by incorporating uncertainty as characterized in Litchenberg and Zilberman [7]. However, Barrett and Segerson do not provide a mechanism for the manager to resolve the uncertainty, and thus our study extends the constrained manager problem by including data collection.

The past research on information acquisition in pollution control problems is rather scanty. Rausser and Howitt [10] examine information acquisition in analyzing regulatory mechanisms. Kolstad [5] considers the role of acquiring information in optimal policy to control global climate change. Cabe and Herriges [2] introduce information acquisition in the regulation of NPS pollution, where information reduces the social cost associated with setting an ambient concentration tax control mechanism as proposed by Segerson [11]. Our problem differs from this past work since the setting for pollution control in our analysis involves incomplete information on the present state of NPS pollution in a constrained management framework.

Our work is also related to Weitzman [14], who provides a diversity theory, which incorporates the degree of differences in genetic characteristics among various species of crane. This difference

[1] Similarly, Garvie and Keeler [3] consider the budgetary tradeoff between data collection and enforcement for a public regulator.

allows the manager to determine which species are most vital for the survival of the genetic base. In extreme cases, allowing some species to go extinct may be optimal, since it serves to increase the probability of survival for the remaining species. In the context of our problem, knowledge of the degree of loading diversity between the sites allows the manager to increase treatment effectiveness by reallocating her budget and further reducing sediment related damage. Once again, in extreme cases, some sites may go untreated because they contribute little or nothing to sediment-related damage.

This paper contributes to the literature by extending the previous research on the management of NPS pollution to include information acquisition and explicit consideration of heterogeneity of sediment loading from the polluting sites. We illustrate the role of information acquisition in reducing NPS pollution uncertainty. To this end, we show that ignoring the difference among the polluting sites results in larger damages than would result if the differences are exploited. We also see that, by increasing the effectiveness of treatment effort, the first unit of information obtained about the sediment loading distribution, makes a particularly large contribution to reducing expected damages. These results lead us to suggest that data collection and other efforts to distinguish the degree of heterogeneity among the polluting sites play an important role in the optimal control of NPS pollution. Furthermore, in our analysis, we derive conditions under which a favorable change in the state of costs in one site can lead to an increase in treatment level at another site, and a higher data collection cost induces more, rather than less, data collection. We also derive conditions under which increased productivity of information acquisition leads to a reduction in the level of information collection.

The rest of the paper is organized as follows. The following section analyzes the sediment control when information is incomplete and data collection is costly. Section III provides the comparative static results for some of the key parameters of the model. Section IV presents the simulation model and numerical results. In this section, we compare the behavior of a perfectly informed manager, a completely uninformed manager, and an incompletely informed manager who acquires information through data collection. Section VI concludes.

II. Costly Information Acquisition Model

Faced with budget and information constraints, the manager can choose to collect data to reduce the uncertainty about the expected treatment productivity. In the case of sediment control, these data samples are taken during the rain season with frequency, a. The information acquired through data collection allows the manager to update the prior uniform distribution to produce a posterior distribution closer to the true underlying distribution. Closeness in this context refers to the distance between the moments of the posterior and true distribution. To properly measure the distance between the prior and posterior distribution, we employ Kullback's divergence statistic [6]:

$$D(q_1 : q_0) = \int_{-\infty}^{\infty} h(q_1) \ln\left(\frac{h(q_1)}{h(q_0)}\right) dq,$$

where $h(q_1)$ and $h(q_0)$ are the posterior and prior distributions on q, a vector of random sediment loading shares, respectively. This information metric measures the level of acquired information (or conversely, the reduction in the level of uncertainty) in the posterior sediment loading share distribution by comparing $I(q_0)$ and $I(q_1)$, the information contained in the prior and posterior distributions, respectively (see Appendix I for details on this information theoretical approach).

For the purposes of this analysis, we only consider the case when the *expected* sediment loading shares from the polluting sites changes with information acquisition.[2] In this simplified model, we can

[2] It may be important to consider higher moments of the distributions, especially if (*i*) we do not know the distribution a priori; or (*ii*) we assume the distribution is not normal, since the first two moments are sufficient for characterizing the normal distribution. However, to properly address the role of information acquisition in NPS pollution control, we reserve these higher moment problems for future analysis.

drop the variance of the sediment load parameter without altering the results by assuming the loss function for ignoring the variance is quadratic (Morgan and Henrion [8]).

After the rain season has ended and the information has been extracted from the data, the manager chooses x^k, the treatment level for each site in order to minimize expected damages. In this formulation, we can look directly at the tradeoff the manager faces between treatment effort and treatment effectiveness, where the latter depends on information about the sediment loading distribution across the sites.

The sequential nature of this problem requires the manager to solve a two-period sequential problem. A single budget is allocated for the total costs incurred over both periods. Figure 1 illustrates the manager's sequential decision tree. In Period 1, the manager collects data on total sediment loading with frequency α and updates the sediment loading distributions. In period 2, having collected data, treatment at the polluting sites (represented by vector X) is decided so as to minimize the expected damage based on the updated knowledge of loading distributions.

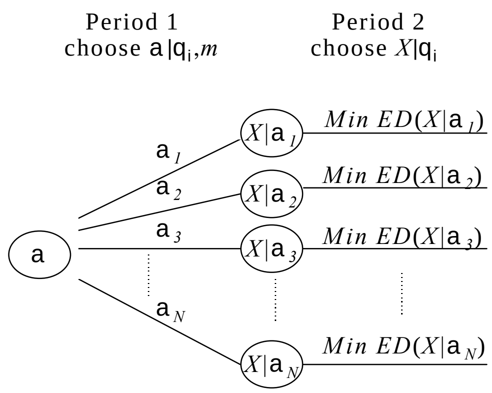

Figure 1. Information Constrained Decision Tree

Solving backwards, the manager knows the minimum expected damage associated with the data collection level, so that, in the first period, the manager only needs to chose the level of data collection that corresponds with the minimum expected damages. The optimization program in the second period is:

$$\underset{X|a,\Omega}{Min}\, ED\left[\left(Q - \sum_{k=1}^{K} f^k(x^k, s_1(a, f, q_0))\right)\right] \qquad (1)$$

$$s.t.\ \sum_{k=1}^{K} c^k(x^k, q^k)x^k + a\,m = B \qquad (2)$$

where, E is the expectation operator taken with respect to s_1, a vector of posterior expected sediment loading shares dependent on a, the data collection frequency, on f, a parameter reflecting the productivity of information acquisition ($s_f > 0$, $s_{af} > 0$), and q_0, the vector of prior expected sediment loading parameters from the underlying sediment loading distribution. The individual shares contained in s_1 are updated by use of the entropy metric described earlier such that as data collection (a) increases the shares representing the dirtier sites will increase and vice versa. We compute the expected sedimen loading shares by dividing the expected posterior sediment loading parameters by Q, the actual cumulative sediment load from the polluting sites in the absence of any treatment.[3] Furthermore, with the new information contained in the collected data, the updating of the expected shares, toward the true underlying distribution of pollution loading increases as f increases. In essence, f represents the underlying human capital held by the manager, which allows her to better utilize the collected data. We assume $D(z)$ is a twice continuously differentiable ($D_z > 0$, $Dzz > 0$) damage function that depends on $z = Q - \sum_k f^k(x^k; s_1)$, the level of excess sediment in the creek. The sediment abatement function $f(.)$ is assumed to be twice continuously differentiable ($f_x > 0, f_{xx} < 0, f_s > 0$ and $f_{xs} > 0$).[4] $c^k(x^k, q^k)$ is the per unit treatment cost at each road crossing, q^k is a parameter reflecting the state of costs at the time decisions are being considered, with $c_x > 0$, $c_{xx} \geq 0$, $c_q < 0$ and $c_{xq} < 0$. m is the per unit data

[3] Because Q is known with certainty, and the shares sum to unity, a change in any sediment-loading share must be offset by an opposite change in at least one other expected sediment-loading share.
[4] This last property implies that the dirtier a particular site the higher the level of marginal abatement productivity at that site.

collection cost, B is the annual budget, and h is the Lagrangian multiplier (the shadow price of the budgeted resources) on (2). Differentiating (1) with respect to x^k, the first order conditions for the optimal treatment levels in Period 2 are

$$-ED_z\left[Q-\sum_k f^k(x^k,s^k(\mathrm{a},\mathrm{f},m))\right]f^k_{x^k}(x^k,s^k(\mathrm{a},\mathrm{f},m))+\mathrm{h}(c^k_{x^k}(x^k)x^k+c^k(x^k))=0,\forall k \quad (3)$$

We can rewrite (3) as

$$\frac{ED_z f^k_{x^k}}{ED_z f^j_{x^j}}=\frac{c^k_{x^k}x^k+c^k}{c^j_{x^j}x^j+c^j},\forall k\neq j \qquad (4)$$

which recasts the optimality condition in the form of the familiar requirement that the marginal rate of transformation across sites should equal the relative marginal treatment cost (the point at which the budget constraint and the iso-expected damage curve are tangent). For a given treatment level, additional data collection changes the uncertainty and hence the ratio on the left-hand side of (4). With less uncertainty associated with the expected sediment loading shares due to data collection, and assuming that the acquired information changes the heterogeneity of sediment loading among the sites, then the expected marginal reduction in damage at the relatively more productive site will increase, and vice versa. This implies that the additional information pivots the iso-expected damage curve, for a given treatment expenditure, towards the more productivity site (see Figure 2).

Prior to data collection, the manager chooses treatment such that the iso-expected damage curve d^0d^0, is tangent to the budget constraint B^0B^0. Note that the iso-expected damage curves decrease the farther away they are from the origin. With data collection, the budget constraint shifts inward to B^1B^1, and the iso-expected damage curve pivots at the pre-data collection allocation to d^1d^1. When x^0 is chosen the actual damage does not change with a change in the iso-expected damage curve from d^0d^0 and d^1d^1. Two cases are shown in Figure 2. In the first case, the pivot in the iso-expected damage curve with data collection is greater than the decrease in resources available for treatment so that the manager can further reduce expected damages by moving to the iso-expected damage curve d^2d^2. The manager will continue to collect data up to the point where the decrease in the resources devoted to treatment are equal to the pivot in the iso-expected damage curve. In the second case, the pivot in the iso-expected damage curve cannot offset the reduction in the budget when data is collected and a higher level of expected damage is reached at d^2d^2. In this latter case, we would expect the manager to forgo data collection.

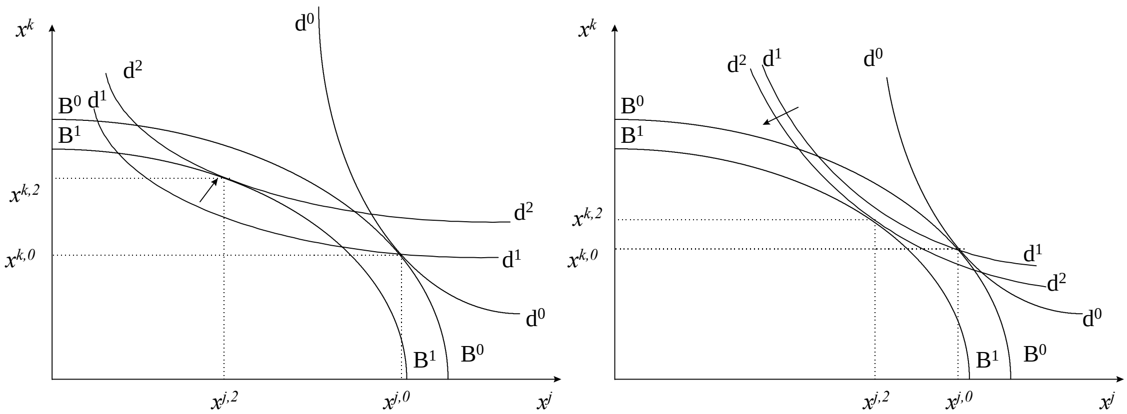

Figure 2. Optimal Budget Allocation with Costly Information

In the first period problem, the manager chooses the level of data collection corresponding with the minimum expected damage. Substituting the optimal treatment as a function of data collection (α) into (1) and differentiating with respect to α, the first order condition for the optimal level of data collection is

$$-ED_z\left[\sum_{k=1}^{K}(f_s^k s_\alpha + f_{x^k}^k x_s^k s_\alpha)\right] + \eta\left(m + \sum_{k=1}^{K}(c_{x^k}^k x^k + c^k)x_s^k s_\alpha^k\right) = 0 \qquad (5)$$

Noting that information acquisition is akin to a public good, Condition (5) has a straightforward interpretation: the manager optimally allocates resources to data collection such that the sum of the marginal productivity of information acquisition over all sites equals the marginal cost of acquiring information.

III. Comparative Static Analysis

The comparative static results for the costly information acquisition model are derived in the standard manner (see Appendix II for complete derivations). Because of the sequential nature of this problem, the timing of any cost change determines the relevant comparative static result. If a change in state of treatment costs occurs in Period 2, then the manager changes the optimal allocation of treatmen across the sites. A decrease in q^1 implies an unfavorable change in the state of costs, represented by an outward shift in the average cost curve. This unfavorable change in the state of costs can arise from several factors including landslides within the treatment area or regulation on treatment effort to preserv endangered species such as the Spotted Owl or the Marbled Murrelet. Table I lists the comparative stat results for both periods.

Table I. Comparative Static Results

	a	q^1	m(Period2)	m(Period1)	f
x^1	?	+	-	?	?
x^2	-	?	-	?	?
a				?	?

The first column of Table I represents the endogenous variables. Optimal data collection (a) will change in Period 1 if either the data collection cost (m) or the information productivity (f) changes. The optimal treatment levels change during Period 2 in response to data collection changes in Period 1, where data collection is now taken to be pre-determined. In addition, optimal treatment levels in Perioc

2 will be affected by changes in the cost of data collection through the budget constraint and through changes in the state of costs (θ).

The sign of the comparative static $\partial x^j / \partial \theta^k$ is ambiguous and depends on the change in total treatment expenditures at site *i* for a given change in θ^i. The change in total treatment expenditure at site *i*, depends on $\varepsilon_{c,x}$, $\varepsilon_{x,\theta}$ and $\varepsilon_{c,\theta}$, the elasticity of average cost with respect to a change in the level of treatment, the elasticity of treatment with respect to a change in the state of costs, and the elasticity of average cost with respect to a change in the state of costs, respectively. Differentiating the budget constraint with respect to θ^l we derive: $\operatorname{sgn}\left(\frac{\partial x^2}{\partial \theta^1}\right) = -\left[\varepsilon_{c^1,\theta^1} + \varepsilon_{x^1,\theta^1}\left(\varepsilon_{c^1,x^1} + 1\right)\right]$.

ε_{c^1,x^1} can take on any sign, whereas $-\varepsilon_{c^1,\theta^1} / \varepsilon_{x^1,\theta^1}$ is positive. However, if we assume that the fiscally constrained manager faces a U-shaped, average cost curve, then we expect the manager to be to the left of the minimum average cost, where $\varepsilon_{c_1,x_1} < 0$.

We thus have:

Proposition I: When, at the optimum, average cost is decreasing (i.e., $\varepsilon_{c_1,x_1} < 0$*), then provided that* $-\varepsilon_{c^1,\theta^1} / \varepsilon_{x^1,\theta^1} > \varepsilon_{c^1,x^1} + 1$*, a favorable change in the state of costs at site 1 will result in a decrease in total treatment expenditure at that site and hence an increase in the treatment at site 2, and vice versa.*

Proposition I implies that when the average cost is falling it is possible for expenditures at site 1 to *decrease* even if the percentage decrease in average cost is less than the percentage increase in treatment at site 1 when the state of costs at site 1 is favorable. Note that a sufficient condition for

$-\varepsilon_{c^1,\theta^1} \big/ \varepsilon_{x^1,\theta^1} > \varepsilon_{c^1,x^1} + 1$ is that $\varepsilon_{x^1,\theta^1} \le -\varepsilon_{c^1,\theta^1}$, i.e., a favorable change in θ^1 must lower c^1 by a greater percentage than it raises x^1 so that c^1x^1 falls and the allocable budget on Site 2 rises.

In Period 2, we also consider the comparative static results for changes in α and *m*. Note that since in Period 2 no data collection occurs, a change in *m* will only affect the treatment levels. Assuming that Site 1 is more productive (i.e., has a higher actual sediment loading), we see the optimal level of treatment at the more productive site will increase for low data collection costs and decrease for high data collection costs. This occurs because the budget constraint limits the manager's ability to improve overall treatment productivity by increasing data collection without paying the price for fewer resources available for allocation to treatment efforts.

If, however, a change in data collection cost occurs in Period 1, then the manager can reallocate the budget to both treatment and data collection. We consider the change in *m* in Period 1 only since that will affect both the treatment expenditure and data collection expenditure and therefore allows us to capture the trade-off between them. A change in *m* in Period 2 affects (x^1, x^2) through the budget and data collection only if data collection occurs also in Period 2. In the sequential case, however, there are no such data collection effects.

Now, in the sequential case, which we are considering, a change in *m* in Period 1 affects α through two channels: directly, as *m* increases, the manager responds by altering α in Period 1, this in turn generates a chain of secondary effects, by changing s^k and hence (x^1, x^2), which in turn changes $\sum_k c^k(x^k)x^k$ and thus $\left[B - \Delta\sum_k c^k(x^k)x^k\right] = \Delta(\alpha\, m)$, and hence α. This chain of effects is depicted below:

$$
\begin{array}{l}
\Delta m \rightarrow \Delta\alpha \rightarrow \Delta s \rightarrow \Delta\left(x^1, x^2\right) \\
\qquad\quad \uparrow \qquad\qquad\quad \downarrow \\
\qquad \left[B - \Delta\alpha m - \Delta\left(C^1 x^1, C^2 x^2\right)\right]
\end{array}
$$

and is formally captured by (See Appendix II for derivation)

$$
\operatorname{sgn}\left(\frac{\partial\alpha}{\partial m}\right) = -\operatorname{sgn}\left(\alpha + (c_x^1 x^1 + c^1)x_m^1 + (c_x^2 x^2 + c^2)x_m^2\right)\left(m + \sum_k (c_x^k x^k + c^k)x_s^k s_\alpha^k\right) \qquad (6)
$$

The second term on the right-hand side (RHS) of equation (6) is the marginal cost of data collection, which is positive from the first order condition (5). The first term on the RHS of (6), which we term the "sequential effect," may take on any sign. The sequential effect represents the tradeoff between higher data costs through the data collection variable α and lower costs through the treatment variables. From (6) we can state the following proposition:

Proposition II: If, with a higher cost of data collection (m), the <u>decrease</u> in treatment expenditures is larger than the <u>increase</u> in data collection expenditure, then the sequential effect is <u>negative</u>, thus inducing an <u>increase</u> in the level of data collection.

This result seems counterintuitive at first. We would expect a decrease in data collection to result from an increase in data collection cost. But, because of the sequential effect from data collection, the net savings from treatment costs in Period 2 exist to enhance data collection efforts.

When data collection occurs and information is acquired on the degree of heterogeneity for sediment loading among the polluting sites, then the response of treatment at these sites will be determined by the size and direction of change in the budget and abatement productivity effects. The budget effect is a decline in treatment at all sites because the resources used to collect data must be taken from the same budget allocated to treatment. In other words, any resources spent on data collection are unavailable for treatment. The abatement productivity effect is a decline in the productivity at sites with lower posterior expected sediment loading shares and a rise in the productivity at sites with higher posterior expected sediment loading shares.

The importance of these effects comes from the shape of the average treatment cost curves. Recall that where the average treatment cost curve is U-shaped, the constrained manager is to the left of the minimum of the average treatment cost. The budget effect will increase the average cost of treatment. The abatement productivity effect either compounds or countervails the budget effect because it moves in opposite direction given the posterior expectations on the sediment loading shares. At the ex-post dirtier sites, the average treatment cost increases due to the budget effect but decreases as the abatement productivity increases. As a result, with a lower data collection cost (*m*), treatment at the dirtier sites will increase when more information is acquired. At the ex-post cleaner sites, however, the average treatment cost will increase because the abatement productivity effect compounds the budget effect to result in a further decrease in the treatment level thereby raising the average cost at the cleaner site.

We have also considered the effect of a change in the productivity of information acquisition (ϕ) on data collection. This is given by

$$\operatorname{sgn}\left(\frac{\partial \alpha}{\partial \phi}\right) = -\operatorname{sgn}((c_x^1 x^1 + c^1)x_s^1 s_\phi^1 + (c_x^2 x^2 + c^2)x_s^2 s_\phi^2)(m + \sum_k (c_x^k x^k + c^k)x_s^k s_\alpha^k) \qquad (7)$$

Once again, we obtain an ambiguous result since treatment at sites 1 and 2 move in opposite directions. The first term on the RHS of (7) we refer to as the "information productivity effect" since it reflects the change in treatment expenditures given a change in the productivity of information acquisition. When information acquisition becomes more productive, the curvature conditions on the abatement functions tells us that, at some level of treatment (x^1, x^2), the cost of treatment at Site 1 increase less than it decreases at Site 2.[5] This leads us to

Proposition III: If a higher productivity of information acquisition (ϕ) raises the treatment expenditures at site 1 by more than it reduces the treatment expenditures at site 2, then the information productivity effect is <u>negative</u>, thus inducing a <u>decrease</u> in the level of data collection.

IV. Model Simulation

To provide a numerical illustration, we consider a simplified pollution control problem modeled after the rehabilitation program for Lower Redwood Creek located in Redwood National Park. Overall, we consider three models. Model I presents the case of the perfectly informed manager; Model II that of the uninformed manager who is assumed to believe that the sediment loading is *uniform* across all polluting sites; and Model III is the case of the data-collecting, incompletely-informed manager. Three polluting sites exist in each model.

The first set of simulations compares model I and II, under a range of scenarios about the heterogeneity of sediment loading across the sites. These scenarios have been selected to cover a wide range of possible sediment loading distributions imposed by the natural environment. For each scenario, the extent of extra damage attributable to the rather implausible assumption of a prior uniform distribution on sediment loading is indicated. Accordingly, the simulations vividly illustrate how the

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[5] From the conditions on the treatment cost functions we know that $c_x^k x^k + c^k > 0$, and $c_{xx}^k x^k + 2c_x^k > 0$. Furthermore, as derived in Appendix II, $x_s^1 s_\phi^1 > 0$, and $x_s^2 s_\phi^2 < 0$. Taken together, these conditions imply that when data collection becomes more productive, the cost of treatment at Site 1 increases but at an increasing rate whereas at Site 2, the cost of treatment decreases at a decreasing rate. Therefore, at some level of treatment (x^1, x^2) the increase in treatment cost at Site 1 will more than offset the decrease at Site 2.

manager can exploit the sediment loading heterogeneity to reduce damages. Further, these scenarios allow us to evaluate the value of perfect information in the choice of treatment levels.

The second set of simulations evaluates Model III to determine the optimal level of data collection for selected heterogeneity scenarios. Once again, we compare each scenario with the assumption of a prior uniform distribution on sediment loading across polluting sites. These latter simulations allow us to see the value of data collection in controlling sediment loading.

To facilitate the simulation, we assume simple functional forms for the various processes involved in controlling sediment within the Park, where all the variables have been defined above. First, the damage function is $D = d\left(Q - \sum_k f^k\right)^2$. The abatement function is $f^k = 900 \cdot \left(x^k\right)^{\frac{1}{2}} \cdot \left(s^k\right)^{\frac{1}{2}}$. The treatment cost function for each site is $C_i = c_i x_i^2$. Finally, the information acquisition function is $A(\alpha) = 0.1\, h(1-(1-\phi)^{\alpha})$, where h is the true state of sediment loading heterogeneity, and $0 < \phi < 1$. This information acquisition function is used to derive a posterior distribution with a more accurate expected value than the prior distribution, where the posterior expected sediment loading share for Site 1 = 0.0001(3333.4 + (A-1)*4000.6), for Site 2 = 0.0001(3333.4 - 0.2(A-1)*4000.6), and for Site 3 = 0.0001(3333.4 - 0.8(A-1)*4000.6). In this sense, $A(\alpha)$ updates the expected sediment loading shares for each of the polluting sites. Notice that as α increases from 0 to ∞, the prior uniform distribution held in the absence of any information acquisition is revised towards the true distribution.

The parameters of the model are assigned crude estimates as follows. The total ambient sediment load is fixed at 10,000 cubic yards. We vary the heterogeneity of loading from each site over eleven scenarios from perfectly homogenous sites with an entropy measure of 1 to heterogeneous sites with an entropy measure of 0.576 to reflect a few of the possible sediment loading distribution that nature imposes on the system (See Table II). The heterogeneity is measured with Shannon's entropy metric [12], $\sum_k s^k \ln s^k$. In the second set of simulations we consider the costly information acquisition and for illustrative purposes we limit the heterogeneity to scenario 7 with an entropy measure of 0.867. Selection of any other scenario does not alter the results. Table III lists the remainder of the parameters

Table II. Heterogeneity Scenarios[6]

Scenario	*Site1*	*Site2*	*Site3*	$\sum s^k \ln s^k$
1	3333	3333	3333	1
2	3733	3253	3013	0.996
3	4133	3173	2693	0.985
4	4534	3093	2373	0.967
5	4934	3013	2053	0.942
6	5334	2933	1733	0.909
7	5734	2853	1413	0.867
8	6134	2773	1093	0.817
9	6534	2693	773	0.755
10	6934	2613	453	0.678
11	7334	2533	133	0.576

Table III. Parameter Values

Parameter	
d	\$0.02/(cubic yd)^2
c_1	\$15,000
c_2	\$10,000
c_3	\$25,000
B	\$500,000
f	0.1, 0.05

The results from the first set of simulations shows the divergence between the actual damage levels for the perfectly informed treatment (PI) and uninformed treatment (UI), for

[6] To derive the heterogeneity scenarios we applied the following formulas:
Site 1 = 3333.4 + 0.1(h-1)*4000.6, Site 2 = 3333.4 - 0.02(h-1)*4000.6, and Site 3 = 3333.4 - 0.08(h-1)*4000.6. For computational convenience and without altering the qualitative results we use h to be the scenario number (i.e., h=1,….,11). Any other derivation of the set of scenarios is equally admissible.

Model I and II. Table IV contains the optimal treatment levels for the perfectly informed manager under the various heterogeneity scenarios. For the *uninformed* manager (UI) who assumes a prior uniform distribution over all heterogeneity scenarios, the first row of Table IV (x^1 = 4.980, x^2 = 7.469, x^3 = 2.988) represents the treatment levels applied to control sediment loading under *every* heterogeneity scenario. Column 1 in Table V shows the damage corresponding to the optimal treatment levels under each scenario (see Table IV) when the manager has perfect information about the distribution of ambient load across sites. Column 2 in Table V shows the damage resulting from uniformed budget allocation. Here the manager's choice of treatment levels always remains the same as presented for scenario 1 in Table IV, but depending on which one of the true scenarios may actually occur, the distribution of sediment loads, s^k, and therefore the actual level of abatement, f^k, will differ. In turn, this leads to a different actual extent of damage for the different scenarios. We see from Table V that when the manager has perfect information less damage results, because of the manager exploits the greater degree of heterogeneity in allocating the budget more efficiently. We also see that the marginal value of perfect information (MV) increases as the difference between the true distribution and the prior distribution grows (i.e., as heterogeneity rises).

Table IV. Optimal Treatment in Miles (Perfect Information)

Scenario	x^1	x^2	x^3
1	4.980	7.469	2.988
2	4.980	7.469	2.988
3	5.260	7.362	2.834
4	5.525	7.263	2.676
5	5.783	7.167	2.512
6	6.037	7.077	2.336
7	6.285	6.991	2.150
8	6.532	6.912	1.945
9	6.781	6.839	1.717
10	7.035	6.775	1.452
11	7.303	6.725	1.119

Table V. Damage ($1,000) for Model I and II[7]

Scenario	Model I (PI)	Model II (UI)	Value of Perfect Information (VPI)	Marginal Value (MV)
1	850.7	850.7	0.0	0.0
2	850.7	850.7	0.0	0.0
3	848.7	849.1	0.4	0.4
4	847.5	849.1	1.6	1.2
5	847.0	850.7	3.6	2.0
6	847.4	853.9	6.5	2.8
7	848.6	858.9	10.3	3.8
8	850.8	866.0	15.2	4.9
9	854.1	875.8	21.6	6.4
10	858.9	889.1	30.2	8.6
11	865.7	908.3	42.6	12.4

[7] Scenario 1 and 2 are identical due to rounding.

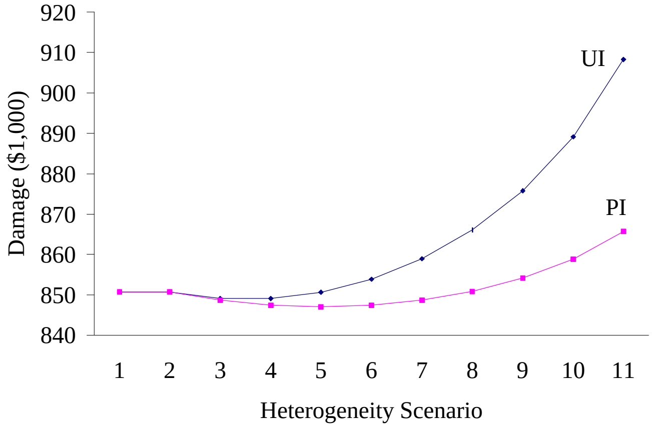

Figure 3. Damage for Perfectly Informed and Uninformed Treatment

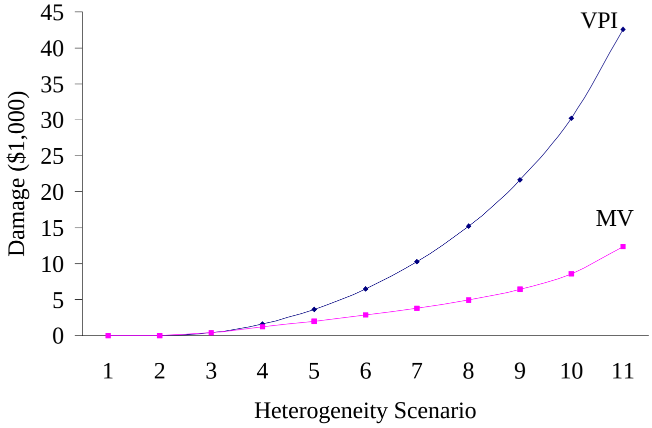

Figure 4. Total and Marginal Value of Perfect Information

Focusing on Scenario 7, the second set of simulations depicts a U-shaped relationship between expected damage and data collection. In each scenario we consider different values for *m*, the data collection cost and different values for f , the information productivity parameter (see Tables VIa and VIb). Since with limited budget and costly information acquisition, it is never optimal to fully resolve uncertainty, we see in these tables that the actual damage will always exceed the expected (minimum) damage corresponding to an efficient level of data collection. We also observe that when it is optimal to acquire information, the marginal value of the first data point is significantly larger than the value for successive data points, which suggests that the slightest improvement in information has a large influence on the control of NPS pollution.

Table VIa. Expected and Actual Damage ($1,000) with Data Collection (f = 0.1)[8]

Expected Damage

a	*m*=500	*m*=750	*m*=1250
0	850.74	850.74	850.74
1	849.92	850.07	850.37
2	849.39	849.69	**850.30**
3	849.08	**849.53**	850.45
4	**848.94**	849.55	850.77
5	848.95	849.71	851.23
6	849.05	849.96	851.80
7	849.23	850.30	852.45
8	849.48	850.70	853.16
9	849.77	851.15	853.91

Actual Damage

a	*m*=500	*m*=750	*m*=1250
0	858.89	858.89	858.89
1	857.27	857.42	857.72
2	856.00	856.30	**856.91**
3	855.04	**855.50**	856.41
4	**854.33**	854.93	856.15
5	853.81	854.57	856.09
6	853.44	854.35	856.18
7	853.19	854.26	856.40
8	853.05	854.27	856.72
9	853.05	854.37	857.13

Table VIb. Expected and Actual Damage ($1,000) with Data Collection (f = 0.05)

Expected Damage

a	*m*=500	*m*=750	*m*=1250
0	850.74	850.74	**850.74**
1	850.45	850.60	850.91
2	850.25	**850.55**	851.16
3	850.11	850.56	851.48
4	850.03	850.64	851.86
5	**850.01**	850.77	852.30

[8] Raising a from 3 to 4 lowers the expected damage by less than the direct cost of doing so. This apparent inconsistency can be explained by condition (5). From (5) we see that h is the value of one dollar from the budget spent on data collection. Thus a unit increase in a will impose two cost components, (*i*) a direct cost equal to h*m* which accounts for the marginal dollar increase in the damage resulting from a diversion of one dollar of the budget from treatment to data collection, and (*ii*) an implicit cost equal to $h\left(\sum_k (c^k_{x^k} x^k + c^k) x^k_s s^k_a\right)$, which accounts for the indirect effect of an additional unit of information on lowering the amount of treatment expenditure through its effect on expected loading shares s_a and hence on the treatment at various sites. This indirect effect is negative and thus the decrease in expected damage does not equal the direct cost of data collection.

6	850.03	850.95	852.78
7	850.10	851.17	853.31
8	850.20	851.42	853.88
9	850.34	851.71	854.48

Actual Damage

α	*m*=500	*M*=750	*m*=1250
0	858.89	858.89	**858.89**
1	858.21	858.36	858.66
2	857.61	**857.91**	858.52
3	857.10	857.56	858.47
4	856.68	857.29	858.50
5	**856.33**	857.09	858.61
6	856.04	856.95	858.78
7	855.81	856.88	859.02
8	855.64	856.86	859.31
9	855.51	856.88	859.61

Further we see from the second set of simulations that when the productivity in information acquisition increases (ϕ increasing from 0.05 to 0.1) data collection effort decreases, which accords with *Proposition III*. Indirectly, the effect of a higher information productivity (ϕ) is like lowering the cost of information, so that in accordance with *Proposition II*, one would expect a reduction in the level of data collection. Furthermore, we observe the role that the optimal information acquisition plays in improving the budget allocation and hence reducing the expected damage when compared with the case of ex ante, uniform prior distribution. When, under heterogeneity scenario 7, data is optimally collected, (with the resulting expected damages of $[848.94, 849.53, 850.30]$ for $\phi = 0.1$ and $[850.01, 850.55]$ for $\phi = 0.05$ respectively) the actual damage, $[854.33, 855.50, 856.91]$ for $\phi = 0.1$ and $[856.33, 857.91]$ for $\phi = 0.05$ respectively, is always *lower* than the damage (D = 858.89) that would result from an uninformed budget allocation only to treatment efforts. In essence, the actual damage under the uninformed manager and perfectly informed manager bound, respectively, from above and below, the actual damages that would occur when data is optimally collected.

V. Concluding Remarks

This paper examines NPS pollution control under incomplete and costly information. We have analyzed the problem within a constrained management framework to bring to light a more realistic setting for studying NPS pollution control. The nature of this problem required a sequential optimization approach. The comparative static results show the manager will vary her decisions, over the two periods, if she has the ability to change all allocation decisions and not just allocation to treatment effort across sites. The numerical results show that the manager can exploit the heterogeneity of the sites to such a level that significant gains in damage reductions can be achieved.

Future analysis should examine the problem in a dynamic setting. Over an arbitrary period of time, the manager chooses investment paths for both information acquisition and treatment efforts. During this time horizon, several factors will influence the dynamics of each path. Principal among these factors is the decline of the productivity of information acquisition as uncertainty about the degree of heterogeneity is reduced. This suggests that the manager will decrease data collection and information acquisition over time. Secondly, as treatment at the site with the largest sediment load occurs early in the time horizon, the system will become increasingly less heterogeneous (Kestenbaum [4]). With a decreasing heterogeneity of sediment loading over time, we expect that at some future date the treatment policy will change from a heterogeneous treatment strategy to a homogeneous treatment strategy. Future research will shed light on this issue.

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Appendix I
Information Theoretical Approach

Information entropy is measured as $I(q) = \int_{-\infty}^{\infty} h(q) \ln h(q) dq$, where $h(q)$ is the probability distribution on q. Originally, Shannon [12] used this metric to measure uncertainty from receiving a noisy signal and we follow by characterizing the uncertainty on sediment loading with this metric. The entropy metric allow us to attribute information acquisition (or a reduction in uncertainty) when the expected shares of sediment loading move away from uniform, since the entropy over the shares is lower for any distribution other than the uniform (or least informative) distribution. The divergence between the prior and posterior distributions is measured by the Kullback divergence statistic [6],

$$K(q_1 : q_0) = \int_{-\infty}^{\infty} h(q_1) \ln\left(\frac{h(q_1)}{h(q_0)}\right) dq .$$

This divergence is the marginal product of data collection when the distributions represent the information from two consecutive observations on Q, the ambient sediment load. The lower bound on K is zero, which occurs when the prior and posterior distributions are identical. The upper bound is the entropy measure on the prior distribution, since in the extreme case of a collapsed posterior distribution with an entropy measure of zero all the information is acquired, and thus the marginal information gain is that of the entropy measure on the prior distribution. The benefit of this information metric is that it allows us to consider changes in *any* moment of the distribution and thus is more general than the traditional standard deviation-mean ratio approach for measuring uncertainty.

Appendix II
Comparative Static Derivation

The appendix shows the comparative static analysis for the above optimization problem. All notation and properties hold as above[9]. To keep the results clearer, we redefine several terms along the way. We consider a two polluting site example since the results are general for a K-site problem. Recursively solving the sequential problem, the optimization problem in Period 2 is

$$\underset{X|\mathrm{a},\Omega}{Min}\, ED\left[\left(Q-\sum_{k=1}^{2} f^{k}(x^{k}, s^{k}(\mathrm{a},\mathrm{f}\ ,q_{0}))\right)\right]$$
$$s.t.\ \sum_{k=1}^{2} c^{k}(x^{k},\mathrm{q}^{k})x^{k} + \mathrm{a}\, m = B$$

E is the expectation operator taken with respect to s, a vector of posterior expected sediment loading shares dependent on a, h, a parameter reflecting the productivity of information acquisition, and q0, the vector of prior expected sediment loading parameters from the underlying sediment loading distribution. The Lagrangian is

$$\mathrm{L} = ED\left[\left(Q-\sum_{k=1}^{2} f^{k}(x^{k}, s^{k}(\mathrm{a},\mathrm{f}\ ,q_{0}))\right)\right] + \mathrm{h}\left(\sum_{k=1}^{2} c^{k}(x^{k},\mathrm{q}^{k})x^{k} + \mathrm{a}m - B\right)$$

The Kuhn-Tucker conditions for Period 2 (assuming an interior solution) are:

$$-ED_{z} f^{k}_{x^{k}}(x^{k}, s^{k}(\mathrm{a},\mathrm{f}\ ,q_{0})) + \mathrm{h}(c^{k}_{x^{k}} x^{k} + c^{k}) = 0, x^{k} > 0, \forall k \qquad \text{(II.1)}$$

$$\sum_{k=1}^{2} c^{k}(x^{k},\mathrm{q}^{k})x^{k} + \mathrm{a}\, m - B = 0, \mathrm{h} > 0 \qquad \text{(II.2)}$$

$$|\overline{H}| = \begin{vmatrix} ED_{zz}(f^{1}_{x^{1}})^{2} - ED_{z} f^{1}_{x^{1}x^{1}} + \mathrm{h}\left(c^{1}_{x^{1}x^{1}} x^{1}x^{1} + 2c^{1}_{x^{1}}\right) & 0 & c^{1}_{x^{1}}x^{1} + c^{1} \\ 0 & ED_{zz}(f^{2}_{x^{2}})^{2} - ED_{z} f^{2}_{x^{2}x^{2}} + \mathrm{h}\left(c^{2}_{x^{2}x^{2}} x^{2}x^{2} + 2c^{2}_{x^{2}}\right) & c^{2}_{x^{2}}x^{2} + c^{2} \\ c^{1}_{x^{1}}x^{1} + c^{1} & c^{2}_{x^{2}}x^{2} + c^{2} & 0 \end{vmatrix}$$

$$|J| = |\overline{H}| = \begin{vmatrix} L_{11} & 0 & L_{1\eta} \\ 0 & L_{22} & L_{2\eta} \\ L_{\eta 1} & L_{\eta 2} & 0 \end{vmatrix} < 0$$

By the implicit function theorem, we can, in principle, solve (II.1) and (II.2) for $x^k = x^k(\beta)$ and $\eta = \eta(\beta)$, where $\beta = (\theta, m, s(\alpha, \phi), B)$. Substituting these functions into equations (II.1) and (II.2) yields:

$$-ED_z f^k_{x^k}(x^k(\beta), s^k(\alpha, \phi, q_0)) + \eta(c^k_{x^k} x^k(x^k(\beta)) + c^k) \equiv 0, \forall k \qquad \text{(II.3)}$$

$$\sum_{k=1}^{2} c^k(x^k, \theta^k)x^k + \alpha m - B \equiv 0 \qquad \text{(II.4)}$$

We can now consider how the optimal data and treatment decisions change with a change in any of the parameters in the Period 2. We do not consider a change in B because the results are not surprising. So, differentiating the identities (II.3) and (II.4) with respect to θ^1 yields:

$$[J]\begin{bmatrix} \dfrac{\partial x^1}{\partial \beta} \\ \dfrac{\partial x^2}{\partial \beta} \\ \dfrac{\partial \eta}{\partial \beta} \end{bmatrix} = \begin{bmatrix} -\eta(c^1_{x^1\theta^1}x^1 + c^1_{\theta^1}) \\ 0 \\ -c^1_{\theta^1} \end{bmatrix}.$$

By Cramer's Rule,

$$x^1_{\theta^1} = \frac{\eta(c^1_{x^1\theta^1}x^1 + c^1_{\theta^1})(c^2_{x^2}x^2 + c^2)^2}{|J|} + \frac{c^1_{\theta^1}(c^1_{x^1}x^1 + c^1)L_{22}}{|J|} > 0$$

$$x^2_{\theta^1} = \frac{(c^2_{x^2}x^2 + c^2)\left[L_{11}c^1_{\theta^1} - \eta(c^1_{x^1}x^1 + c^1)^2\right]}{|J|}$$

---

9 We assume damages are additively separable.

To resolve the ambiguity for $x^2_{\theta^1}$, we return to identity (II.3). Differentiating (II.3) with respect to θ^1 yields,

$$\left[\frac{\partial c_2}{\partial x_2} c^2_{x^2} x^1 + c^2\right] x^2_{\theta^1} = -c_1\left[c^1_{\theta^1}\frac{x_1}{c_1} + x^1_{\theta^1}\left(c^1_{x^1}\frac{x_1}{c_1} + 1\right)\right]$$

or, $$\operatorname{sgn}\left(x^2_{\theta^1}\right) = -\frac{c^1 x^1}{\theta^1}\left[\varepsilon_{c^1\theta^1} + \varepsilon_{x^1\theta^1}\left(\varepsilon_{c^1x^1} + 1\right)\right] \tag{II.5}$$

Equation (II.5) provides the necessary condition, in terms of elasticity, for determining the change in treatment at Site 2 given a change in the state of costs at Site 1.

Differentiating the identities (II.3) and (II.4) with respect to *m* yields:

$$[J]\begin{bmatrix} \frac{\partial x^1}{\partial \beta} \\ \frac{\partial x^2}{\partial \beta} \\ \frac{\partial \eta}{\partial \beta} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -m \end{bmatrix}$$

$$x^1_m = \frac{mL_{22}(c^1_{x^1}x^1 + c^1)}{|J|} < 0$$

$$x^2_m = \frac{mL_{11}(c^2_{x^2}x^2 + c^2)}{|J|} < 0$$

For the next set of comparative static results we will assume, without loss of generality, that when information is acquired the expected sediment load share increases for site 1 and decreases for site 2. In addition, since the shares must sum to unity, then it follows that $s^1_\alpha + s^2_\alpha = 0$.

Differentiating the identities with respect to α yields:

$$[J]\begin{bmatrix}\dfrac{\partial x^1}{\partial \beta}\\ \dfrac{\partial x^2}{\partial \beta}\\ \dfrac{\partial \eta}{\partial \beta}\end{bmatrix}=\begin{bmatrix}-\left(ED_{zz}\left[f^1_{s^1}s^1_\alpha - f^2_{S^2}s^2_\alpha\right]f^1_{x^1} - ED_z f^1_{x^1s^1}s^1_\alpha\right)\\ -\left(ED_{zz}\left[f^1_{s^1}s^1_\alpha - f^2_{S^2}s^2_\alpha\right]f^2_{x^2} + ED_z f^2_{x^2s^2}s^2_\alpha\right)\\ -m\end{bmatrix}=\begin{bmatrix}-\alpha^{11}\\ -\alpha^{22}\\ -m\end{bmatrix},$$

The first terms in α^{11} and α^{22} are zero because it is assumed that the concavity of the abatement functions implies that the reduction in expected sediment loading share at site 2 decreases the abatement productivity by the same amount as the increase in abatement productivity at Site 1.

$$x^1_\alpha = \frac{m(c^1_{x^1}x^1 + c^1)L_{22} + \alpha^{11}(c^2_{x^2}x^2 + c^2)^2 - \alpha^{22}(c^1_{x^1}x^1 + c^1)(c^2_{x^2}x^2 + c^2)}{|J|}$$

$$x^2_\alpha = \frac{m(c^2_{x^2}x^2 + c^2)L_{11} + \alpha^{22}(c^1_{x^1}x^1 + c^1) \;\; - \alpha^{11}(c^1_{x^1}x^1 + c^1)(c^2_{x^2}x^2 + c^2)}{|J|} < 0$$

We can see that when m is sufficiently large $x^1_\alpha < 0$.

Differentiating with respect to φ yields:

$$[J]\begin{bmatrix}\dfrac{\partial x^1}{\partial \beta}\\ \dfrac{\partial x^2}{\partial \beta}\\ \dfrac{\partial \eta}{\partial \beta}\end{bmatrix}=\begin{bmatrix}-\left(ED_{zz}\left[f^1_{s^1}s^1_\varphi - f^2_{s^2}s^1_\varphi\right]f^1_{x^1} - ED_z f^1_{x^1s^1}s^1_\varphi\right)\\ -\left(ED_{zz}\left[f^1_{s^1}S^1_\varphi - f^2_{s^2}s^2_\varphi\right]f^2_{x^2} + ED_z f^2_{x^2s^2}s^2_\varphi\right)\\ 0\end{bmatrix}=\begin{bmatrix}-\varphi^{11}\\ -\varphi^{22}\\ 0\end{bmatrix}$$

Again, the first terms in φ^{11} and φ^{22} are zero.

$$x^1_\varphi = \frac{-\varphi^{22}(c^2_{x^2}x^2 + c^2)(c^1_{x^1}x^1 + c^1) + \varphi^{11}(c^2_{x^2}x^2 + c^2)^2}{|J|} > 0$$

$$x^2_\varphi = \frac{-\varphi^{11}(c^2_{x^2}x^2 + c^2)(c^1_{x^1}x^1 + c^1) + \varphi^{22}(c^2_{x^2}x^2 + c^2)^2}{|J|} < 0$$

$$\mathrm{h}_{\mathrm{f}\,2} = \frac{-\mathrm{f}\,^{11}L_{22}(c^{1}_{x^{1}}x^{1} + c^{1}) + \mathrm{f}\,^{22}L_{11}(c^{2}_{x^{2}}x^{2} + c^{2})}{|J|} < 0$$

For calculating the comparative static results in Period 1, the Jacobian is

$$|J| = \begin{vmatrix} \frac{\partial}{\partial \mathrm{a}}\left(-E\sum_{k=1}^{K} D_Z(f^{k}_{s^{k}} + f^{k}_{x^{k}}x^{k}_{s})s^{k}_{\mathrm{a}} + \mathrm{h}(m + \sum_{k=1}^{K}(c^{k}_{x^{k}}x^{k} + c^{k})x^{k}_{s}s^{k}_{\mathrm{a}})\right) & \mathrm{h}m + \mathrm{h}\sum_{k=1}^{2}(c^{k}_{x^{k}}x^{k} + c^{k})x^{k}_{s}s^{k}_{\mathrm{a}} \\ \mathrm{h}m + \mathrm{h}\sum_{k=1}^{2}(c^{k}_{x^{k}}x^{k} + c^{k})x^{k}_{s}s^{k}_{\mathrm{a}} & 0 \end{vmatrix}$$

where $|J| < 0$. In the standard manner we can in principle derive the following functions $\mathrm{a} = \mathrm{a}(\mathrm{c})$ and $\mathrm{h} = \mathrm{h}(\mathrm{c})$ where $\mathrm{c} = (m, \mathrm{f}\,)$. Substituting these functions into the Kuhn-Tucker Conditions yields the identities:

$$-E\sum_{k=1}^{2} D_Z(f^{k}_{s^{k}} + f^{k}_{x^{k}}x^{k}_{s})s^{k}_{\mathrm{a}} + \mathrm{h}(m + \sum_{k=1}^{2}(c^{k}_{x^{k}}x^{k} + c^{k})x^{k}_{s}s^{k}_{\mathrm{a}}) \equiv 0$$

$$\sum_{k=1}^{2} c^{k}(x^{k}, \mathrm{q}^{k})x^{k} + \mathrm{a}\,m - B \equiv 0$$

Differentiating the identities with respect to *m* yields:

$$[J]\begin{bmatrix} \frac{\partial \mathrm{a}}{\partial m} \\ \frac{\partial \mathrm{h}}{\partial m} \end{bmatrix} = \begin{bmatrix} -\mathrm{h}(1 + \sum_{k=1}^{2}(c^{k}_{x^{k}x^{k}}x^{k}_{m}x^{k} + 2c^{k}_{x^{k}}x^{k}_{m})x^{k}_{s}s^{k}_{\mathrm{a}}) \\ -\mathrm{a} - \sum_{k=1}^{2}x^{k}_{m}(c^{k}_{x^{k}}x^{k} + c^{k}) \end{bmatrix}.$$

By Cramer's Rule,

$$\mathrm{a}_{m} = \frac{\mathrm{h}\left(\mathrm{a} + \sum_{k=1}^{2}x^{k}_{m}(c^{k}_{x^{k}}x^{k} + c^{k})\right)\left(m + \sum_{k=1}^{2}(c^{k}_{x^{k}}x^{k} + c^{k})x^{k}_{s}s^{k}_{\mathrm{a}}\right)}{|J|}$$

Now, differentiating with respect to f yields:

$$[J]\begin{bmatrix} \frac{\partial a}{\partial f} \\ \frac{\partial h}{\partial f} \end{bmatrix} = \begin{bmatrix} L_{aa} \\ -\sum_{k=1}^{2} x_{f}^{k} (c_{x^k}^{k} x^{k} + c^{k}) \end{bmatrix}$$

$$a_{f} = \frac{h\left(\sum_{k=1}^{2} x_{s}^{k} s_{f}^{k} (c_{x^k}^{k} x^{k} + c^{k})\right)\left(m + \sum_{k=1}^{2} (c_{x^k}^{k} x^{k} + c^{k}) x_{s}^{k} s_{a}^{k}\right)}{|J|}$$