Endogenous Uncertainty and Market Volatility

by

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Summary.
Endogenous Uncertainty is that component of economic risk and market volatility which is propagated within the economy by the beliefs and actions of agents. The theory of Rational Belief (see Kurz [1994]) permits rational agents to hold diverse beliefs and consequently, a Rational Belief Equilibrium (in short, RBE) may exhibit diverse patterns of Endogenous Uncertainty. This paper shows that *most of the observed volatility in financial markets is generated by the beliefs of the agents and the diverse market puzzles which are examined in this paper, such as the equity premium puzzle, are all driven by the structure of market expectations.* To make the case for this theory we present *a single RBE model,* which builds on developments in Kurz and Beltratti [1997] and Kurz and Schneider [1996], with which we study a list of phenomena that have been viewed as "anomalies" in financial markets. The model is able to predict the correct order of magnitude of:

(i) the long term mean and standard deviation of the price\dividend ratio;
(ii) the long term mean and standard deviation of the risky rate of return on equities;
(iii) the long term mean and standard deviation of the riskless rate;
(iv) the long term mean equity premium.

In addition, the model predicts

(v) the GARCH property of risky asset returns;
(vi) the Forward Discount Bias in foreign exchange markets.

We also conjecture that an adaptation of the same model to markets with derivative assets will predict the appearance of "smile curves" in derivative prices.

The common economic explanation for these phenomena is the existence of heterogenous agents with diverse but correlated beliefs. Given such diversity, some agents are optimistic and some pessimistic. We develop a simple model which allows agents to be in these two states of belief but the identity of the optimists and the pessimists fluctuates over time since at any date any agent may be in these two states of belief. In this model there is a unique parameterization under which the model makes all the above predictions *simultaneously*. That is, although the parameter space of the RBE is large, all parameterizations outside a small neighborhood of the parameter space fail significantly to reproduce some subset of variables under consideration. Any parameter choice in this small neighborhood requires the optimists to be in the majority but the rationality of belief conditions of the RBE require the pessimists to have a higher intensity level. This higher intensity has a decisive effect on the market: it increases the demand for riskless assets, decreases the equilibrium riskless rate and increases the equity premium. In simple terms, the large equity premium and the lower equilibrium riskless rate are the result of the fact that at any moment of time there are agents who hold extreme pessimistic beliefs and they have a relatively stronger impact on the market. The relative impact of these two groups of agents who are, at any moment of time, in the two states of belief is a direct consequence of the rationality of belief conditions and in that sense it is unique to an RBE.

As for the correlation among the beliefs of agents, the paper shows that the dynamics of asset prices are strongly affected by such correlation. The pattern of correlation which was used in the model can be explained intuitively in terms of its effect on the dynamics of prices. The model correlation causes periods of price rises (i.e. bull markets) to develop slower than periods of decline (i.e. bear markets) hence the model dynamics do not permit prices to shoot directly from the bottom to the top but the opposite is possible and takes the form of *market crashes*.

Note: Both the RBE model developed in this paper as well as the associated programs used to solve it are available to the public on Professor Kurz's web page at http://www.stanford.edu/~mordecai/

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The theory of Rational Belief Equilibrium (in short, RBE; see Kurz [1994], [1997]) was developed with the view of studying the effects of beliefs and expectations of economic agents on the volatility of economic variables and on social risk. Application of the theory to various markets were reported by Kurz and Beltratti [1997], by Kurz and Schneider [1996] and by Kurz [1997a],[1998]. These papers advanced the idea that the "equity premium puzzle" (due to Mehra and Prescott [1985]) can be resolved by the theory of rational beliefs. This is in contrast with recent attempts to resolve the equity premium puzzle by the use of a "habit forming" utility function (see Abel [1999], Campbell and Cochrane [1995] and Constantinides [1990]). Other approaches to the equity premium puzzle were reported by Brennan and Xia [1998], Epstein and Zin [1990], Cecchetti, Lam and Mark [1990],[1993], Heaton and Lucas [1986], Mankiw [1986], Reitz [1988], Weil [1989] and many others. Most of the work on the equity premium concentrated on the analysis of the premium, the riskless rate, the risky rate and their second moments. We note that apart from the price volatility controversy generated by Shiller [1981], the "calibration" literature has mostly ignored the comparison between the model's volatility of stock prices and the historical record; such a comparison is one more test of the model's ability to explain the data. Also, financial markets exhibit other dynamical patterns for which standard

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models have failed to give a satisfactory explanation. Examples include the GARCH phenomenon in asset returns, the "Forward Discount Bias" in foreign exchange markets and the various "smile curves" in derivative markets. Hence the validity of any equilibrium theory should not be judged only by its ability to match the five statistics mentioned above, but also by the range of other anomalies that the theory is capable of explaining.

This paper is broader in scope than previous papers on RBE and its purpose is to make the case that most of the observed volatility in financial markets is expectationally generated and many "anomalies" observed in these markets such as the equity premium puzzle, the GARCH pattern of asset returns, the Forward Discount Bias in foreign exchange markets, *are all driven by the structure of heterogenous beliefs in the market.* In support of this view we present a unified model of market volatility which is relatively simple and demonstrate that the RBE of the model is able to explain a wide range of these anomalies. First, it predicts the correct order of magnitude of (i) the equity premium, (ii) the first and second moments of the price\dividend ratio, (iii) the first and second moments the risky return, and (iv) the first and second moments of the riskless rate. Second, the time series of stock returns exhibit a GARCH phenomenon, and third, an extension of the model to a two countries model exhibits a "forward discount bias" in its foreign exchange market. Our model extends ideas in Kurz and Beltratti [1997], Kurz and Schneider [1996] and Kurz [1997a] in a manner to be explained. We are able to give both a simple economic interpretation to the empirical record as well as provide a unified theory for the dynamics of financial markets.

Before turning to the description of our OLG model in Section 2, we provide a justification for our modeling strategy which is based on heterogenous beliefs. This explanation will be linked to the methodology with which we propose, in Section 3, to test the validity of any model with heterogenous beliefs. Section 4 will close the paper.

1. **Why a Paradigm of Heterogenous Beliefs**?

The theory of Rational Beliefs and RBE starts from the empirical observation that intelligent economic agents hold diverse beliefs even when there is no difference in the information at their disposal. Indeed, the center of their disagreement is often the diverse *interpretations* of the available information. By adopting axioms which allow rational agents to hold diverse beliefs, our theory does not lead, in general, to a Rational Expectations Equilibrium (in short, REE). However, an REE is also an RBE since the theory of RBE is an extension of the theory of REE. These observations suggest that it would be constructive to explain first why the REE model is an unsatisfactory special case and why the paradigm of diverse beliefs offers a satisfactory alternative in situations when economic volatility is a dominant phenomenon.

1a. Is Rational Expectations a Reasonable Assumption?

The assumption of rational expectations takes various forms. In the Arrow-Radner equilibrium theory with securities agents are required to know the map between future equilibrium prices and exogenous states. However, in Radner [1979] where agents have asymmetric information, the ability to invert the equilibrium map requires agents to know each other's probability beliefs of the exogenous states. To satisfy this condition one may as well assume that all agents hold the same belief and that all agents know this fact. In most dynamic models in economics and finance, rational expectations takes the more familiar requirement that agents know the *true* equilibrium probability distribution of all variables. Most economists agree that both conditions impose unreasonable requirements on what an agent must know in order to be viewed as "rational". Kurz [1994] refers to both forms of knowledge as "structural knowledge" to be distinguished from "empirical knowledge" or "information" about the state of the world.

Probability distributions and equilibrium maps are not observable but one may speculate that learning could provide some foundation for the assumption of rational expectations. This view proposes that heterogeneity may vanish as data becomes available since added data could enable agents to learn the true structure of the economy. Most of the work along this line adopted the Bayesian perspective which was inspired by martingale convergence theorems. A corresponding heated debate took place in the statistical literature under the heading of "Bayes Consistency" (see Diaconis and Freedman [1986]). The essential conclusion of the debate is that the convergence of the posterior to the true distribution is a rare occurrence. In two influential papers, Freedman [1963], [1965] shows that even when the statistician has a controlled experiment so that the data is generated i.i.d., if the true distribution is complex, the convergence of the posterior is a rare event. The problem is further compounded in typical learning situations in markets and games where the data is generated by an unknown process and the convergence of the posterior is even less likely (see Feldman [1991]).

If learning cannot provide a foundation for rational expectations, it does not make sense to regard as "irrational" an agent who does not know what he cannot know. Hence, a more general model than an REE would be desirable simply on the elementary ground that an REE is based on unrealistic assumptions. In addition, those who are opposed to REE often note the ample empirical evidence for the presence of diverse opinions in the market. Hence, it is important to explore how the REE literature explains this diversity of beliefs observed in the market.

1b. Diversity of Information or Diversity of Beliefs?

Starting with financial markets, the most common explanation given in the REE literature for the observed persistent heterogeneity of beliefs is the diversity of private information. It is argued that agents do not possess different prior beliefs but, rather, that they have different private information resulting in different conditional beliefs. The theoretical and applied literature adopting this approach is extensive (for example see Kyle [1985], Wang [1993] [1994] and references there). This explanation is unsatisfactory from both theoretical as well as empirical perspectives. Theoretical considerations lead to the information revelation of rational expectations (e.g. Grossman [1981], Radner [1979]) which implies that prices make public all private information and therefore the introduction of asymmetric information, *by itself*, is not sufficient. It simply transforms the problem into other paradoxes. These include the problem of explaining why agents trade at all (e.g. Milgrom and Stokey [1982]); why asset prices fluctuate more than could be explained by "fundamentals" (e.g. Shiller [1981]), indirectly generating an equity premium puzzle (see Mehra and Prescott [1985]); and why any resources are ever used for the production of information (see Grossman and Stiglitz [1980]). To explain the observed heterogeneity and avoid such paradoxes researchers had, therefore, to introduce some additional assumptions of market structure that would remove the information revelation property of rational expectations. Consider, for example, the explicit introduction of uninformed noise traders or general "noise" which leads to a theory of "noisy rational expectations equilibrium." This is a negation of rational expectations since the assumption of noise in prices explicitly introduces *irrationality* of uninformed traders into the theory. The artificial, and unsatisfactory, assumption of irrationality is then the one driving all the important conclusions.

We turn now to the empirical considerations making the assumption of asymmetric information in financial markets unsatisfactory. We note first the ample empirical evidence for the opposite view that equally informed agents interpret differently the same information (see, for example, Frankel and Froot [1990], Frankel and Rose [1995] and Kandel and Pearson [1995]). This implies that the agents have different probability beliefs which they condition on *the same public information*. However, focusing on asymmetric information, is there any empirical evidence to support the assumption of widespread use of private information in financial markets? We think that the evidence is not there. Moreover, since it is illegal to trade on inside private information, are we to conclude that the high volatility of financial markets is a result of widespread and persistent *criminal* behavior by traders? The majority of firms whose securities are traded on public exchanges are monitored carefully by a professional community of regulators, brokers, financial managers etc. Hence there is ample evidence that, on the whole, the majority of firms avoid letting any market participant either obtain private information or trade on it if he has such information. Furthermore, since modern financial markets are dominated by large institutions with vast resources which can be used to process all available information, elementary competitive behavior should lead us to conclude that all will possess the same information.

We conclude that, apart from insurance markets where asymmetric information plays a central role, in most financial markets the assumption of asymmetric information has a dual problem. First, on its own, this assumption has little explanatory power due to the revelation mechanism of REE. Consequently, asymmetric information must be supplemented by unreasonable additional assumptions about "friction", "noise" or other "stories". Second, it is difficult to find an empirical justification for the validity of this assumption in securities markets.

Turning to macroeconomics, recall that the critique of the Keynesian theory by the rational expectations approach was associated with the rejection of the wage and price rigidities implicit in the Keynesian system. However, under the classical assumptions of price and wage flexibility and market clearing in equilibrium, rational expectations implies the usual conclusions of neo-classical analysis: in equilibrium the economy operates at full employment, GNP grows at the potential level, etc. This classical framework cannot explain the observed structure of aggregate fluctuations and, in particular, the observed cyclical correlation among economic variables such as the positive correlation between the price level and aggregate output (the "inflation - output tradeoff"). In order to explain the data, the New Classical Theory introduced complex assumptions of asymmetric information which became the driving force of the theory. More specifically, although agents are assumed to have rational expectations, they have asymmetric information and *are unable to obtain information which is public in other parts of the economy*. This rigidity in the transmission of public information leads to diverse models of Phelpsian or Lucasian "islands" (see Phelps [1970] and Lucas [1973]). The important *Lucas supply curve* (Lucas [1973]) is deduced from the assumption that firms confuse price level fluctuations for relative price fluctuations since they are also assumed not to be able to observe the normally observable aggregate price level.

The essential point is that in standard macroeconomic REE models, heterogeneity across agents is caused by assumptions about agents not being able to make rather simple observations and needing to form expectations about what they do not know. The validity of the "islands" assumptions and the rigid information structure which they impose are hard to accept. Using a term proposed by Lucas [1982], the models are "rigged" to generate the heterogeneity which induces the desired empirical implication.

The arguments presented here highlight the fact that although the rational expectations assumption insists on a common belief of agents, the empirical implications of the common belief assumption - by itself - are rather absurd. The crucial empirical implications of models incorporating such assumption are generated by an *added set of assumptions*. These may include asymmetry of information, lack of adequate knowledge, irrational behavior of some agents, etc. Most of these added assumptions introduce "stories" with questionable theoretical and empirical foundations but *these questionable assumptions are the ones which drive the results!*

This paper, the papers included in Kurz [1997] and others (e.g. Garmaise [1998], Kurz [1998], Motolese [1998], Nielsen [1997], Wu and Guo [1998]) suggest that in many situations, particularly in the study of market risk and economic fluctuations, it is more plausible to accept an alternative paradigm. This paradigm is based on the hypothesis that agents *do not have structural knowledge* and, as a natural consequence, conclude that *rational agents may have diverse beliefs about what they do not know.* The empirical evidence for these *two* components of our approach is substantial. Moreover, the scientific merit of this alternative paradigm is derived mostly from the fact that it offers new and useful economic insights with which we can answer difficult economic questions. Hence, we conclude this Section by presenting brief arguments in support of this new paradigm. We use the terminology of "the diversity of beliefs theory" to refer to the combination of the hypothesis that agents do not have structural knowledge and the related theory of RBE which demonstrates that rational agents may have diverse beliefs.

We observe at the outset that REE *assumes* that expectations do not matter and, having done so, insists that other *exogenous* "fundamental" factors drive the real conclusions of the model. In contrast, under the diversity of belief theory "expectations matter" and the distribution of beliefs can have an important effect on the time series generated by the economy. On a more fundamental level, the diversity of beliefs theory rejects the validity of the formulation of uncertainty as being only an *exogenous* phenomenon. It insists that economic uncertainty and fluctuations have a large endogenous component which is propagated *within* the economy rather than being caused by exogenous shocks. Following Kurz [1974] we call it *Endogenous Uncertainty*. This uncertainty, which is probably the dominant form of uncertainty in our society, is *indirectly* the uncertainty about the beliefs and actions of other agents. Price uncertainty is, perhaps, the central form of Endogenous Uncertainty in a sequential economy.

Keep in mind that the common belief assumption is *a special case* of a model with diverse beliefs. Also, the assumption of asymmetric information is entirely compatible with diverse prior beliefs. Hence the diversity of beliefs theory is a more general paradigm than the model of common belief. Yet, the idea of diverse beliefs has been controversial. It would thus be constructive to review some arguments against models with diverse beliefs and, by implication, make the case in favor of such a paradigm.

Those who object to the introduction of *any* diversity of beliefs insist that it reduces the predictive value of equilibrium analysis since it enlarges the set of individual actions which are viewed as optimal. We reject this criticism on the ground that it is based on a misunderstanding of the function of the diversity of beliefs theory. We have stressed that an important purpose of introducing models with diverse beliefs is *to replace the artificial "rigging" of REE based models* when the added assumptions are of questionable validity. The diversity of beliefs does, indeed, enlarge the set market outcomes which are viewed as rational *as an alternative* to the way in which the assumptions of asymmetric information and noise trading add outcomes that would have otherwise been impossible in an REE[2].

Returning to the enlarged set market outcomes which may be explained with diverse beliefs, we claim that the enlarged set *is the main virtue of our new approach*! The two paradigms offer profoundly different explanations for the observed facts. The REE perspective proposes that the sources of all risk and economic fluctuations are *exogenous to the economy*. This is also true of REE with sunspots where the sunspot process is exogenous to the economy and in no sense is endogenously selected by the agents. The problem is that in most studies the level of volatility of the exogenous shocks is insufficient to explain the observed market volatility. The diversity of beliefs paradigm points to endogenous uncertainty as the *additional* component of social risk that has been missed in these studies. In general equilibrium terms, it insists that the state space be endogenously expanded to include the "state of beliefs" so that variations in this component of the state space have a real impact on economic allocations. Since endogenous uncertainty entails added fluctuations on a microeconomic level, the presence of such uncertainty necessitates a larger set of individual actions. Hence the enlarged set of outcomes is exactly why

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[2] The last point is central to our perspective and at the risk of repetition we sharpen its statement. We have shown that without the "extra" assumptions, rational expectations based models cannot explain a large array of observed phenomena. Hence, if any theory under consideration is to have explanatory power, one must make a choice in which direction to proceed. One direction in which contemporary analysis has gone is to introduce "fundamental" assumptions such as private information, rigidity in the transfer of public information, noise trading, etc. What we propose is that the diversity of beliefs paradigm *is an alternative new direction that one may take*. The papers in Kurz [1997] demonstrate that there is an extensive range of problems which can be studied with the tools of this paradigm, leading to new insights with implications for positive analysis and for collective actions.

endogenous uncertainty is a useful theory!

Finally, our rejection of the criticism is also methodological. Recall the comments regarding the availability of empirical evidence that equally informed agents disagree. Hence, it is a sound scientific procedure to explore the implications of a competing theory which explains the empirical evidence. Indeed, any such theory should have the REE model of common belief with full structural knowledge as a special case and comparisons with this reference case should be important in determining which approach provides a deeper insight. Ultimately, it is scientific usefulness which should be the basis for a choice between the two approaches.

We have cited recent work to show that the diverse beliefs paradigm has been a productive scientific tool. This paper adds an important dimension to support the diverse beliefs paradigm: *a unified theory of market volatility*. It demonstrates that a long array of "anomalies" in financial markets are all driven by expectations and consequently can all be explained by a single theory in which the diverse beliefs paradigm is the central component.

2. The RBE of an OLG Stock Market Economy

Our stock market economy is a relatively standard two-agent, OLG, economy with a single, homogenous, consumption good. Each agent lives two periods, the first when he is "young" and the second when he is "old." Each young agent is a replica of the old agent who preceded him, where the term "replica" refers to *utilities* and *beliefs*, and hence this is a model of two infinitely lived "dynasties" denoted by k = 1, 2. One can think of k as the identity of the pair of young and old agents of the dynasty at date t. We often use the term "agent k" but the context should make it clear whether the agent is the young or the old of dynasty k. Only young agents receive an endowment Ω_t^k, t = 1, 2, ... of the single consumption good. We view Ω_t^k as the labor income of agent k at date t and the stochastic processes $\{\Omega_t^k, t = 1, 2, ...\}$ for k = 1, 2 will be specified below. Additional net output is supplied by a firm which produces exogenously, as in Lucas [1978], the strictly positive profit process $\{D_t , t = 1, 2, ...\}$ with no input. These net outputs are paid out to the shareholders of the firm as dividends at the date at which the output is produced. The ownership shares are traded on a stock market and their aggregate supply is 1.

The stock market economy has three markets: (i) a market for the consumption good with an aggregate supply equaling the total endowment plus total dividends, (ii) a stock market with total supply of 1, and (iii) a market for a zero net supply, short term riskless debt instrument which we call a "bill". Since the stochastic growth rate of dividends is Markovian with two states, *the economy has a complete financial structure* in the sense that the number of financial instruments equals the number of states. To ensure intergenerational efficiency, the financial sector is initiated at date 1 by distributing the unit supply of shares among the old of that date.

The above assumptions are the same as in Kurz and Beltratti [1997] and Kurz and Schneider [1996]. However, Kurz and Schneider [1996] did not assume that the economy grows and did not calibrate their results to any empirically known facts. Kurz and Beltratti [1997] allowed growth into the model and in that sense our model is the same as theirs. The main difference is in the parametrization of the model and in the economic interpretation of the results. The notation which we employ is as follows: for k = 1, 2

C_t^{1k} - consumption of k when young at t;

C_{t+1}^{2k} - consumption of k when old at t + 1 (implying that the agent was born at t);

D_t - total amount of dividends produced exogenously at t;

$d_{t+1} = \frac{D_{t+1}}{D_t}$ - the random growth rate of dividends;

θ_t^k - amount of stock purchases by young agent k at t;

B_t^k - amount of one period bill purchased by young agent k at t;

Ω_t^k - endowment of young agent k at t;

P_t - the price of the common stock at t;

$p_t = \frac{P_t}{D_t}$ - the price/dividend ratio of the common stock at t;

q_t - the price of a one period bill at t. This is a discount price;

I_t - history of all observables up to t;

2.1 The Equilibrium Concept.

We normalize prices by using consumption as a numeraire. Given this, the optimization problem of agent k has the following structure at all $t = 1, 2, ...$:

(1a) $$\max_{(C_t^{1k}, \theta_t^k, B_t^k, C_{t+1}^{2k})} E_{Q_t^k}\left\{u^k(C_t^{1k}, C_{t+1}^{2k}) | I_t\right\}$$

subject to

(1b) $$C_t^{1k} + P_t\theta_t^k + q_t B_t^k = \Omega_t^k$$

(1c) $$C_{t+1}^{2k} = \theta_t^k(P_{t+1} + D_{t+1}) + B_t^k.$$

Q^k is a probability belief of agent k on all future variables which he does not know. To enable us to compute equilibria we take the utility function agent k to be

(2) $$u^k(C_t^{1k}, C_{t+1}^{2k}) = \frac{1}{1-\gamma_k}(C_t^{1k})^{1-\gamma_k} + \frac{\beta_k}{1-\gamma_k}(C_{t+1}^{2k})^{1-\gamma_k}, \quad \gamma_k > 0, \quad 0 < \beta_k < 1.$$

With this specification the Euler equations for agent k are

(3a) $$-P_t(C_t^{1k})^{-\gamma_k} + \beta_k E_{Q_t^k}((C_{t+1}^{2k})^{-\gamma_k}(P_{t+1} + D_{t+1}) | I_t) = 0$$

(3b) $$-q_t(C_t^{1k})^{-\gamma_k} + \beta_k E_{Q_t^k}((C_{t+1}^{2k})^{-\gamma_k} | I_t) = 0.$$

The parameters of the model are selected to equal the values of the corresponding estimates for the real economy. We thus aim to calibrate the model and test its ability to generate solutions which are of the same order of magnitudes as the observed endogenous variables.

(2.1a) *__The dividend process and the equilibrium map__*. The simulation model is Markovian where the exogenous process of dividends is as specified in Kurz-Beltratti [1997] who follow the estimates of Mehra and Prescott [1985]. It takes the following form

(4) $$D_{t+1} = D_t d_{t+1}.$$

where $\{d_t , t = 1, 2, ...\}$ is a stationary and ergodic Markov process. The state space of the process is $J_D = \{d^H , d^L\}$ with $d^H = 1.054$ and $d^L = .982$ and a transition matrix

(5) $$\begin{bmatrix} \phi, & 1 - \phi \\ 1 - \phi, & \phi \end{bmatrix}$$

with $\phi = .43$. Hence, over time agents experience a secular rise of dividends and it is therefore convenient to focus on growth rates. To do that let

$\omega_t^k = \frac{\Omega_t^k}{D_t}$ is the endowment/dividend ratio of agent k at date t;

$b_t^k = \frac{B_t^k}{D_t}$ is the bill/dividend ratio of agent k at date t;

$c_t^{1k} = \frac{C_t^{1k}}{D_t}$ is the ratio of consumption when young to aggregate capital income;

$c_{t+1}^{2k} = \frac{C_{t+1}^{2k}}{D_{t+1}}$ is the ratio of consumption when old to aggregate capital income;

We assume that $\omega_t^k = \omega^k$ for k = 1, 2 are constant. This implies that if we define $\nu = \omega^1 + \omega^2$ then $(\Omega_t^1 + \Omega_t^2) = \nu D_t$ for all t . We do not allow endowments to fluctuate in part because production and labor markets are not the focus of this paper and in part because of computational feasibility. We now divide the budget constraints (1b) by D_t and (1c) by D_{t+1}, equation (3a) by $D_t^{1-\gamma_k}$ and equation (3b) by $D_t^{-\gamma_k}$ to obtain, for k = 1, 2

(6a) $$c_t^{1k} = -p_t\theta_t^k - q_t b_t^k + \omega^k,$$

(6b) $$c_{t+1}^{2k} = \theta_t^k(p_{t+1}+1) + \frac{b_t^k}{d_{t+1}},$$

(6c) $$-p_t(c_t^{1k})^{-\gamma_k} + \beta_k E_{Q_t^k}((c_{t+1}^{2k}d_{t+1})^{-\gamma_k}(p_{t+1}+1)d_{t+1}) \mid I_t) = 0,$$

(6d) $$-q_t(c_t^{1k})^{-\gamma_k} + \beta_k E_{Q_t^k}((c_{t+1}^{2k}d_{t+1})^{-\gamma_k} \mid I_t) = 0.$$

(6a) - (6b) imply demand functions which take the general time dependent form, for k = 1, 2

(7a) $$b_t^k = b_t^k(p_t, q_t, d_t, I_t)$$

(7b) $$\theta_t^k = \theta_t^k(p_t, q_t, d_t, I_t)$$

Equilibrium requires the market clearing conditions

(7c) $$\theta_t^1 + \theta_t^2 = 1$$

(7d) $$b_t^1 + b_t^2 = 0;$$

The equilibrium in (7a)-(7d) depends upon the beliefs of the agents and upon what they condition on. In this paper we restrict our attention to stable Markov equilibria.

Definition 1: Beliefs (Q^1,Q^2) and a stochastic process $\{(p_t, q_t, (\theta_t^1, b_t^1), (\theta_t^2, b_t^2), d_t), t = 1, 2, ...\}$ with initial portfolios $((\theta_0^1, b_0^1 = 0), (\theta_0^2, b_0^2 = 0))$ and with true probability Π constitute a *stable Markov competitive equilibrium* if

(i) $(p_t, q_t, \theta_t^1, b_t^1, \theta_t^2, b_t^2, d_t)$ satisfy conditions (7a) - (7d) at all dates t;

(ii) $b_t^k = b_t^k(p_t, q_t, d_t, I_t)$ and $\theta_t^k = \theta_t^k(p_t, q_t, d_t, I_t)$ are independent of the history I_{t-1}.

(iii) (Q^1, Q^2, Π) are stable measures in the sense of Kurz [1994][3].

It follows from (7a)-(7d) that the price process $\{(p_t, q_t), t = 1, 2, ...\}$ of a stable Markov equilibrium is defined by an equilibrium sequence of maps

(7e) $$\begin{bmatrix} p_t \\ q_t \end{bmatrix} = \Phi_t(d_t)$$

where the time dependence of the equilibrium map represents the potential time dependence of the beliefs of agents. In an REE, $Q^1 = Q^2 = \Pi$ where Π is the true probability induced by (5) and by the stationary equilibrium map (7e). In an REE the states of beliefs of agents have no effect on prices and all demand functions are time independent. We review this case first.

(2.1b) *__Rational expectations equilibria.__* In a Markov REE $Q^1 = Q^2 = \Pi$ and, deduced from (5), the probabilities of $(p_{t+1}, q_{t+1}, d_{t+1})$ in (6c) - (6d) are conditioned only on the realized value of d_t. It then follows that the demand functions must take the form

(8a) $$b_t^k = b^k(p_t, q_t, d_t)$$

(8b) $$\theta_t^k = \theta^k(p_t, q_t, d_t).$$

(8a)-(8b) and the market clearing conditions (7a)-(7b) imply a stationary equilibrium map

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[3] Let $\{x_t, t = 1, 2, ...\}$ be a stochastic process under the probability Π. For any event B denote by $m_n(B)(x)$ the relative frequency out of n draws at which the process visits B. Note that the event B may be complex and multidimensional. Then,
__Definition__: A stochastic process is called *stable (or statistically stable)* if for any finite dimensional set (i.e. cylinder) B

$$\lim_{n \to \infty} m_n(B)(x) = m(B)(x) \qquad \text{exists } \Pi \text{ a.e.}$$

A non-stationary process may be unstable just due to the fact that it grows without bound. In that case the definition would be applied to some transformation of the process.

(9a) $$\begin{bmatrix} p_t \\ q_t \end{bmatrix} = \hat{\Phi}(d_t).$$

In the special case postulated in (5) the growth rate of dividends takes two values. In this case Equation (9a) shows that a stable Markov equilibrium is, in fact, a stationary equilibrium with two prices and two optimal portfolios.

2.2 *The Structure of Beliefs and Rational Belief Equilibrium (RBE)*

Our development here uses assumptions and concepts from the theory of Rational Beliefs (see Kurz [1994] [1997]) and the tools of "assessment variables" used to construct Markov RBE as developed in Kurz and Schneider [1996] and Nielsen [1996]. For completeness of exposition we briefly explain below how these tools are used here.

The theory of Rational Beliefs assumes that agents do not have "structural knowledge": they do not know market excess demand functions or equilibrium maps and hence cannot compute equilibria or invert equilibrium maps. In addition, they do not know the *true* equilibrium probability of any observed variable in the economy. The theory assumes that agents have costless access to all past economic data and hence know all that can be deduced from the empirical distribution of the observed variables. *Rational Beliefs* are then probability beliefs on sequences of observed equilibrium variables which are compatible with the known empirical distribution of the equilibrium process. The probability defined by the empirical distribution is called "the stationary measure" induced by the equilibrium dynamics. The sense in which the term "compatible" is used here requires us to think of a rational belief Q^k *as if* it was the true probability of the equilibrium process. This may be false and we denote the true, but *unknown,* probability of the equilibrium process by Π. Under the hypothetical probability Q^k, the equilibrium process will generate an empirical distribution which may be different from the known empirical distribution generated under Π. A belief Q^k is a *Rational Belief* if the empirical distribution under Q^k is the same as the one generated under the true equilibrium probability Π.

The main theorem in Kurz [1994] shows that if agent k adopts a belief Q^k which is different from the stationary measure, he must believe that the economic environment is non-stationarity. However, a non-stationary probability of a Markov process with finite number of states is represented by a time varying sequence of Markov matrixes $(F_1^k, F_2^k, F_3^k, ...)$. This is interpreted to say that at date t the process is defined by the transition matrix F_t^k. If the set of possible Markov matrices is $\{G_1, G_2, ..., G_M\}$ one can represent the non-stationary probability with a time function g_t^k taking values in $\{1, 2, ... , M\}$. This is then used to represent the sequence of transition matrices as $\{ G^k_{g_t^k}, \ t = 1, 2, ... \}$.

The above problem, of selecting a sequence of matrices to describe a non-stationary probability measure, is exactly the same problem of describing Rational Beliefs Q^k for k = 1, 2 as needed in (6c)-(6d) and (7e). It turns out that the complicating factor is the determination of the *rationality of belief conditions* which the sequence of matrices must satisfy. The method of "assessment variables" is our tool to describe the non stationarity of such rational beliefs.

(2.2a) *<u>Assessment Variables and the State Space</u>*. Assessment variables are sequences of random variables $\{y_t^k, t = 1, 2, ...\}$ for k = 1, 2 generated by the agents. In this paper $y_t^k \in Y = \{0, 1\}$. A belief Q^k is then a probability on the joint process $\{(p_t, q_t, d_t, y_t^k), t = 1, 2, ...\}$ which is assumed to be Markov. Hence, under Q^1 and Q^2, the assessment variables are *jointly distributed*

with the real market variables and hence their distribution may depend upon other economic variables. Also, they may be correlated with future economic variables and hence conditioning on them alters the predictions of future economic variables.

From an economic perspective, assessment variables are parameters indicating how an agent interprets current information and heeee are tools for the description of stable and non-stationary processes (see Kurz and Schneider [1996] pages 491-495 on this point). These variables have purely subjective meaning and should not be taken to be objective and transferable "information". Their impact on the real economy arises from the fact that conditioning on them by the agents *alters their probability beliefs* about future values of economic variables. We explain this now.

(i) Assessment variables and the equilibrium map. In (6c) - (6d) agent k uses the probability of $(p_{t+1}, q_{t+1}, d_{t+1}, y^k_{t+1})$ *conditional* on (p_t, q_t, d_t, y^k_t). It follows from our Markov assumptions that the demands of agent k for stocks and bills are time-independent functions of the form

(10a) $$b^k_t = b^k(p_t, q_t, d_t, y^k_t)$$

(10b) $$\theta^k_t = \theta^k(p_t, q_t, d_t, y^k_t).$$

Consequently we can write the market clearing conditions as

(10c) $$\theta^1(p_t, q_t, d_t, y^1_t) + \theta^2(p_t, q_t, d_t, y^2_t) = 1$$

(10d) $$b^1(p_t, q_t, d_t, y^1_t) + b^2(p_t, q_t, d_t, y^2_t) = 0.$$

The system (10c)-(10d) implies that the equilibrium map of this economy takes the form

(11) $$\begin{bmatrix} p_t \\ q_t \end{bmatrix} = \Phi^*(d_t, y^1_t, y^2_t).$$

The equilibrium map (11) reveals that prices are determined by the exogenous shock d_t and by the "*state of belief*" represented by the vector (y_t^1 , y_t^2).

To clarify the role of the assessment variables in (11) note that in (10a) - (10b) we specified that the demand functions are not time dependent and hence the assessment variables completely determine the conditional probabilities of the agents. From the assumption of a Markov equilibrium it must be that y_t^k determines completely the transition matrix from (p_t, q_t) to (p_{t+1} , q_{t+1}) which is used by agent k at date t. Moreover, $y_t^k \in \{ 0 , 1\}$ implies that the agent has *at most two Markov matrices* and at each date the value taken by his assessment variable determines which of these two the agent uses. We shall later define the beliefs in such a manner that "1" is a state of *optimism* while "0" is a state of *pessimism*.

The equilibrium map implies that there are at most 8 distinct price vectors (p_t , q_t) that may ever be observed and these correspond to the 8 combinations of (d_t , y_t^1 , y_t^2). Moreover, due to our Markov assumption, the *true* equilibrium transition probability from the 8 prices (p_t , q_t) to the 8 prices (p_{t+1} , q_{t+1}) is determined entirely by the transition probability from (d_t , y_t^1 , y_t^2) to $(d_{t+1} , y_{t+1}^1 , y_{t+1}^2)$. In all applications below we select the joint process $\{(d_t , y_t^1 , y_t^2), t = 1, 2, ...\}$ to be a stationary Markov process with a transition matrix Γ. This implies[4] that the true equilibrium process of prices has a *fixed* transition probability from (p_t, q_t) to (p_{t+1}, q_{t+1}) defined by Γ. The agents, who compute the empirical distribution, will discover Γ and this matrix will be used to construct the stationary measure. However, the agents do not know that this is the true equilibrium probability and for this reason they form rational beliefs

[4] The choice of the equilibrium dynamics being generated by a fixed, stationary, matrix is a matter of convenience and simplicity in this paper. In general the process $\{(d_t , y_t^1 , y_t^2), t = 1, 2, ...\}$ could have been selected to be any stable process with a Markov stationary measure induced by the empirical distribution. In such a case the fixed transition matrix Γ would characterize only the *stationary measure* of the equilibrium dynamics rather than be the matrix of the *true* probability of the equilibrium dynamics of prices.

relative to Γ. Indeed, the fact that they form rational beliefs in accord with their assessment variables *is what rationalizes Γ to be the equilibrium probability of the implied RBE.*

(ii) Assessment variables and the state space. The state space for prices is $(J_D \times Y \times Y)^\infty$ but one may also consider the state space to be S^∞ where S is the index space S = {1, 2, ..., 8}. We can then define a new equilibrium map Φ between the *indices* of prices and the states of dividends and assessment variables (indexed by a number from 1 to 8 rather than by t) by

$$(12) \qquad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} = \Phi \begin{bmatrix} d_1 = d^H, \; y_1^1 = 1, \; y_1^2 = 1 \\ d_2 = d^H, \; y_2^1 = 1, \; y_2^2 = 0 \\ d_3 = d^H, \; y_3^1 = 0, \; y_3^2 = 1 \\ d_4 = d^H, \; y_4^1 = 0, \; y_4^2 = 0 \\ d_5 = d^L, \; y_5^1 = 1, \; y_5^2 = 1 \\ d_6 = d^L, \; y_6^1 = 1, \; y_6^2 = 0 \\ d_7 = d^L, \; y_7^1 = 0, \; y_7^2 = 1 \\ d_8 = d^L, \; y_8^1 = 0, \; y_8^2 = 0 \end{bmatrix}.$$

d^H is the "high dividends" and d^L is the "low dividends" states. (11)-(12) highlight the idea of Endogenous Uncertainty which identifies the variability of prices at each state of the exogenous variables. It shows that the volatility of prices depends upon the states of belief of the agents.

(iii) The exogenous variables. A belief Q^k was defined as a probability on the space of sequences $\{(p_t, q_t, d_t, y_t^k), \; t = 1, 2, ...\}$. We have also shown that the belief of an agent was reduced to selecting transition matrices from (p_t, q_t) to (p_{t+1}, q_{t+1}). This appears to ignore the probability of the exogenous variable d_t. To see that this distribution is not ignored consider the map Φ in (12). It shows that the probability of d^H equals the probability of prices from 1 to 4 and the probability of d^L equals the probability of prices from 5 to 8. Thus, the distribution of d_t is defined by the *partition* of the state space. The agents discover this partition in the empirical

distribution and for simplicity *we have assumed that agents believe this partition to be the truth*.[5]

(2.2b) <u>*The Rationality of Belief Conditions*</u>. A *stable* process is a process which has, with probability 1, a non-trivial empirical distribution on all finite dimensional sets. The *stationary measure* of the process is the unique extension of the empirical frequencies of finite dimensional sets to a probability on the entire space. A *Rational Belief Equilibrium* (RBE) is a stable equilibrium in which agents hold rational beliefs. Hence, to establish an RBE we need to construct an equilibrium process which is stable from which agents compute the empirical distribution that is used to construct the stationary measure. The rationality conditions require the beliefs of the individual agents to imply a stationary measure which is equal to the one computed from the data. In the case of our model here, these concepts are significantly simplified. We now explain why.

The beliefs Q^1 and Q^2 of the two agents are probabilities on the space of sequences $\{(p_t ,q_t , d_t , y_t^k), t = 1, 2, ...\}$. However, the probability used in (6c)-(6d) is $Q^k((\bullet)| y^k)$, the probability which is *conditional on the assessment variable of agent k*. The rationality of belief conditions must then apply to this last conditional probability. These conditions require that

(i) $Q^k((\bullet)| y^k)$ is a stable measure and the dynamics of the economy under it has an empirical distribution with probability 1;

(ii) that the stationary measure of $Q^k((\bullet)| y^k)$ equals the probability on infinite sequences induced by the true transition matrix Γ.

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[5] By studying the relationship between prices and d_t agents discover the partition in the long run data. This happens to be the truth at all dates but an agent may not believe it. Instead he may form a rational belief about this variable. This issue has little significance to our study and we chose the simpler assumption.

Since $Q^k((\bullet)| y^k)$ is represented by two Markov matrices used by the agent at different times, we need to specify the joint distribution of (p_t, q_t, y_t^k) and the implied rationality conditions which are consistent with these Markov matrices. To do that we use the "Conditional Stability Theorem" (see Kurz and Schneider [1996] page 492 - 494). It says that if the probability Q^k of the *joint* process $\{(p_t, q_t, y_t^k), t = 1, 2, ...\}$ is stable, then $Q^k((\bullet)| y^k)$ which is the probability Q^k *conditional* on the index y_t^k, is a stable probability on $\{(p_t, q_t), t = 1, 2, ...\}$ and the stationary measure of $Q^k((\bullet)| y^k)$ is the *marginal* of Q^k on (p_t, q_t) obtained by integrating on y^k.

To simplify the procedure above we assume that the marginal distribution of Q^k on y_t^k is i.i.d. and we denote these unconditional probabilities by $Q^k\{y_t^k = 1\} = \alpha_k$ for k = 1, 2. By the Conditional Stability Theorem there exist two pairs of matrices, (F_1, F_2) for agent 1 and (G_1, G_2) for agent 2, such that Q^1 and Q^2 are characterized by the following conditions:

(13a) Q^1 *for agent 1:* adopt F_1 if $y_t^1 = 1$ Q^2 *for agent 2*: adopt G_1 if $y_t^2 = 1$

adopt F_2 if $y_t^1 = 0$ adopt G_2 if $y_t^2 = 0$.

(13b) $$\alpha_1 F_1 + (1 - \alpha_1) F_2 = \Gamma \quad , \quad \alpha_2 G_1 + (1 - \alpha_2) G_2 = \Gamma.$$

An intuitive interpretation starts by noting that these rational agents believe that the price-dividend process is not stationary, and their beliefs are parametrized by (y_t^1, y_t^2). (13b) insists that the sequence of matrices which they adopt is compatible with the true price-dividend process (i.e. generating the same empirical distribution) which is a Markov process with transition matrix Γ. α_1 is the frequency at which agent 1 uses matrix F_1 and α_2 is the frequency at which agent 2 uses matrix G_1. This leads to a formal definition of the equilibrium which we construct below:

Definition 2: *A Markov Rational Belief Equilibrium* (RBE) is a Markov Competitive Equilibrium in which the (Q^1 , Q^2) are defined by (13a) and satisfy the rationality conditions (13b)[6].

(2.2c) *The Stationary Measure*. We now assemble the conditions which Γ must satisfy. Recall that the probability of d^H is equal to the probability of the first four prices and the probability of d^L is equal to the probability of the last four prices. Now, (5) specfied the dividend process and since in an RBE the driving mechanism of prices is the distribution of (d_t , y_t^1, y_t^2), t = 1, 2,..., it follows that the marginal of Γ with respect to d_t (or equivalently with respect to a random variable taking the value 1 when one of the first four prices occurs and 0 when one of the last four prices is realized) must equal the dividend matrix in (5). Similarly with respect to (y_t^1 , y_t^2) : the marginal of Γ with respect to each of the y_t^k must be i.i.d. with probability α_k.

We observe that each agent has a marginal distribution on his own assessment variable, hence the i.i.d. requirement on the marginals of Γ with respect to each one of the two assessments is a consistency condition between the market observations and what each agent perceives. No such conditions apply to the *joint* distribution of the assessments. This joint effect of the assessment variables, as distinct from the individually perceived effect, is that part of Γ which describes *the externalities of beliefs in the market performance.* These externalities cannot

[6] In some presentations of this paper we were asked if our RBE can be viewed as a sunspot equilibrium. We note first that assessment variables are not observbles, the joint distribution of (y_t^1 , y_t^2) is not known and the agents do not have the structural knowledge needed to invert an equilibrium map. Hence an RBE cannot be a fully revealing equilibrium. But the issue is deeper. Even if we assumed, for the sake of discussion, that (y_t^1 , y_t^2) is observable and that the agents know the equilibrium map, the RBE is not a sunspot equilibrium because the "sunspot" variable (y_t^1 , y_t^2) *alters the real economy*. That is, for the RBE to be a sunspot equilibrium the different values of (y_t^1 , y_t^2) must be associated with exactly the same fundamentals of the economy. This is not the case in an RBE. For example, in the "sunspot" state (1, 1) the von-Neumann Morgenstern preferences of the agents are defined by the probabilities (F_1 , G_1) while in the "sunspot" state (0, 1) they are defined by (F_2 , G_1). These changes in the fundamentals f the economy show that *endogenous uncertainty has real effects on the economy* induced by the states of belief (y_t^1 , y_t^2). We also note that given each state of belief (y_t^1, y_t^2) the equilibrium at date t is unique.

be found in each of the marginal distributions of the beliefs Q^k. They are, however, reflected in the equilibrium process. They specify that interactions among the agents which reflect the structure of communication in society, the manner in which agents influence each other and how the real variables in the economy (i.e. the dividends) affect this interaction.

In sum, the matrix Γ must satisfy the following:

(14a) the marginal on y_t^k is i.i.d. with $P\{y_t^k = 1\} = \alpha_k$ for $k = 1, 2$;

(14b) the marginal on d_t is Markov as specified by the dividend process (5);

The family of matrices which satisfy these conditions is rather limited and the main criterion for selecting from this family is flexibility in parameterization of equilibria. The following matrix Γ satisfies all the conditions specified in (14a) - (14b):

$$\Gamma = \begin{bmatrix} \phi A, & (1-\phi)A \\ (1-\phi)B, & \phi B \end{bmatrix} \tag{15}$$

where A and B are 4×4 matrices which are characterized by the 10 parameters α_1, α_2, and (a, b) where $a = (a_1, a_2, a_3, a_4)$, $b = (b_1, b_2, b_3, b_4)$:

$$A = \begin{bmatrix} a_1, & \alpha_1 - a_1, & \alpha_2 - a_1, & 1 + a_1 - \alpha_1 - \alpha_2 \\ a_2, & \alpha_1 - a_2, & \alpha_2 - a_2, & 1 + a_2 - \alpha_1 - \alpha_2 \\ a_3, & \alpha_1 - a_3, & \alpha_2 - a_3, & 1 + a_3 - \alpha_1 - \alpha_2 \\ a_4, & \alpha_1 - a_4, & \alpha_2 - a_4, & 1 + a_4 - \alpha_1 - \alpha_2 \end{bmatrix}, \quad B = \begin{bmatrix} b_1, & \alpha_1 - b_1, & \alpha_2 - b_1, & 1 + b_1 - \alpha_1 - \alpha_2 \\ b_2, & \alpha_1 - b_2, & \alpha_2 - b_2, & 1 + b_2 - \alpha_1 - \alpha_2 \\ b_3, & \alpha_1 - b_3, & \alpha_2 - b_3, & 1 + b_3 - \alpha_1 - \alpha_2 \\ b_4, & \alpha_1 - b_4, & \alpha_2 - b_4, & 1 + b_4 - \alpha_1 - \alpha_2 \end{bmatrix} \tag{16}$$

If $A \neq B$ then the distribution of (y^1_{t+1}, y^2_{t+1}) depends upon d_t. (16) implies that $P\{y^k_t = 1\} = \alpha_k$ for k = 1, 2 as required in (14a). Note, however, that although each process $\{y^k_t, t = 1, 2, \ldots\}$ for k = 1, 2 is very simple, the joint process $\{(d_t, y_t^1, y_t^2), t = 1, 2, \ldots\}$ may be complex: it allows correlation among the three central variables and these effects are important. If we set $\alpha_1 = \alpha_2 = .5$ and $a_i = b_i = .25$ for i = 1, 2, 3, 4 then all correlations are eliminated. In this case the stationary distribution $(\pi_1, \pi_2, \ldots, \pi_8)$ implied in (16) is $\pi_i = .125$ for all i. If, in addition, the agents adopt the stationary measure as their belief (i.e. $F_1 = G_1 = \Gamma$), then we have exactly an REE.

For simplicity of parameterization, we set in almost all simulations the parameter values $\alpha_1 = \alpha_2 = .57$, $a = (a_1 \neq a_2 = a_3 = a_4)$ and $b = (b_1 \neq b_2 = b_3 = b_4)$. It is clear, however, that there are natural restrictions which the parameters must satisfy and these will be discussed later. We specify now the family of rational beliefs which we use in the simulations.

(2.2d) *Rational Beliefs: the Family of Optimism/Pessimism Beliefs*. We now use two parameters λ and μ to select two pairs of matrices: (F_1, F_2) of agent 1 and (G_1, G_2) of agent 2 satisfying the rationality conditions (13b). To do that denote the row vectors of A and B by:

$$A^j = (a_j,\ \alpha_1 - a_j,\ \alpha_2 - a_j,\ 1 + a_j - (\alpha_1 + \alpha_2)) \qquad j = 1, 2, 3, 4$$

$$B^j = (b_j,\ \alpha_1 - b_j,\ \alpha_2 - b_j,\ 1 + b_j - (\alpha_1 + \alpha_2)) \qquad j = 1, 2, 3, 4.$$

With this notation we define the 4 matrix functions of a real number z as follows:

$$(17) \quad A_1(z) = \begin{bmatrix} zA^1 \\ zA^2 \\ zA^3 \\ zA^4 \end{bmatrix}, \quad A_2(z) = \begin{bmatrix} (1-\phi z)A^1 \\ (1-\phi z)A^2 \\ (1-\phi z)A^3 \\ (1-\phi z)A^4 \end{bmatrix}, \quad B_1(z) = \begin{bmatrix} zB^1 \\ zB^2 \\ zB^3 \\ zB^4 \end{bmatrix}, \quad B_2(z) = \begin{bmatrix} (1-(1-\phi)z)B^1 \\ (1-(1-\phi)z)B^2 \\ (1-(1-\phi)z)B^3 \\ (1-(1-\phi)z)B^4 \end{bmatrix}.$$

Finally we define

$$(18) \qquad F_1 = \begin{bmatrix} \phi A_1(\lambda), & A_2(\lambda) \\ (1-\phi)B_1(\lambda), & B_2(\lambda) \end{bmatrix} \qquad G_1 = \begin{bmatrix} \phi A_1(\mu), & A_2(\mu) \\ (1-\phi)B_1(\mu), & B_2(\mu) \end{bmatrix}.$$

By the rationality conditions (13b), $F_2 = \frac{1}{1-\alpha_1}(\Gamma - \alpha_1 F_1)$, $G_2 = \frac{1}{1-\alpha_2}(\Gamma - \alpha_2 G_1)$.

To motivate this construction, note that the parameters λ and μ are proportional revisions of the conditional probabilities of states (1, 2, 3, 4) and (5, 6, 7, 8) *relative to* Γ. $\lambda > 1$

and $\mu > 1$ imply *increased* probabilities of states (1, 2, 3, 4) in matrix F_1 of agent 1 and matrix G_1 of agent 2 where the first four prices are associated with the states when $d_t = d^H$. Since these are the states of the higher prices, $\lambda > 1$ implies that agent 1 is optimistic about high prices at $t + 1$. Similarly for $\mu > 1$. In all simulations below we set $\lambda \geq 1$ and $\mu \geq 1$ and hence the assessment variables y_t^k have a simple interpretation: when $y_t^k = 1$ agent k is optimistic (*relative to* Γ) at t about high prices at t + 1. The special case of $\lambda = 1$, $\mu = 1$ and $a_i = b_i = .25$ identifies an REE. Finally, it turns out that the concepts of "agreement" and "disagreement" between the agents are useful. We then say that *the agents agree if* $y_t^1 = y_t^2$ *and disagree if* $y_t^1 \neq y_t^2$.

(2.2e) *Markov Rational Belief Equilibrium (RBE)*. Conditions (6c) - (6d) require each agent to forecast prices (p_{t+1}, q_{t+1}). A rational agent should be able to perform this task since there is a set of 8 prices $\{(p_{t+1}, q_{t+1})\}$ that can occur at date t + 1 and all agents know this set from past history. We can then use the index set S with the map (12) to define equilibrium consumptions, portfolios and prices in terms of the transitions from state s to state j in the set S. To state the equilibrium conditions in these terms denote by $Q^k(j \mid s, y_s^k)$ agent k's probability of price state j given price state s and the value of y_s^k which he perceives at state s but *under the competitive assumption that k knows neither the map (12) nor the fact that he influences prices*. Conditions (6) - (7) are then restated for k = 1, 2 and j, s = 1, 2, ..., 8:

(19a) $$c_s^k = \omega^k - \theta_s^k p_s - b_s^k q_s$$

(19b) $$c_{sj}^{2k} = \theta_s^k (p_j + 1) + \frac{b_s^k}{d_s}$$

(19c) $$-(c_s^{1k})^{-\gamma_k} p_s + \beta_k \sum_{j=1}^{8} (c_{sj}^{2k} d_j)^{-\gamma_k} (p_j + 1) d_j Q^k(j|s, y_s^k) = 0$$

(19d) $$-(c_s^{1k})^{-\gamma_k} q_s + \beta_k \sum_{j=1}^{8} (c_{sj}^{2k} d_j)^{-\gamma_k} Q^k(j|s, y_s^k) = 0 .$$

(19e) $$\theta_s^1 + \theta_s^2 = 1 \quad \text{for all } s$$

(19f) $$b_s^1 + b_s^2 = 0 \quad \text{for all } s.$$

A stable Markov RBE is then a solution of equations (19a)-(19f) for feasible parameters.

We now review the feasibility conditions which the model parameters are required to satisfy. The parameters (a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4), α_1, α_2 and ϕ must satisfy

(20) $$\begin{gathered} a_i, b_i \le \alpha_1 < 1 \quad \text{for} \quad i = 1, 2, 3, 4 \\ a_i, b_i \le \alpha_2 < 1 \quad \text{for} \quad i = 1, 2, 3, 4 \\ 0 \le \phi \le 1. \end{gathered}$$

The selection of (λ, μ) is restricted by 10 inequality constraints:

(21) $$\begin{gathered} \lambda \le \frac{1}{\phi} \qquad \mu \le \frac{1}{\phi} \qquad \lambda \le \frac{1}{1-\phi} \qquad \mu \le \frac{1}{1-\phi} \\ \lambda \le \frac{1}{\alpha_1} \qquad \mu \le \frac{1}{\alpha_2} \qquad \lambda \ge \frac{\alpha_1 + \phi - 1}{\phi\alpha_1} \qquad \mu \ge \frac{\alpha_2 + \phi - 1}{\phi\alpha_2} \\ \lambda \ge \frac{\alpha_1 - \phi}{(1-\phi)\alpha_1} \qquad \mu \ge \frac{\alpha_2 - \phi}{(1-\phi)\alpha_2} \end{gathered}$$

The RBE's in our simulations are solutions of the 48 equations (19a) - (19f) in prices and quantities which satisfy the feasibility constraints (20) - (21). The particular family of RBE which we shall study in the simulations is *drastically* simplified by the following criteria:

(i) a *single* intensity variable $\lambda = \lambda_s = \mu = \mu_s$ for all $s = 1, 2, \ldots, 8$;

(ii) $\alpha = \alpha_1 = \alpha_2$;

(iv) $a = (c_1, c_2, c_2, c_2)$ and $b = (c_1, c_2, c_2, c_2)$ for two parameters (c_1, c_2);

(iii) γ_1 and γ_2 in the realistic interval [2.5 , 3.5] (see results of Kurz and Beltratti [1997]) ; β_1 and β_2 are in the empirically plausible interval [.85 , .95].

It follows from these specifications that once we select empirically realistic γ_1 , γ_2 , β_1 and β_2, *the model is entirely determined* by the four parameters : λ, α , c_1 , c_2 .

We comment on the difference between our treatment and Kurz and Beltratti [1997]. The earlier paper introduced the vectors $\lambda = (\lambda_1 , \lambda_2 , .,.,., \lambda_8)$ and $\mu = (\mu_1 , \mu_2 , .,.,., \mu_8)$ and allowed the agents to select 16 parameters. This permits the agents to select parameters (λ_s , μ_s) which vary with prices. Since there are only two agents in the model they obviously have an effect on prices but are required to act competitively and ignore such effect. By allowing the agents to select different (λ_s , μ_s) for different s, we permit the agents to take into account their effect on prices and thus violate the condition of anonymity (see Kurz [1998] for more details). Our procedure of selecting only two parameters (λ , μ) ensures anonymity.

3. Endogenous Uncertainty and Volatility: Simulation Results

We now examine the model's success in simulating the real economy. For this reason we first review the empirical averages in the U.S. of the key seven variables of the model:

p - the long term price/dividend ratio. Mehra and Prescott [1985](in short M&P [1985]) used the data base compiled by Shiller [1981] for 1889-1978. We used the updated version of the same Shiller's data base for 1889 - 1998 and estimated this variable to be 22.84;

σ_p - the standard deviation of the price/dividend ratio p. For the period 1889 - 1998 we estimated it to be 6.48 using the updated version of Shiller's [1981] data base;

R - the average risky return on equities was estimated by M&P [1985] to be 6.98%. Using the updated Shiller [1981] data for 1989 - 1998 our estimate is 8.34% suggesting that 6.98% is on the low side. We thus record the mean risky rate to be around 8.00%;

σ_R - the standard deviation of R was estimated by M&P [1985] to be 16.67%. Using the updated data for 1989 - 1998 our estimate is 18.08%;

r^F - The mean riskless interest rate was estimated by M&P [1985] to be .80% for 1889 - 1978. It is based on the 90 day treasury bill rate for 1931 - 1978. However, for 1889 - 1931 one may estimate the riskless rate by using various alternative securities. We do not offer an independent estimate and suggest that the evidence places the mean riskless rate around 1.00%. There is some evidence that this low rate has prevailed mostly since the Great Depression and that prior to 1931 the rate was higher (see Siegel [1994]);

σ_{r^F} - the standard deviation of r^F was estimated by M&P [1985] to have an average of 5.67% during the period of 1889 - 1978;

ρ - the premium of equity return over the riskless rate. Given that we set the estimate of the mean riskless rate around 1.00%, and given the evidence regarding the mean risky rate which we record as 8.00%, we conclude that the empirical evidence places the mean equity premium around 7.00%;

3.1 The Scaling Problem of OLG Models

Before proceeding we resolve the issue of scaling an RBE. The problem arises from the fact that in our OLG economy agents live for two periods and the young purchase from the old the capital stock of the economy using their labor endowment. Hence, the equilibrium $p_t = \frac{P_t}{D_t}$ in the model depends entirely on the endowment of labor income of the young. Since in the real economy it takes a generation for the capital stock to change ownership from the old to the young, our OLG model faces a problem. If the labor income of the young is of the same order of magnitude as dividend income in the economy, the model will never be able go generate a price\dividend ratio of 23. Hence, the young's labor endowment must be a large multiple of D_t *in any one year* in order to attain an equilibrium price\dividend ratio close to the historical average of about 23. To highlight the point, Table 1 below presents the equilibrium values of (p, r^F, R, ρ) in a sequence of REEs in which $\omega = \omega^1 = \omega^2$ take different values. Other parameter

Table 1: REE Solutions for Varying Values of $\omega = \omega^1 = \omega^2$

	$\omega = 12$	$\omega = 14$	$\omega = 18$	$\omega = 22$	$\omega = 23$	$\omega = 24$	$\omega = 25$	$\omega = 26$
p	11.39	13.35	17.26	21.17	22.15	23.13	24.11	25.09
r^F	10.24%	8.93%	7.21%	6.13%	5.92%	5.72%	5.54%	5.38%
R	10.75%	9.44%	7.71%	6.62%	6.41%	6.21%	6.04%	5.87%
ρ	.51%	.51%	.50%	.49%	.49%	.49%	.49%	.49%

choices in Table 1 are: $a_i = b_i = .25$ for all i; $\lambda = \mu = 1$; $\alpha_1 = \alpha_2 = .5$; $\gamma_1 = \gamma_2 = 3.25$; $\beta_1 = \beta_2 = .90$. Table 1 shows that variations in the endowment of the young acts as a scaling factor which determines the *level* of prices (p, q) and hence the average returns on securities. Note that when ω reaches the range of 24, p is close to 23 and the mean risky return is 6.21%. Both means are close to the historical average.

Our procedure is then to select that value of the endowment which results in a price\dividend ratio of approximately $p = 23$. For the RBE below, this value is $\omega = 26$. We view this as a pure scaling of the OLG model and in this sense the model does not reproduce the empirical evidence of $p = 23$, *it is scaled to that level.*

The problem of scaling the OLG model raises a deeper question, which may have already occurred to the reader: why should we expect the unrealistic OLG model to be an appropriate model for the study of market volatility? Since the discount rate is around 10%, *the unit of time* *is a year* and hence the model length of life of an agent is not an approximation of real human work life. Our answer to this question consists of two parts. First, note the fact that once the model was scaled, the predictions of the REE of the model reproduced very closely the predictions of the M& P [1985] model of infinitely lived agents. This analytical fact is the main reason why we have postponed the discussion of the question at hand until this point.

Turning to the second answer we note first that the Euler equations of an OLG agent are exactly the same as the Euler equations of an infinitely lived agent. The crucial differences between them are the definitions of their budget constraint, their consumption and wealth. Since our model assumptions imply that *aggregate* consumption is proportional to total dividends, it follows that the growth rates of dividends and aggregateconsumption are identically the same in the OLG model and in the M& P [1985] infinite horizon model, and obey the exogenous Markov process defined by (5). Given this fact we need to assess why might one expect the models to have different predictions. If the equity premium and other "anomalies" in financial market are determined by *real factors* such as the horizon of the agents' optimization and by the life cycle saving patterns over the very long horizon, then the OLG model and the infinite horizon models would yield drastically different results. Alternatively, if the characteristics of market volatility under study are *essentially* driven by expectations, then, given the Markov structure of the model, it would not make any difference whether the agents trade infinite number of times over their own life-time or only once; their expectations for one date at a time will drive the results. Hence, if our theory is right and the phenomena under study are primarily expectation driven, then the OLG model is an entirely useful model for the study of market volatility.

3.2 Results for REE: the Equity Premium Puzzle

Focusing on the case $\omega^1 = \omega^2 = 24$ we now study further the REE which is defined by the parameter choices in Table 1: $a_i = b_i = .25$ for all i ; $\lambda = \mu = 1$; $\alpha_1 = \alpha_2 = .5$; $\gamma_1 = \gamma_2 = 3.25$ and $\beta_1 = \beta_2 = .90$. The results in Table 2 represent the main components of what is known as "the equity premium puzzle." In the narrow sense, the puzzle is the observation that the model prediction of ρ is .49% while the historical average is 7.00%. Note that the REE predicts reasonably well the mean rate of return on equities but errs in predicting a riskless rate of 5.72% when the empirical average is 1.00%. Hence, the puzzle of the low equity premium is the puzzle of the *very high riskless rate* predicted by the REE of the model. M& P [1985] noted this fact.

Table 2: REE Results

variable	REE	Empirical Record
p	23.13	23
σ_p	.069	6.48
R	6.21%	8.00%
σ_R	4.12%	18.08%
r^F	5.72%	1.00%
σ_{r^F}	.88%	5.67%
ρ	.49%	7.00%

An inspection of Table 2 reveals that the equity premium is not the only problem which the REE of the model presents; *all volatility measures in the table are low relative to the historical record*. The empirical value of σ_p is 94 times larger than the REE prediction, the value of σ_R is more than 4 times larger than the REE prediction and the value of σ_{r^F} is over 6 times larger than the model prediction. One objective of Kurz [1994] and of the papers in Kurz [1997] was to demonstrate that the theory of RBE points to *Endogenous Uncertainty* as the explanation of this high volatility. Endogenous Uncertainty is that component of economic risk which is *propagated within the market* by the beliefs of the agents. This is our next subject.

3.3 A Family of RBE with Optimists/Pessimists

We study a family of "optimists\pessimists" RBE. For this family we scale the model by selecting $\omega^1 = \omega^2 = 26$ and the four parameters which characterize this family are as follows:

(i) $\lambda = \mu = 1.7542$. This is a model where an agent is optimistic when his assessment variable takes the value 1; in that state he adjusts the probabilities of high prices in the next period by a factor of 1.7542 *which is approximately the maximal feasible value.*

(ii) $\alpha_1 = \alpha_2 = .57$ hence in the majority of dates (57%) an agent is optimistic while only in 43% of the time he is pessimistic. In a large economy this assumption means that the optimists are always in the *majority* but we shall see later that this also means that the pessimists are more intense in their outlook than the optimists;

(iii) Correlation of belief parameters: a = b with $a_1 = b_1 = c_1 = .50$ and $a_i = b_i = c_2 = .14$ for i = 2, 3, 4. These parameters regulate the correlation of the states of beliefs of the agents. The state of belief is defined by a random variable L_t which takes three values: (i) $L_t = 1$ if $y_t^1 = y_t^2 = 1$ is the state OO when both agents are optimistic; (ii) $L_t = 0$ if $y_t^1 = y_t^2 = 0$ is the state PP when both agents are pessimistic and (iii) $L_t = 2$ if $y_t^1 \neq y_t^2$ is the state DIS when theagents disagree. The stochastic process $\{ L_t , t = 1, 2, ...\}$ is a Markov process with the transition matrix:

	$(OO)_{t+1}$	$(PP)_{t+1}$	$(DIS)_{t+1}$
$(OO)_t$	.50	.36	.14
$(PP)_t$	.14	0	.86
$(DIS)_t$	.14	0	.86

The condition $a_i = b_i = c_2 = .14$ for i = 2, 3, 4 means that if at t the state of belief is PP or DIS, PP cannot occur at t + 1: the agents are either optimistic or they disagree. The condition $a_1 = b_1 = c_1 = .50$ implies that a state of total optimism at date t can be followed by any state at date t+1. Hence, the structure of correlation takes the following form:

(i) unanimous optimism at t may leads to any state of belief at t + 1;

(ii) unanimous pessimism or disagreement at t *prevents total pessimism* at t + 1.

We shall see that the emergence of asymmetries in an otherwise symmetric economy is the key to understanding the structure of endogenous volatility. Observe that the transition matrix of the states of belief is not symmetric. To understand later results caused by this matrix we observe that the very high bull market prices and the very low crash stock prices result from an interaction between the growth rate of the economy determined by the states of d_t and the states of belief. This implies that the asymmetry in the transition matrix of the states of belief will translate into asymmetry in the dynamics of stock prices. We shall explore the exact pattern later.

In Table 3 we report the simulation results for $\gamma = \gamma_1 = \gamma_2$ from 2.5 to 3.5 and $\beta = \beta_1 = \beta_2$ from .85 to .95 hence these results apply to a reasonably wide range of values of β and γ. Table 3 shows that under the given parameterization, the model predicts well the historical record. If we compare the results in Tables 3 with the empirical record, we note a small difference only in two variables. The mean risky return R is close to the historical average of 8.00% and its standard deviation σ_R is clearly close to 18.08%; the riskless rate is within range of the historical average of 1.00%, and the equity premium is clearly close to the historical record of 7.00%. In the case of σ_p the historical average is 6.48 while the model prediction is around 2.5 - 3.4 and in the case of σ_{r^F} the average is 5.67 while the model predictions are

Table 3: Results for RBE with optimists\pessimists

		γ = 2.5	γ = 2.75	γ = 3.00	γ = 3.25	γ = 3.50
β = .85	p	23.06	23.12	23.19	23.26	23.34
	σ_p	2.53	2.78	3.00	3.20	3.36
	R	7.85%	8.19%	8.51%	8.80%	9.05%
	σ_R	18.76%	20.63%	22.27%	23.69%	24.89%
	r^F	2.36%	1.79%	1.22%	.66%	.12%
	σ_{r^F}	14.62%	16.12%	17.41%	18.48%	19.35%
	ρ	5.49%	6.40%	7.29%	8.14%	8.93%
β = .90	p	23.36	23.38	23.43	23.48	23.54
	σ_p	2.52	2.77	2.99	3.18	3.34
	R	7.75%	8.08%	8.39%	8.70%	8.93%
	σ_R	18.48%	20.32%	21.94%	23.35%	24.55%
	r^F	2.37%	1.81%	1.25%	.71%	.18%
	σ_{r^F}	14.40%	15.89%	17.17%	18.24%	19.11%
	ρ	5.38%	6.27%	7.14%	7.97%	8.75%
β=.95	p	23.64	23.63	23.66	23.69	23.74
	σ_p	2.51	2.76	2.97	3.16	3.28
	R	7.65%	7.98%	8.29%	8.57%	8.61%
	σ_R	18.22%	20.03%	21.64%	23.03%	23.40%
	r^F	2.37%	1.83%	1.29%	.75%	.04%
	σ_{r^F}	14.20%	15.67%	16.95%	18.02%	19.04%
	ρ	5.28%	6.15%	7.00%	7.82%	8.57%

around 14.2 - 19.4. In both cases the predictions are not accurate but the *order of magnitudes* of the model predictions and the historical record are close.

3.4 The Explanatory Neighborhood as a Method of Analysis

Why is the RBE able to explain the data? Since our model offers a resolution of the equity premium puzzle, what is the *economic interpretations* of the conditions which define the model and what are the theoretical reasons that these conditions enable the model to explain the historical record? The methodology which we use to answer these questions is central to the theory of RBE and consists of three parts:

Part 1: Our model identifies a relatively small "explanatory neighborhood" in the parameter space under which the RBE's prediction matches the empirical record;

Part 2: The explanatory neighborhood which is defined by the values of (α_1, α_2, λ , μ) has a simple economic meaning: in these RBE optimists are in the majority but the intensity of the pessimists is relatively stronger than the intensity of the optimists;

Part 3: There is no other neighborhood in the feasible parameter space that identifies RBE which match the historical record.

We start by the examination of a small neighborhood in the parameter space. In Table 4

Table 4: Results for the Explanatory Neighborhood

		$\alpha_1 = .56$	$\alpha_1 = .57$	$\alpha_1 = .58$
$\alpha_2 = .56$	p	23.56	23.59	23.97
	σ_p	2.69	2.79	2.11
	R	7.95%	8.09%	7.19%
	σ_R	19.86%	20.60%	15.84%
	r^F	3.32%	1.56%	1.41%
	σ_{r^F}	17.09%	16.42%	12.12%
	ρ	4.63%	6.53%	5.78%
$\alpha_2 = .57$	p	23.59	23.48	23.90
	σ_p	2.79	3.18	2.22
	R	8.09%	8.70%	7.31%
	σ_R	20.60%	23.35%	16.52%
	r^F	1.56%	.71%	.92%
	σ_{r^F}	16.42%	18.24%	12.70%
	ρ	6.53%	7.97%	6.39%
$\alpha_2 = .58$	p	23.97	23.90	23.87
	σ_p	2.11	2.22	1.94
	R	7.19%	7.31%	7.00%
	σ_R	15.84%	16.52%	14.35%
	r^F	1.41%	.92%	1.89%
	σ_{r^F}	12.12%	12.70%	10.96%
	ρ	5.78%	6.39%	5.11%

we report the results of varying the values of the parameters α_1 and α_2 over the range of .56,

.57 and .58. It is clear from the restrictions on the parameter space in (20) - (21) that once we change these parameters, we must also change other parameters in accord with the feasibility conditions. For example, if α_1 is changed from .57 to .58, the *maximal* value of λ which is feasible changes to 1.7241, the value of c_2 to .15 but the value of c_1 remains equal to .50. In all cases $\beta = .90$ and $\gamma = 3.25$. In examining the results in Table 4 note that within this narrow neighborhood there are model configurations which predict σ_{r^F} in the range of 11.0 - 12.7 which is only twice the historical average. We observe that the explanatory neighborhood is relatively a very small set in the parameter space.

Moving on to the Part 3 of our methodology we assert that *there is no other neighborhood in the parameter space yielding predictions which are close to the empirical record.* Given that the dividend process and the parameter values of (β_k, γ_k) for k = 1, 2 have been fixed at the specified realistic values, the small size of the neighborhood of the other four parameter is a striking fact! It implies that our RBE has a *unique* theoretical explanation of the historical record[5] which we now explore.

We thus turn to Part 2: the *economic interpretation* of the family of models which is defined by four parameters: α, λ, c_1 and c_2. We recall first that $\alpha_1 = \alpha_2 = .57$ means that both agents are optimistic in 57% of the dates.

The second parameter is $\lambda = 1.7542$ which is approximately the maximal value of the

[5] In reference to the discussion in Section 1 we note that some who oppose the use of heterogenous beliefs have argued that such models allow for too many equilibria and hence give a researcher too much freedom in explaining any empirical phenomena. The conclusion here shows that this is a superficial argument since the isolation of a small neighborhood in the parameter space which is compatible with the historical record acts exactly as identification in any econometric model. In fact, we shall argue below that the *existence* of such a set of parameters arises directly from the rationality conditions of the RBE and to that extent the method employed supports the RBE theory employed.

adjustment, by an optimist, of the probability of $((p_1, q_1), (p_2, q_2), (p_3, q_3), (p_4, q_4))$ next period. To see the implication of this choice recall the transition matrix (5) for the growth rate of dividends and the feasibility conditions (21). In the neighborhood of $\alpha_1 = \alpha_2 = .57$, we have $\alpha = 1 - \phi = .57$ and the binding feasibility constraints are $\lambda \le \frac{1}{1-\phi}$, $\lambda \le \frac{1}{\alpha_1} = \frac{1}{\alpha_2}$. Suppose that agent 1 is an optimist using F_1. As λ in F_1 rises, the rationality conditions $\alpha F_1 + (1-\alpha) F_2 = \Gamma$ require a *downward* adjustment of the probability of $((p_1, q_1), (p_2, q_2), (p_3, q_3), (p_4, q_4))$ in the pessimistic matrix F_2. Although the changes of the probabilities in F_2 are made to correspond to the change of probabilities in F_1, the rationality conditions, which regulate the relation between them, induce a *fundamental asymmetry between the intensities* of the two.

To explain the asymmetry in intensities, note that the matrix in (5) implies that the upper limit of the feasible λ is almost reached at 1.7542 when some probabilities in the matrix F_2 are close to 0. Symmetry appears to dictate an exact correspondence between the 0 entries in the matrix F_2 and the entries of 1 in F_1. At ($\alpha_1 = \alpha_2 = .57$, $\lambda = 1.7542$) *this symmetry does not hold*. If f^1_{ij} is the (ij) entry of F_1, then $f^1_{ij} = \lambda \Gamma_{ij}$ for j = 1, 2, 3, 4 and if f^2_{ij} is the (ij) entry of F_2, then $f^2_{ij} = \frac{1}{1-\alpha}[\Gamma_{ij} - \alpha\lambda\Gamma_{ij}]$. In the neighborhood of $\alpha_1 = \alpha_2 = .57$ and $\lambda = 1.7542$ we have the following *asymmetric conclusion*:

(22a) For all i = 1, 2, ..., 8, $f^2_{ij} \approx 0$ for j = 1, 2, 3, 4.

(22b) For only i = 5, 6, 7, 8, $f^1_{ij} \approx 0$ for j = 5, 6, 7, 8.

(22a) says that in the neighborhood pessimistic agents are *almost certain* that a recession will occur at date t + 1. This extreme degree of pessimism holds *for all* states of the economy at date t. Now, (22b) says that optimistic agents at date t are *almost certain* that a recession will not occur at t + 1 *only if at t the economy is in a recession* (i.e. $d_t = d^L$) . If the economy is in an expansion mode at t, the optimistic agent thinks that the probability of a recession at t + 1 is about 25%. We thus propose to view the pessimists in this configuration as being *more intensely pessimistic than the optimists* and because of this difference in intensity, they have a greater effect on the security markets. We stress that the asymmetry discussed here *is the result of the rationality of belief conditions and hence it is an essential characteristic of an RBE.*

We finally turn to the economic interpretation of the parameters a = b = (.50 , .14, .14, .14). These regulate the correlation between the assessment variables (y_t^1 , y_t^2) defined by the transition matrix of the states of belief. The central impact of this matrix is on the *dynamics* of prices: the correlation of (y_t^1 , y_t^2) implies that bull and bear markets are asymmetric. For the market to transit from the lowest price of the crash states (in the recession $d = d^L$ and the state of belief in DIS) to the highest prices of the bull market states (which occur in PP) it needs to take several steps since *it cannot go directly from the low to the high prices.* The opposite, however, is possible since at the bull market states *there is a positive probability of reaching the crash states in one step*. Thus a bull markets which reaches the highest price must evolve in several steps but a crash can occur in one step.

To sum up this section, we offer a simple and intuitive reason why the RBE generates a low riskless rate and a high equity premium. *Relative to Γ* there are, at any time, optimists and pessimists in the population of investors but on average there are more optimists than pessimists. Since over the entire population the average belief must correspond to Γ, the rationality of belief conditions imply that the intensity level of the pessimists dominates and their high demand for the riskless asset raises its price, leading to a low equilibrium riskless rate and high equity premium. This effect is further amplified by the dynamics of prices and rates of return.

3.5 The Dynamics of Asset Price and Return

We turn now to the examination of some of the dynamic characteristics of asset prices under the RBE theory. We discuss three characteristics: (i) the structure of asset price volatility, (ii) the property of time dependent variance of asset returns (i.e. the GARCH property) and (iii) the forward discount bias in foreign exchange markets.

(i) The Structure of Asset Price volatility. We have noted that little attention has been paid in recent literature to the question of price volatility and the problem of ensuring that price volatility in the model has the same structure as the volatility realized in the market. In Figures 1a, 1b we present time series of model simulation. Each contains 200 realized price\dividend ratios (which

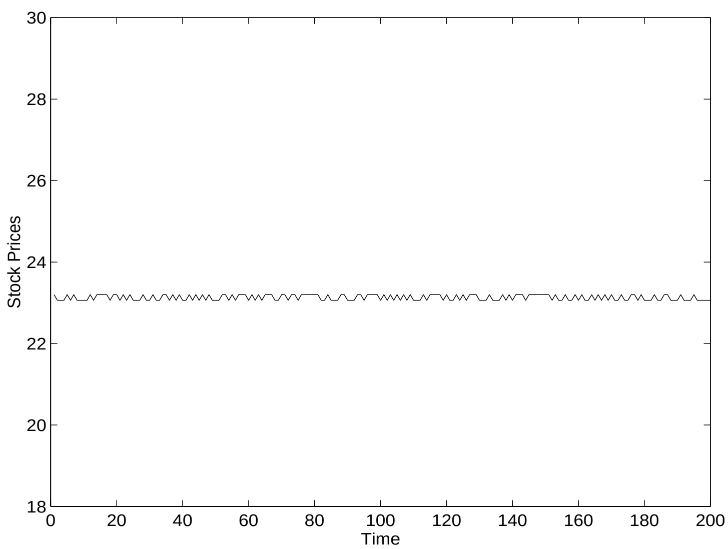

Fig. 1a: REE Simulation

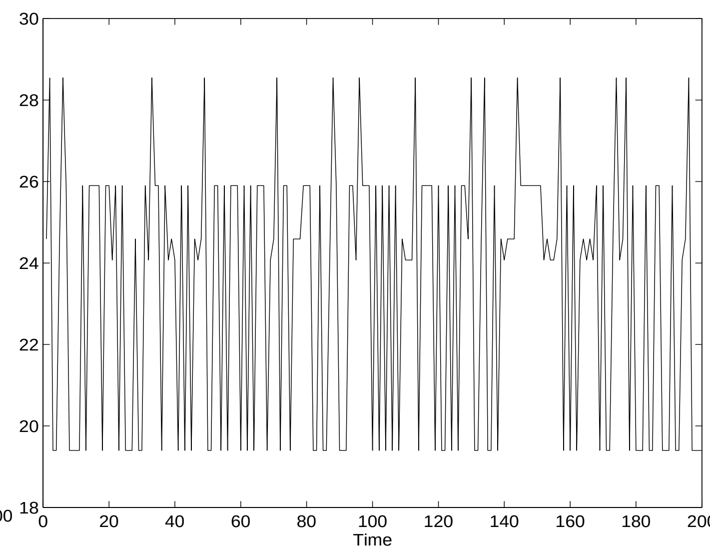

Fig. 1b: RBE Simulation

we call "the" price) generated by the REE of Table 2 and the RBE of Table 3 with $\beta_1 = \beta_2 = .90$ and $\gamma_1 = \gamma_2 = 3.25$. The standard deviation of the price\dividend ratio is .069 in the REE and 3.18 in the RBE. There are two *distinct* prices in the REE: 23.20 and 23.06 with a mean of 23.13. In the RBE there are 6 *distinct* prices with a conditional mean of 25.82 given d^H, a mean of 21.14 given d^L and with an unconditional mean of 23.48. It is natural to decompose the standard deviation of prices in the RBE into two components. The first component, which is an *amplification* of the effect of dividends on prices, is measured by the standard deviation of a random variable which takes the values of 25.82 when $d_t = d^H$ and 21.14 when $d_t = d^L$. Hence, keeping the REE functional relation between prices and exogenous variables, *amplification increases the impact of exogenous variables on prices.*

The second component of volatility is the pure effect which *the states of belief have on price volatility*. This component is uncorrelated with the exogenous dividend process and represents pure Endogenous Uncertainty which takes the form of additional prices induced by the states of beliefs and by the variability of the states of beliefs over time. To define this effect let $z_t^1 = 1$ when $d_t = d^H$ and 0 otherwise, and let $z_t^2 = 1$ when $d_t = d^L$ and 0 otherwise. Now define $e_t = p_t - 25.82\, z_t^1 - 21.14\, z_t^2$. In Figure 2 we exhibit 200 values of e_t computed from the simulated values of the RBE in Figure 1b. What is interesting about Figure 2 is the asymmetry in the distribution of e_t which is generated by the basic asymmetry in the causal structure of volatility in this model. We conclude by noting that if we take the volatility of the price\dividend ratio in the REE to be approximately the volatility that can be justified by the dividends, our analysis demonstrates that *most of the volatility of stock prices is generated by the*

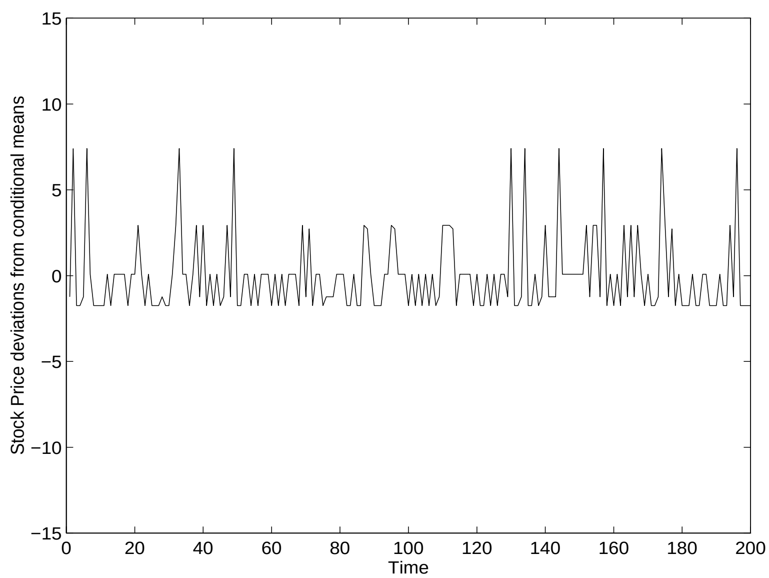

Fig. 2: RBE Simulation

beliefs of the agents either in the form of price amplification or in the form of pure endogenous volatility. Thus, most of the volatility of asset prices is endogenously generated.

(ii) The GARCH Property of Asset Returns. In Figure 3 we exhibit the R_t^2 - the *square* of the risky returns - associated with the prices generated by the RBE of Figure 1b. Note that the bursts of price volatility in Figure 1b reappear as a GARCH property of asset returns; that is, Figure 3 shows that the variance of the risky rates of return changes over time. Since the growth of dividends is a stationary Markov process, the time variability of the risky return is the result of the dynamical properties of the states of belief in the market.

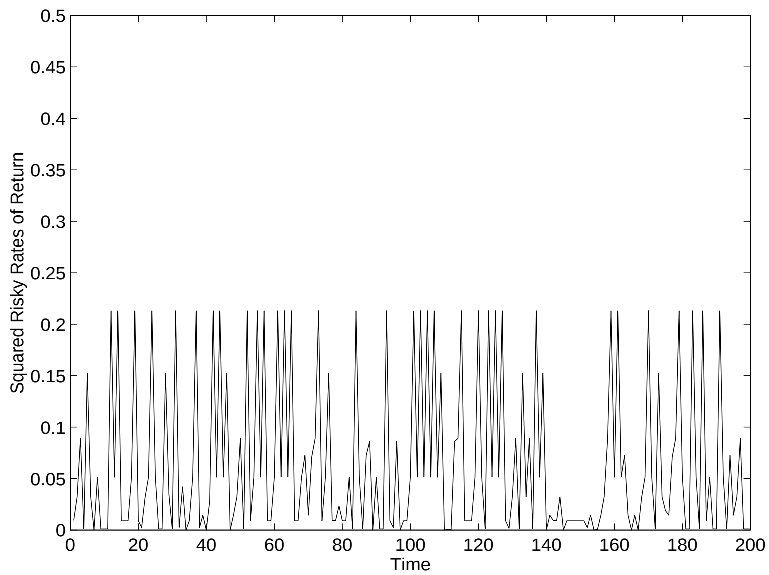

Fig. 3: RBE Simulation

What is the cause for the GARCH property of the risky return? To answer this question recall the transition matrix of the state of beliefs which we reproduce here. We observe first that a regime of "agreement" (when $y_t^1 = y_t^2$ in states OO or PP) generates price variability which is *sharply different* from the price

	$(OO)_{t+1}$	$(PP)_{t+1}$	$(DIS)_{t+1}$
$(OO)_t$	.50	.36	.14
$(PP)_t$	.14	0	.86
$(DIS)_t$	.14	0	.86

variability in the regime of disagreement (when $y_t^1 \neq y_t^2$ in state DIS). Now suppose that at some date the state of belief is the agreement state OO. From OO the economy can move to all states of beliefs. If it moves to PP it remains in the regime of agreement and if from PP it moves back to OO it completes a cycle within the regime of agreement. If, however, the economy moves from PP to DIS, a regime of disagreement is started with sharply different price volatility characteristics. Note the sharp spikes in Figure 2. The very highest price occurs only in the regime of agreement when the state of belief is in PP while the lowest "crash" price occurs in the recession when $d_t = d^L$ and beliefs are in DIS. As the states of beliefs change over time, market prices and returns move among different volatility regimes. Indeed, the volatility regimes of returns is a Markov process with varying degrees of persistence because the states of belief is a Markov process with varying degrees of persistence. We conclude that the GARCH property of asset return is caused by *the dynamic properties of the different regimes of belief*.

To further examine the GARCH property of asset returns we simulated 100,000 observations of R_t^2 in the RBE. Estimating the regression $R_t^2 = \xi_0 + \xi_1 d_t + \varepsilon_t$, we report in Table 5 the first 10 terms of the autocorrelation function of the residual of R_t^2. Note that the first three terms are large and the majority of terms are positive but decline rapidly, a result which

Table 5: The Autocorrelation Function of the Residuals of the Squared Return Regression

lag	1	2	3	4	5	6	7	8	9	10
	.026 (.003)	.044 (.003)	.016 (.003)	.007 (.003)	-.003 (.003)	-.005 (.003)	.0007 (.003)	.0003 (.003)	.001 (.003)	.004 (.003)

is compatible with the evidence (see Brock and LeBaron [1996]). We have explored several models that may best describe the behavior of the data over time. Following the Akaike Information Criterion, we found that the following E-GARCH(1, 1) model fits the data best:

$$R_t^2 = \underset{(.0003)}{-.3192} + \underset{(.0002)}{.3541}\, d_t + \varepsilon_t \ , \qquad \varepsilon_t \sim N(0, h_t)$$

where

$$\log(h_t) = \underset{(.0216)}{-5.8139} - \underset{(.0040)}{.2873} \log(h_{t-1}) - \underset{(.0064)}{1.6924} \left| \frac{\varepsilon_{t-1}}{\sqrt{h_{t-1}}} \right| + \underset{(.0038)}{.4938} \frac{\varepsilon_{t-1}}{\sqrt{h_{t-1}}} .$$

(iii) *The Forward Discount Bias in Foreign Exchange Markets*. Kurz [1997a] and Black [1997] developed a model which is similar to ours except for the addition of a second country and two more short term nominal debt instruments. To define the problem that was addressed in these papers suppose that you estimate a regression of the form

$$\frac{ex_{t+1} - ex_t}{ex_t} = c + \zeta (r_t^D - r_t^F) + \varepsilon_{t+1} \tag{23}$$

where $(ex_{t+1} - ex_t)$ is the change of the exchange rate between date t and date t + 1 while $(r_t^D - r_t^F)$ is the difference between the short term *nominal* interest rates in the domestic and the foreign economies. Under rational expectations the differential of the interest rates at date t should provide an unbiased predictor of the depreciation of the currency between date t and date t + 1. This means that apart from a technical correction for risk aversion the parameter ζ should be close to 1. In 75 empirical studies in which equation (23) was estimated, the estimates of the parameter ζ are significantly less than 1. Indeed, in many studies this parameter was estimated to be *negative* (see Froot [1990], Engel [1996] for an extensive survey). The failure of this parameter to exhibit estimated values close to 1 is known as the "Forward Discount Bias" in foreign exchange markets. Applying the RBE theory to this market, Kurz [1997a] and Black [1997] estimated ζ to be .152. However, the specifications in their models were different from ours and violated the condition of anonymity which we have imposed on our model. We have thus reformulated the model so as to satisfy our narrow parameter specification. There are, however, several issues that need to be evaluated first.

If we think of the first agent as the "domestic U.S." and the second agent as a "foreign economy" then we need to reformulate the model so as to allow the introduction of two nominal interest rates, two different monetary policies and a different stochastic structure. We thus assume that there is only one stock market in the home currency and the stochastic process of dividends is as in (5). As in our model above we also assume that the endowment\dividend ratio of the domestic agent is a constant ω and the domestic economy has a real bill which is traded by both agents. But then, how should we model the second country? What is the meaning of an exogenous shock in the foreign country? With such difficulties we (along with Kurz [1997a] and Black [1997]) model a *hypothetical* foreign economy which is characterized as follows:

(i) the endowment\dividend ratio ω^* of the foreign agent is a random variable with two states $(\omega^{*H}, \omega^{*L})$ which is i.i.d with the probability of $\omega^* = \omega^{*H}$ being .8;

(ii) the shocks to endowment are small, say of 2% - 3% hence in the REE $\omega^{*H} = 24.6$ and $\omega^{*L} = 23.4$ and in the RBE $\omega^{*H} = 26.6$ and $\omega^{*L} = 25.4$. Monetary policy in the home economy is responsive to the dividend shocks and monetary policy in the foreign country is responsive to the endowment shock in the foreign economy. The main reason for the endowment shock in the foreign economy is to allow the determination of the exchange rate in any REE;

(iii) an RBE requires a selection of a Γ^* matrix to generate the stationary measure of the equilibrium dynamics. A matrix that satisfies the requirements specified is

(24) $$\Gamma^* = \begin{bmatrix} .8\phi A & .8(1-\phi)A & .2\phi A & .2(1-\phi)A \\ .8(1-\phi)B & .8\phi B & .2(1-\phi)B & .2\phi B \\ .8\phi C & .8(1-\phi)C & ,2\phi C & .2(1-\phi)C \\ .8(1-\phi)D & .8\phi D & .2(1-\phi)D & .2\phi D \end{bmatrix}.$$

where A, B, C, and D are matrices of the form (16). The crucial ingredient of the Explanatory Neighborhood is the assumption $\alpha_1 = \alpha_2 = .57$ and $\lambda_s = \lambda = \mu = \mu_s = 1.7542$ *and we shall continue to maintain this assumption.*

(iv) in our basic domestic model we set A = B and a = b = (.50, .14, .14, .14) which we shall continue to assume. Given that the probability of $\omega^* = \omega^{*H}$ is .8, it follows from the structure of the matrix Γ^* that 80% of the time, the international economy will look very much like our domestic economy when the second agent has endowment of ω^{*H}. But now, how should we select C and D? What about the other 20% of the time when the lower part of Γ^* is realized? To consider this point note that the *arbitrary* stochastic structure introduced by the i.i.d process of $\{\omega_t^*, t = 1, 2, \ldots\}$ introduces into Γ^* a new and arbitrary element which may have nothing to do with the way the international economy *actually* works. This change must have some effect on the dynamics of the states of beliefs. The effect that we found was entirely minimal and is represented by the simple specification c = a = b = (.50, .14, .14, .14) but d = (.57, .14, .57, .14). Hence we can view the international model as a proper extension of our earlier model.

Summary of specification: $\phi = .43$, $\beta_1 = \beta_2 = .90$, $\gamma_1 = \gamma_2 = 3.25$, $\alpha_1 = \alpha_2 = .57$, $\lambda_s = \lambda = \mu = \mu_s = 1.7542$, a = b = c = (.50, .14, .14, .14), d = (.57, .14, .57, .14). In the REE ($\omega = 24$, $\omega^{*H} = 24.6$, $\omega^{*L} = 23.4$) ; in the RBE ($\omega = 26$, $\omega^{*H} = 26.6$, $\omega^{*L} = 25.4$).

Table 6 presents the simulation results for the REE and the RBE of the specified international model. In Table 6 ex denotes the "exchange rate" and σ_{ex} is the standard deviation of the exchange rate. Note first that the results for the REE are essentially the same as the results in Table 2 and the parameter ζ is computed to be .95, as is expected. From the point of view of comparing the RBE with the REE the only new result is the much larger variance of the foreign exchange rate in the RBE relative to the REE. Since the foreign economy is hypothetical we do not suggest any particular value for ex and σ_{ex}. Turning finally to the RBE, we observe that the

Table 6: Results for the Reformulated International Model

variable	REE	RBE	Empirical Record
p	23.31	23.94	23
σ_p	.37	2 .70	6.48
R	6.21%	7.80%	8.00%
σ_R	4.72%	19.34%	18.08%
r^F	5.64%	1.52%	1.00%
σ_{r^F}	1.89%	16.37%	5.67%
ρ	.57%	6.28%	7.00%
ex	.68	.67	----
σ_{ex}	1.29%	9.93%	----
ζ	.95	.47	diverse < 1

results here are essentially the same as in Tables 3 or 4 but the new result is the simulated equilibrium value of $\zeta = .47$ which is significantly less than 1. We thus can conclude that the *Forward Discount Bias is one more anomaly which is explained by the same model*. We note that sharper results for the parameter ζ could be obtained without any expected effect on the other parameters by formulating the foreign sector in a more realistic way.

Why does the RBE predict a value for ζ which is much lower than 1? Start by recalling the REE argument in favor of ζ close to 1. If $\zeta < 1$ then in an REE agents can make an *expectational arbitrage*: they can borrow in one currency and invest in the other, expecting that the net return on their investment will be larger than the depreciation of the currency. In a stationary world in which all agents hold the same rational expectations the possibility of such a riskless arbitrage cannot be an equilibrium. Note that in world of securities (rather than an

Arrow-Debreu world of contingent claims) this is not an arbitrage in the strict sense of the term since the trades *do not take place at the same time.*

In an RBE agents hold diverse beliefs and borrow and invest based on their own beliefs. In such a world a differential nominal interest rates across countries offers an investment opportunity but now such investment is subjected to endogenous uncertainty. This results in a true, equilibrium, process of the exchange rate which exhibits excessive fluctuations in part due to variability in the states of belief of the agents. Hence, at almost all dates the nominal interest differential between the two countries is a *biased* estimate of the rate of depreciation of the exchange rate one period later. Why should we expect that under rational beliefs $\zeta < 1$? To see why, consider first an REE in which the difference between the domestic and foreign nominal rates is z%. In that equilibrium you do not need to form expectations on currency depreciation. It is sufficient for you to believe that other investors or currency arbitrageurs know the true probability of currency depreciation and they have already induced the interest differential to be equal to the average rate of currency depreciation *which will be z%.* Now consider an RBE. All agents know that no one knows the true probability distribution of the exchange rate and therefore the exchange rate is subject to endogenous uncertainty. Being risk averse, agents who invest in foreign currency would demand a risk premium on endogenous uncertainty and over the long run the difference $(1-\zeta)$ *is the premium received by currency speculators* for being willing to carry foreign currency positions. For a positive premium it follows that $\zeta < 1$.

We close this discussion with a conjecture. A phenomenon known as "smile curves" appears in many markets for derivative assets. An application of our theory to such markets faces significant technical difficulties and for this reason we have not studied the effect of endogenous uncertainty on derivative markets. However, given what we know about the phenomenon in question we would conjecture that the present model would predict the emergence of smile curves in derivative markets.

4. **Conclusions**

In this paper we advance the proposition that most of the observed volatility of asset prices and returns is driven by the beliefs of agents and we thus question the basic proposition of REE or the Efficient Markets theory that asset prices are determined only by fundamental values. The alternative theory of RBE predicts the emergence of Endogenous Uncertainty which is the additional component of social risk and volatility which is propagated within the economy by the beliefs and actions of agents.

By implication, the theory of RBE proposes that a large number of phenomena which are viewed as REE "anomalies" in financial markets such as the equity premium puzzle, the GARCH property of asset returns, the Forward Discount Bias in foreign exchange markets, the "smile curves" in derivative pricing and many others, are all *expectational phenomena* in the sense that they are entirely the consequences of the dynamics of the distribution of beliefs (i.e. the state of beliefs) in our markets. They have nothing to do with "fundamental" causes or exogenous variables. In support of this claim we present in this paper *a single RBE model* which is calibrated to the long term U.S. statistics and with which we study some of these "anomalies". The model predicts the empirically observed order of magnitude of:

(i) the long term mean and standard deviation of the price\dividend ratio;

(ii) the long term mean and standard deviation of the risky return on equities;

(iii) the long term mean and standard deviation of the riskless rate;

(iv) the long term mean equity premium.

In addition, the model is able to predict

(v) the GARCH property of risky asset return;

(vi) the Forward Discount Bias in foreign exchange markets.

The common economic explanation for these phenomena is the heterogeneity of the beliefs of agents. Given such diversity, some agents are optimistic and some pessimistic. In a simple model which allows for these two states of belief there is a unique parameterization under which the model makes all the above predictions *simultaneously*. This parameter choice requires the optimists to be in the majority but the the RBE rationality conditions of requires the pessimists to have a higher intensity level. This intensity has a decisive effect which increases the demand for riskless assets, decreases the equilibrium riskless rate and increases in the equity premium.

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