Stable Demands and Bargaining Power in Majority Games

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Abstract

This paper provides a cooperative as well as a non-cooperative analysis of weighted majority games. The cooperative solution concept introduced here, the *Stable Demand Set*, yields a meaningful selection within the Mas-Colell Bargaining Set, it contains the Core, it eliminates the "dominated" coalition structures, and has sharp implications for weighted majority games: for such games it is *non-empty*, it predicts a *unique* stable demand vector for every homogeneous representation, and every agent within the winning coalition is expected to obtain a payoff share proportional to her relative bargaining power. The set of stable demand vectors coincides with the set of balanced aspirations defined in Bennet (1983), but it is obtained in the space of individually rational payoff configurations, rather than restricting attention to the aspirations domain. I then define two different kinds of non-cooperative coalitional bargaining games, showing that the set of Symmetric Stationary Subgame Perfect Equilibria of one of them, and the set of Subgame Perfect Equilibria of the other, have a one-to-one correspondence with the Stable Demand Set for homogeneous weighted majority games.

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1 Introduction

Weighted majority games are an important class of games. They have been one of the favorite grounds of confrontation for cooperative game theorists and social choice theorists, because they present very clearly most of the difficulties of studying the stability properties of coalitions when payoff division is endogenous. Moreover, they are very relevant for applications, especially in formal political theory: the formation of a coalitional government or any other type of agreement/decision in a parliament have to be studied with game-theoretic models (cooperative or non-cooperative) of coalitional bargaining. A solution concept for majority games can be helpful to the modeler also in problems like the determination of corporate governance, the resolution of international negotiations, the allocation of scarce resources in activity analysis.

The existing cooperative solution concepts in the literature focusing on imputations fail to provide an adequate prediction of the outcomes of weighted majority games. The Core of many voting games and all constant-sum essential games is empty, and cannot be used as a guideline. Value concepts yield only an *ex ante* evaluation, hence they cannot offer predictions about the equilibrium coalition structure, nor about the prevailing payoff distribution within the prevailing coalitions. Solution concepts like the Bargaining Set, the Stable Set, and the Kernel, avoid the "existence" problems of the Core and yield some different predictions, but the set of solutions is often too large. Most solution concepts determine the distribution of gains *within* given coalitions or coalition structures, and hence are very helpful to model arbitration problems, where coalitions are formed before the bargaining over the distribution of payoffs begins. On the other hand, as pointed out in Bennet & Zame (1988), the Aspirations approach seems natural for situations where "players can negotiate over payoffs before committing themselves to particular coalitions." The aspirations approach seems to be more appropriate to capture the features of majority games and make predictions about them. This paper takes great inspiration from the aspirations approach, and especially from the work on *Balanced Aspirations* (see Bennet 1983). However, rather than limiting the attention to the aspirations domain, we will obtain stable aspiration vectors (which we will call stable demand vectors in order to avoid confusion with the axiomatic concept of aspirations) in the space of individually rational payoff configurations. In other words, while on one hand the traditional imputation approach determines payoff distribution after having fixed a coalition structure, and while on the other hand the aspirations approach determines coalitional outcomes after having fixed a payoff distribution, we allow payoff distribution and coalition formation to be simultaneously determined, and inspite of this larger strategic space, we obtain balanced aspirations as distributional outcomes.

The cooperative solution concept introduced in this paper, the *Stable Demand Set*, is a subset of the Mas-Colell Bargaining Set and contains the Core. The Bargaining Set is non-empty for every coalition structure, even for unreasonable ones; the "dominated" coalition structures[1] are instead never part of a solution in the Stable Demand Set. With respect to the Bargaining Set, the two innovations yielding such a "selection" are the following:

1. we make use of the fact that every allocation (or imputation) can be viewed as a *pair* consisting of a *demand vector* and a *coalition structure*;

2. for every proposed pair, the set of counter-objections (to the objections to such a proposal) is restricted to include only those pairs that use the same demand vector as in the original proposal.

The Stable Demand Set is *non-empty* in every weighted majority game, and predicts a *unique* payoff distribution for every equilibrium coalition structure if the game has an equivalent homogeneous representation. For every vector of weights the bargaining power of each

[1]See Shenoy (1979) for a first study of the possibility to eliminate dominated coalition structures.

player is obtained endogenously (looking at the number of winning coalitions each player can belong to), and players who turn out to have the same bargaining power are said to be of the same type. Thus, the Stable Demand Set provides a complete characterization of weighted majority games with an explicit treatment of bargaining power. Every player participating in a winning coalition is expected to receive a payoff share *proportional* to her bargaining power, which is uniquely defined for every vector of weights. For games that admit a representation where all the minimal winning coalitions have the same number of votes, known in the literature as "homogeneous" weighted majority games, the payoff share going to a player in a winning coalition is *exactly* equal to the ratio between the number of votes she contributes to the coalition, and the number of votes that constitutes a majority. For games that do not admit an homogeneous representation, the payoff can differ slightly from the previous one, due to the presence of minimal winning coalitions with different numbers of votes.

The non-cooperative analysis of the same class of games requires the solution of an n-player coalitional bargaining game with heterogeneous endowments. We will show that there exist at least two game forms which give a non-cooperative foundation to the Stable Demand Set, with rules very similar to those followed in the formation of government coalitions in many parliamentary democracies. In the first game form a player, selected to be the first mover, proposes a coalition and a distribution of payoffs. If the players, to whom the offer is made, accept, responding sequentially, the game is over; if one rejects, he becomes the new proposer, but he cannot include the previous proposer in his new offer. The game ends when some offer is accepted by all the members of a proposed coalition.[2] If

[2] Payoff shares are bargained upon in the process of coalition formation. There is very little in the literature on coalitional bargaining with endogenous payoff division. Ray & Vohra (1996) produced interesting results in this important direction, but mainly for symmetric games, i.e., for games where players are *ex ante* identical. In majority games, as in most other games, players have instead heterogenous endowments. Other

the game admits a homogeneous representation, its unique Symmetric Stationary Subgame Perfect Equilibrium outcome has a one-to-one correspondence with the Stable Demand Set, while when the game does not admit any homogeneous representation the set of equilibria is larger.
The second non-cooperative game considered is a sequential demand game, similar to that introduced by Selten (1992). The Subgame Perfect Equilibrium outcome of such a game displays the same payoff distribution as in the Stable Demand Set, without having to use any stationarity refinement.

The paper is organized as follows. In section 2 we define the games we are most interested in, and introduce the problem of heterogeneous endowments; in section 3 we present the cooperative solution concept and some of its general properties; in section 4 we provide a full characterization for weighted majority games; section 5 contains the non-cooperative alternate-offer analysis of majority games, and the full implementation result for homogeneous games; section 6 presents an algorithm to compute the Symmetric SSPE outcomes for any endowment vector; section 7 shows the implementation result using demand games, and section 8 contains some concluding remarks. All the proofs are in the Appendix.

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authors, like Hart & Kurz (1983) and Aumann & Myerson (1988), keep the coalition formation and the payoff distribution problems separate, obtaining predictions that do not seem appropriate for the class of games we are interested in (see discussion on example 6). Finally, there is a non-cooperative legislative bargaining literature in political science, where the order of play has too big a role, affecting payoff distribution even within the prevailing winning coalition. All the game forms proposed in this paper, on the other hand, have the common feature of yielding distributional outcomes that do not depend on the order of play, at least when the weighted majority game admits an equivalent homogeneous representation.

2 Simple Games with Heterogeneous Types

2.1 Definitions

Let us consider a finite set $N \equiv \{1, 2, \ldots, i, \ldots, n\}$ of players ($n \geq 3$). Let us denote a generic coalition of players by S and the set of all possible coalitions by $\mathcal{S} \equiv \{\mathcal{S} \subseteq \mathcal{N}\}$. Σ will denote the set of partitions, i.e., coalition structures, of N; $\sigma \in \Sigma$ will denote an element of such a set, i.e., a specific coalition structure. Consider then the characteristic function $V : \mathcal{S} \longrightarrow R_+$.

Definition 1 *A coalitional game* (N, V) *is a* simple game *iff*

$$V(\emptyset) = 0, \ V(N) = 1, \ V(S) = 0 \ \textit{or} \ 1,$$

and

$$V(S) = 1 \quad \textit{whenever} \ V(T) = 1 \quad \textit{for some} \ T \subset S.$$

Denote by $\Omega \equiv \{S : V(S) = 1\}$ the set of *winning coalitions* (WC) and by $\Omega^m \equiv \{S : V(S) = 1, V(T) = 0 \ \ \forall T \subset S\}$ the set of *minimal winning coalitions* (MWC).

Definition 2 *A simple game is called* proper *if and only if* $\forall S \in \Omega, \quad N \setminus S \notin \Omega.$

Definition 3 *A simple game* (N, V) *is a* weighted majority game *if and only if there exists a vector of non-negative weights* w *and a number* q *(*$0 < q \leq \sum_{i=1}^{n} w_i$*) such that*

$$S \in \Omega \iff \sum_{i \in S} w_i \geq q.$$

Assumption 1 $q = \lfloor \frac{\sum_{i=1}^{n} w_i}{2} + 1 \rfloor$, *so that a weighted majority game is always a proper simple game and majority means simple majority. For simplicity, let us also assume that* *the total number of votes $p \equiv \sum_{i=1}^{n} w_i$ is an odd number.*[3]

Assumption 2 $w_i < q \quad \forall i.$[4]

Definition 4 *A weighted majority game admits a* homogeneous *representation iff $\exists$ a vector w (and an induced quota q) such that*

$$\sum_{i \in S} w_i = q \quad \forall S \in \Omega^{m5} \tag{1}$$

Example 1 *Consider a four-player game where player1 holds 3 votes, player2 holds 2, and players 3,4 hold 1; this representation is not homogeneous: there is one MWC with 5 votes, and three MWCs with 4 votes; however, an equivalent representation of the same characteristic function game is one where player1 has 2 votes, players 2,3,4 have one vote each, and this is homogeneous.*

2.2 Heterogeneous Types

Denote by W^i the set of winning coalitions (WC), for a given game (N, V), containing player i. Notice that $W^i \subseteq \Omega$, and $W^i = \Omega$ only if i is a veto player. Denote by μ^i the number of WCs in W^i. Similarly, denote by M^i the set of MWCs containing player i, and m^i is the number of MWCs in M^i.

Definition 5 *Player i and player j are of the same* type *if and only if the characteristic function is unchanged when permuting them.*

[3] In parliaments p is the total number of seats, and is almost always an odd number. In any case, if p were an even number one could always take an equivalent representation of the game where each player i would have a renormalized weight $\omega_i = w_i - \frac{w_i}{p}$ (and where therefore the total number of votes would be $p-1$).

[4] Otherwise the coalitional game would be irrelevant.

[5] The term "homogeneous weighted majority game" was first introduced by Von-Neumann and Morgenstern (1944).

For weighted majority games the definition of type relates in particular to the number of winning coalitions players can be in and hence an equivalent definition is, for this class of games, that two players are of the same type if $\mu^i = \mu^j$. The simple number of votes (weight) each player has is instead not a correct measure of bargaining power. To see this, consider the following example:

Example 2 *Three players, one with* 5 *votes, one with* 4 *votes, and one with* 3 *votes;* 7 *votes are needed to win, and hence every pair of players can make it. Every player has the same number of MWCs (and of WCs), and permuting them does not change the characteristic function, even though every player has a different weight. Here it is clear that there is no reason why the player with* 5 *votes should have more bargaining power than the other two.*

In general it is therefore the number of WCs that determines bargaining power and hence the different types of players. While it is always true that if $w_i = w_j$ then $\mu^i = \mu^j$, the converse is true (as established below) if the game is homogeneous.

Lemma 1 *If the coalitional game* (N, V) *admits a homogeneous representation through some vector* w*, then* $\mu^i = \mu^j \longrightarrow w_i = w_j$*,* $\forall i : M^i \neq \emptyset,\ \forall j : M^j \neq \emptyset$*.*

Lemma 2 *If the coalitional game* (N, V) *admits a homogeneous representation through some vector* w*, then* $m^i = m^j$ *iff* $w_i = w_j$*,* $\forall i, j :\ M^i \neq \emptyset,\ M^j \neq \emptyset$*.*

Lemma 3 *Consider a homogeneous weighted majority game satisfying Assumptions 1 and 2; for every player* i*, either* $m^i = 0$ *or* $m^i \geq 2$*.*

3 Cooperative Solution

In this section we introduce the cooperative solution concept, defining it for any TU game and discussing some general properties. In section 4 we will use it and show all its implications for weighted majority games.

3.1 Aspirations and Feasible Assignment

According to Bennet's definition,[6] a payoff vector x is an aspiration iff

1. $x(S) \geq V(S) \;\; \forall S \subseteq N$;

2. $\forall i \in N \; \exists S$ containing i such that $x(S) = V(S)$.

Limiting attention to the space of aspirations, also called *aspirations domain*, it has been possible to define a bunch of solution concepts, all of which are non-empty because proposals and objections are restricted to belong to the aspirations domain. Our solution concept will be instead defined on an unrestricted domain, and will show that some subset of the set of aspirations is stable even when the objections are not restricted to the aspirations domain.

Consider the unrestricted set R^n_+. A demand vector $\alpha \in R^n_+$ specifies what the n players would like to obtain from the game. No feasibility constraint is imposed directly on these demands. For any given pair (α, σ) (recall that σ denotes a coalition structure), a feasible allocation is obtained using the following *payoff assignment rule*:

$$\begin{array}{rcll} \alpha_i^\sigma & = & \alpha_i & \textit{if } i \in S \in \sigma : \; \sum_{j \in S} \alpha_j \leq V(S) \\ & & 0 & \qquad\qquad\qquad \textit{otherwise} \end{array} \tag{2}$$

In words, the demands are assigned as actual payoffs to the members of a coalition S only if their sum is feasible given the worth of S, otherwise all the members of S are assigned 0. We will denote by $\alpha_i(S)$ the payoff assigned to i when i belongs to S, which can take one of the two values depending on the feasibility of α for S.

[6]See for example Bennet 1981 and 1983.

3.2 Acceptable Objections

Consider a proposed allocation (α, σ).

Definition 6 *We say that a coalition T can* block *(or,* make an objection to*) the proposal* (α, σ) iff *there exists an allocation vector y such that*

$$y_i > \alpha_i^\sigma \ \forall i \in T$$

and $\sum_{i \in T} y_i \leq V(T)$ (i.e., y is feasible for T).

Definition 7 *A coalition Z can* counter *(or,* make a counter-objection to*) the objection pulled out by T against the proposal* (α, σ) iff *(1) $Z \cap T \neq \emptyset$ and (2) the* original *demand vector α is such that*

$$\begin{aligned} \alpha_i(Z) &> y_i \qquad \forall i \in T \cap Z \\ &\geq \alpha_i^\sigma \qquad \forall i \in Z. \end{aligned} \tag{3}$$

Definition 8 *An objection to* (α, σ) *is* acceptable *iff it cannot be countered using α itself.*

Notice that with respect to the standard definition of a counter-objection (as used in most versions of the Bargaining Set) we restrict the set of possible counter-objections to include only those that can be derived using the same demand vector of the original proposal. We also require the inequality within $Z \cap T$ to be strict.

3.3 The Stable Demand Set

We say that an objection (y, S) is *anonymous* iff $y_i = y_j$ whenever i and j (both in S) are of the same type. This restriction on objections clearly bites only if there exist players of the same type, which generically does not happen outside the realm of simple games.

Definition 9 *The* Stable Demand Set *is the set of pairs* (α, σ) *such that there is no anonymous acceptable objection to it.*

If players can counter-object using an allocation that can be derived from the same demand vector proposed at the beginning, it means that the demand vector itself is *stable*.[7]

Limiting the counter-objections to those that have some relationship with the original proposal, like we do here, seems to have a lot to do with reality, since the demands of people do not seem to change much during the bargaining process. Intuitively, trying to obtain larger shares (with an objection) does not pay if there is the risk of being simply excluded from a coalition that distributes the unchanged demands of the other players. The anonymity requirement generically does not bite, but if there are players of the same type, then it seems reasonable to ask that those players should be treated equally when they belong to the same coalition. Requiring that objections be anonymous is irrelevant for theorem 1, but it plays some role when the weighted majority game is not homogeneous.

Proposition 1 *The Stable Demand Set is contained in the Mas-Colell Bargaining Set and contains the Core.*

The comparison with other relevant solution concepts is hinted in example 4.

[7]The term "stable demand vector" was first introduced by Selten (1981): according to Selten's axiomatic definition, a demand vector is stable if it satisfies *Maximality, Feasibility*, and *Balancedness*. The first two axioms correspond to the two used to define aspirations, and in Selten's terminology the vectors that satisfy just those two axioms are called "semi-stable". Denoting by $\mathcal{S}_i(\alpha)$ the set of coalitions containing i that are feasible given α, Balancedness requires that for any pair of players i, j either $\mathcal{S}_i(\alpha) = \mathcal{S}_j(\alpha)$, or $\mathcal{S}_i(\alpha) \setminus \mathcal{S}_j(\alpha) \neq \emptyset$ and $\mathcal{S}_j(\alpha) \setminus \mathcal{S}_i(\alpha) \neq \emptyset$. This definition of stable demand vectors is very similar to that of "Partnered Aspirations, given in Bennet (1983). The set of stable demand vectors that we *obtain* is not always stable in the Selten sense. To see this, consider the three-player game where $V(1,3) = V(2,3) = 10$, $V(1,2,3) = 15$, $V(S) = 0$ otherwise. In this example it is easy to check that $\alpha = 5, 5, 5$ is a stable demand vector in our sense, but not in the Selten sense, nor it is in the set of partnered aspirations. Another example would be a game with four players, where $V(1,2,3) = 9$, $V(i) = 0$, $V(S) = 1$ for every other S: in this case the vector $3, 3, 3, x$ (together with the coalition structure $\{(1,2,3); 4\}$) is in the SDS, for every x. Instead, the feasibility requirement imposed in Selten's and Bennet's definitions forces x to be equal to 0.

The SDS may be empty in non symmetric games. To see this, consider a four-player game where $V(1,2,3) = 1,\ V(1,2,4) = 1+\epsilon,\ V(1,3,4) = 1-\epsilon,\ V(2,3,4) = 1+\epsilon/2$; $V(S) = 0$ otherwise. If $\epsilon \neq 0$ all players are of different types, and thus anonymity does not bite. In this example, whatever (α, σ) one starts from, it is possible to find an acceptable objection where two players are given more than α, making it impossible to counterobject using α itself.

3.4 The Selection of "Undominated" Coalition Structures

It is well known that the two main problems of the Bargaining Set are (1) the fact that it is *"too large"*; (2) the fact that it is sometimes difficult to compute.[8] We will show that the SDS may constitute an improvement on both dimensions: as far as the computability problem is concerned, section 4 indicates that it is possible, at least for some classes of games, to construct algorithms to generate the allocations in the SDS. On the other hand, we can now show that the SDS does not allow "dominated" coalition structures, and therefore constitutes a meaningful selection of the Bargaining Set.[9] To show why such selection is meaningful let us first define what a dominated coalition structure is in our context and then give an example.

Definition 10 *A coalition structure σ is* dominated *given α if either*

$$\sum_{i \in T} \alpha_i(S) < V(T),\ \textit{for some } T \subset S \ \ \textit{for some } S \in \sigma \tag{4}$$

or when $\exists T : V(T) > \sum_{i \in T} \alpha_i^\sigma$ and T is the union of elements of σ.

Example 3 *Consider the players' set $N = \{1,2,3\}$ with*

$$V(1,2) = 20,\ V(2,3) = 40,\ V(1,3) = 30,\ V(1,2,3) = 42,\ V(i) = 0\ \forall i.$$

[8]See Maschler (1992) for a lucid discussion of these and other pros and cons of the Bargaining Set.

[9]The problem of dominated coalition structures was first discussed in Shenoy (1979).

In this example (taken from Maschler (1992), example 3.4*) there exists an allocation in the Bargaining Set* for any *coalition structure. However, the grandcoalition and the "all singletons" structure are dominated. The pair formed with the grandcoalition and the payoff allocation* $(4, 14, 24)$ *is in the Bargaining Set, but it is dominated (by any of the coalitions of two players). On the other hand, the same pair cannot belong to the SDS; the coalition (*$1, 2$*), for instance, has an acceptable objection using the allocation* $(5, 15)$*.*

The SDS only *contains pairs with one of the three undominated structures:* $(\{1, 2\}, \{3\})$, $(\{1, 3\}, \{2\})$, $(\{2, 3\}, \{1\})$*. The* unique *stable demand vector is:*

$$\alpha_1 = 5,\ \ \alpha_2 = 15,\ \ \alpha_3 = 25.$$

Remark 1 *It is true* for every *coalitional game* (N, V) *that pairs* (α, σ)*, where* σ *is dominated given* α*,* cannot *belong to the SDS. In fact, if (4) holds for some* T*,* T *itself can form an acceptable objection.*[10]

4 Characterization for Majority Games and Proportionality of Payoffs

Remark 1 has the following implication for the selection of coalition structures in weighted majority games:

Lemma 4 *The only "candidate pairs"* (α, σ) *for the SDS of any proper weighted majority game are those where*

(1) σ *always includes a MWC* $S \in \Omega^m$*, unless there are dummy players;*[11]

[10]The objection would be acceptable because any coalition Z trying to block the objection by T would have to give the player(s) in the intersection strictly more than their αs.

[11]In principle a dummy player, i.e., a player that is not crucial for any coalition, could be part of a winning coalition and receive 0. In this case the set of possible winning coalitions that could be part of a solution

(2) α is such that $\sum_{i\in S}\alpha_i = V(S)$.

We can now characterize the SDS of weighted majority games with two theorems. They could be combined, but we prefer to keep them separate because the first one has an instructive direct proof while the second requires a programming algorithm.

Theorem 1 *Consider a weighted majority game (N, V) that admits a homogeneous representation, satisfying Assumptions 1 and 2; the SDS of any such game is* non-empty *and* only *contains pairs (α^*, σ) where the* unique *stable demand vector has*

$$\alpha_i^* = \frac{w_i}{q}, \quad \forall i.$$

Example 4 *Consider, as an example, the so called Apex Game.*[12]

There are $n = 5$ players, where $\{1,2,3,4\}$ have one vote each and player 5 has three votes. Thus, $q = 4$; the MWCs are:

I: $\{5, i\}$;

II: $\{1,2,3,4\}$.

The SDS predicts what we believe to be the most reasonable thing, i.e., that if **I** *forms then player 5 receives $3/4$, and the other player gets $1/4$ (proportional payoffs); if* **II** *forms they share equally ($1/4$ each).*

The proof that the only stable demand vector (for this example) is the proportional one can be summarized as follows.

Consider first the proposal where the demand vector is $\alpha^* = (1/4, 1/4, 1/4, 1/4, 3/4)$ and the coalition structure includes the WC **I**;

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doesn't necessarily coincide with the set of MWCs.

[12] Davis & Maschler (1965) used this example to contrast the predictions of the existing solution concepts. The name "apex game" came later.

one possible objection to this proposal is coalition **I** but with a different small player, and where y is $(3/4+\epsilon, 1/4-\epsilon)$; however, these objections are not *acceptable*, because there exists an objection to each of them, with the vector α^* and the WC **II**. The second (and last) kind of objection to the pair $(\alpha^*, \mathbf{I})$, would be one with the four small players together, where the "blocker" receives $1/4+\epsilon$ and the others share the rest; but then at least one of these other three small players must be receiving less than $1/4$, and can therefore counter by offering again α^* to player 5.[13]
Consider then the pair $(\alpha^*, \mathbf{II})$;
the objection to be considered here is one with **I** and payoffs $(3/4-\epsilon, 1/4+\epsilon)$; to this, however, there exists a counter-objection with $(3/4, 1/4)$ and **I** (with a simple replacement of the small agent in the WC), and hence there is no acceptable objection. It is finally easy to see that no other pair can be in the SDS.

The Core of the Apex Game is empty, as it would be for any other constant sum essential game. The Shapley Value only looks at the ex ante balance of power, which here implies 3/5 for the big player and 1/10 for each small player. Similarly, the Nucleolus with respect to the grandcoalition has the allocation $(3/7, 1/7, 1/7, 1/7, 1/7)$ (the homogeneous representation of the game itself)[14] and can again be interpreted as an ex ante evaluation. While this solution assigns each player the appropriate relative bargaining power, this property disappears when the Nucleolus is computed with respect to coalition structures containing a MWC: for example, if MWC **I** forms, the Nucleolus gives each of the two players a payoff of 1/2, inspite of the very different endowments. Since we have established that the coalition structures including a MWC are the only meaningful ones, the Nucleolus and the Shapley Value are therefore not appropriate solutions for these games if one wants an *ex*

[13] Notice that here, and more generally in theorem 1, we did not need to use anonymity.

[14] See Peleg (1968).

post prediction, i.e., a prediction of the payoff distribution *contingent* on one of the possible coalition structures prevailing, i.e., contingent on which MWC prevails.

The Bargaining Set, on the other hand, not only allows for dominated coalition structures (like the grand-coalition), but it also gives an uninformative prediction about the payoff distribution even when contingent on an undominated coalition structure. In fact, it contains **I** as one of the possible coalitions in a solution, but with a payoff vector $x = (x_5, 1-x_5)$, where x_5 can take any value between 3/4 and 1/2. The latter extreme ($x = (1/2, 1/2)$) is the Kernel of the Apex Game. The only existing solution concept which yields the same prediction as the SDS for the Apex Game (but not for homogeneous or non- homogeneous weighted majority games in general), is the Main Simple solution in the Stable Set.

When the game does not admit any homogeneous representation, the SDS is characterized by the following result:

Theorem 2 *Consider a weighted majority game* (N, V) *satisfying Assumptions 1 and 2. Any such game has a* non-empty *SDS. For some* σ *satisfying Lemma 4, the pairs* $(\hat{\alpha}, \sigma)$ *are in the SDS, where* $\hat{\alpha}$ *solves the program:*

$$\min_{\alpha} \sum_{i \in N} \alpha_i \tag{5}$$

subject to

$$\sum_{i \in S} \alpha_i \geq 1 \quad \forall S \in \Omega \tag{6}$$

The program above is the same as the one used by Bennet (1981-1983) to characterize the set of *Balanced Aspirations*; thus, even though the SDS allows to consider objections outside the aspirations domain, the obtained set of stable demand vectors coincides with the set of Balanced Aspirations of the game.

When the game is not homogeneous the linear programming algorithm may be satisfied by a range of values of α (between two corners of the polytop) and all those values can generate pairs in the SDS.

Example 5 *To see this, consider a game with* 5 *players: one with* 3 *votes - call her player* k *- one with* 1 *vote - call her player* j *- and three with* 2 *votes* ($p = 10, q = 6$)*. In this example any* α *with* $\alpha_i = 1/3$ *for every player* i *with* 2 *votes, with* $\alpha_k \in [1/3, 2/3]$*, and with* $\alpha_j + \alpha_k = 2/3$*, does satisfy (5) subject to (6). The coalition structure including the MWC with one player of each type, paired with any* α *in that range, is in the SDS.*[15]

To see the connection between the direct proof of theorem 1 and the algorithm, consider example 1: the two players with one vote and the player with two votes are of the same type (they also have the same number of MWCs). The solution in the SDS has $\frac{2}{3}$ for the player with the most votes and $\frac{1}{3}$ for the others. With a demand by the big player greater than $\frac{2}{3}$, α_i would have to be less than $1/3$ for at least some i (because otherwise the total sum would be greater than for $\hat{\alpha}$, violating (5)), but then this violates (6) unless some other $j \neq i$ has $\alpha_j > \frac{1}{3}$, which again would imply a violation of (5). Similarly, for any demand by the big player smaller than $\frac{2}{3}$, (5) is violated whenever (6) is satisfied. The same solution $\hat{\alpha}$ is obtained when taking the equivalent homogeneous representation and applying theorem 1.

Remark 2 *Notice that the SDS depends only on the characteristic function, and hence it is invariant to the particular representation chosen.*

Remark 3 *In every* 3*-player weighted majority game where Assumptions 1 and 2 are satisfied, the unique Stable Demand vector is* $\hat{\alpha} = (1/2, 1/2, 1/2)$*.*

[15] In this example anonymity is needed, and it would be needed even if we took an equivalent representation of the game with an odd number of votes. With anonymity all the possible objections must include j or k or both, and at least one player of type i; the latter must therefore (for fesibility) be offered less than $1/3$; hence all these objections are countered by the coalition with the three players of type i.

Remark 4 *The results on the SDS found for weighted majority games can be extended to activity analysis and production problems with indivisibilities. Take any production problem with indivisibilities (e.g. where output is some fixed task or is produced one unit after the other); if Y is the value of such a production or task, we could normalize it to* 1 *and apply the same reasoning followed for majority games. In a production economy the distribution still reflects bargaining power and the latter depends on relative scarcity, i.e., on the whole description of endowments and technology. We will deal with this extension in a separate paper.*

5 Alternate Proposals Bargaining

The vector w can be interpreted, for example, as the vector of weights that the parties have when they bargain in the parliament. The problem of forming a coalitional government (common to most parliamentary democracies) is an important case of n-player coalitional bargaining with heterogeneous endowments. In this and in the next section we will study a model of such situations based on alternate *offers*, while in section 7 we will study a *demand* game, and both game forms can be seen also as attempts to give different non-cooperative foundations to the SDS.

5.1 The Rules of the Game

Let us denote by $S^i \in \mathcal{S}^i$ the coalition *proposed* by player i (obviously including i). The second component of a proposal is the payoff vector $x^i(S^i)$, specifying a payoff for every player in S^i.[16] The order of response β^i (chosen by i if i is the proposer) determines the order in which the proposed members of S^i are called to respond. r^i_l denotes the l-th

[16] A vector $x^i(S)$ has of course as many components as the number of players in S.

responder, l going from 1 to $|S^i| - 1$.

Let us call the first proposer P_k;[17] her proposal can be defined as follows:

Definition 11 Proposal: *The agent chosen to be the first proposer (P_k) makes a proposal which specifies:*

1. *A coalition S^k;*
2. *A payoff vector $x^k(S^k)$;*
3. *An order of response β^k.*

Bargaining process: The player in r_1^k is called to respond to the proposal; if he accepts, then the play moves to the next responder; if he rejects, the whole proposal is eliminated from the table, and the player rejecting the proposal becomes the next proposer. The only restriction on the space of possible proposals for the new proposer - say P_i - is that S^i cannot contain P_k.[18] If the first responder to the proposal by P_k accepts, then the player in r_2^k is called to respond, and again there are the same two possibilities as above. If all the responders accept, S^k gets formed, and the game is over.[19] If some h-th responder rejects, then the proposal is removed, and the same player, i.e., the one in r_h^k, is selected to be

[17] In the literature on coalition formation games the protocol is usually a random variable (see for example Ray & Vohra (1996) and Bloch (1996)). Here too, we can think of the first proposer as being selected randomly. In real problems like the formation of coalitional governments, the first proposer is usually one of those with the largest number of votes.

[18] The role of this rule will become clear below. It is however a plausible mechanism for negotiating the formation of a government coalition. A Head of State (or even a constitution) might find this kind of mechanism useful to force the first proposer to make the best possible proposal, knowing that otherwise the rejecter would exclude her from the next iteration.

[19] For simplicity, coalitional proposals are implicitly restricted to be winning coalitions, so that we can avoid the useless complication of defining the branches of the game tree where players make coalitional proposals that even when accepted do not conclude the game.

the next proposer; and P_k cannot be included in the new proposal. Similarly, whenever the proposal of some P_i is rejected by a player, in some responding position r_h^i, the new proposer cannot include P_i in his proposal.

It should be obvious that a proposal by a generic proposer P_i has some chance of being accepted only if it satisfies

$$\begin{aligned} \sum_{h \in S^i} x_h^i &\leq 1 \\ S^i &\in \Omega \end{aligned} \tag{7}$$

In fact, if it does not include a winning coalition it cannot distribute any payoffs, and if it proposes an unfeasible distribution we can assume that even if they accept they will all receive 0 due to the unfeasibility itself, and hence the responders would always reject in the first place. If a responder j does not have any feasible alternative to propose, then he accepts the proposal on the table.

The individual *strategies* for the game in extensive form $\gamma(n, w)$ that we are describing are as follows: denoting by a_i the action that player i prescribes for herself at a node and by A_i the actions space, such action can only take the form of a *proposal* (see definition 11) at the nodes where i is a proposer (i.e., when $i = P_i$), or of a *response*, that we denote by R^i, which can simply be *yes* or *no*, to a proposal of some other player. In other words:

$$\begin{aligned} a_i = \quad & \{S^i, x^i(S^i), \beta^i\} \quad \textit{when } i \textit{ is a proposer } (i = P_i) \\ & R^i \text{ (yes or no)} \quad \textit{when responding to a proposal} \end{aligned} \tag{8}$$

We will concentrate only on Stationary strategies. The only thing that could make two nodes different is the identity of the previous proposer, since she cannot be included in the new proposal. The actions space when i is a proposer and j was the previous proposer (which can be denoted by $\mathcal{S}^i(j)$) depends on who j was.

5.2 Equilibrium

The set of equilibria considered here is a subset of the set of Subgame Perfect Equilibria.

Definition 12 *A strategy profile is a* Stationary Subgame Perfect Equilibrium *(SSPE) iff*

1. *it is Nash in every subgame, there is no profitable deviation for any player at any node where (s)he plays, given the other players' strategies;*
2. *the strategy for each player i prescribes identical actions a_i at nodes with the same characteristics, i.e., where i has the same actions space and the payoff structure is the same; time does not matter.*

The restriction to consider only Stationary equilibria is only for simplicity, but it is a logical one and without much loss of generality. In fact, even though there are definitely many non-Stationary equilibria of the game (since there is no discounting), it is true on the other hand, that for any equilibrium allocation corresponding to a non-Stationary strategy profile, there exists a Stationary one that could determine the same equilibrium allocation.

Definition 13 *A strategy profile satisfies Symmetry if and only if players of the same type have the same strategy.*

Existence of SSPE is never a problem, but in some games the set of equilibria may be large. However, restricting attention to *Symmetric* SSPE, the set of equilibria has a one-to-one correspondence with the SDS for homogeneous weighted majority games. Before moving to this implementation result in theorem 3, let us give the intuition through the Apex Game.

Example 6

$$N \equiv \{1, 2, 3, 4, 5\},\ w_i = 1\ (i = 1, 2, 3, 4),\ w_5 = 3.$$

In this example the unique Symmetric SSPE strategy profile is the following: (1) player 5 *accepts offers* $\geq \frac{3}{4}$ *at the nodes where she is a responder (if the proposal is feasible), and proposes MWC* **I** *with allocation* $x_5^5 = \frac{3}{4}$ *and* $x_i^5 = \frac{1}{4}$ *when she is a proposer; (2) any other player i's strategy is to accept offers* $\geq \frac{1}{4}$ *(if the proposal is feasible) when she is a responder, and to propose MWC* **II** *with equal sharing when she is a proposer and the previous proposer was player* 5*; (3) i proposes MWC* **I** *with allocation* $x_i^i = \frac{1}{4}$ *and* $x_5^i = \frac{3}{4}$ *when she is a proposer and some other player of the same type was the previous one. Given this equilibrium profile, if the first proposer is* P_5 *the equilibrium MWC is* **I**.[20] *To see why there are no other symmetric SSPE profiles, suppose player* 5 *is the first proposer; can there be a Symmetric SSPE where* P_5 *offers* $x_i^5 = 1/4 + \epsilon$ *and all the other players reject any offer below* $1/4 + \epsilon$*? The answer is negative:* P_5 *can deviate by offering* $1/4$ *to any player i chosen at random and i will accept, because there is no continuation equilibrium where i could obtain more than* $1/4$ *after rejecting the proposal by* P_5.[21]

Similarly, no Symmetric SSPE exists where the proposed payoff is $x_i^5 = 1/4 - \epsilon$ *(and the strategy of the other players is to accept* $\forall x_i \geq 1/4 - \epsilon$*). The first responder would reject*

[20] Among other things, this example illustrates why it is not appropriate to separate the distributive problem from the coalition formation process: in fact, by doing so, Aumann & Myerson (1988) find that the only stable coalition structure in the Apex Game is the one where the MWC is **II**. They obtain this prediction by assuming that there is an exogenous mapping from the set of cooperation structures to payoff assignments, so that players cannot affect payoff sharing within a coalition at all. Instead, we believe that even though the worth of coalitions may well depend on the prevailing coalition structure, payoff shares are bargained upon ex ante and are therefore endogenous. In reality it is difficult to observe cases of a party with 3/7 of the votes being left out, and in general we believe that our approach takes bargaining power more explicitly into account.

[21] In fact, if he rejects, given the other players' strategies, no proposal by i where i gets more than $1/4$ could ever be accepted. Given the deviation by P_5 (who now wants $3/4$), no acceptance would occur in any subsequent round either.

and propose the coalition with the other three players of his type, which would be in fact feasible given the other strategies.

Example 7

$$N = \{1, 2, 3, 4\},\ w_i = 1\ (i = 1, 2, 3),\ w_4 = 3.$$

This is the same as in example 6 but with 3 *instead of* 4 *players of the same type; q is still* 4*, but now the only feasible MWC is the one containing the big player and one small player. Here the small players have no outside option and the only equilibrium payoff is* $x_4^4 = 1$ *and* $x_i^4 = 0$*, which is also the unique allocation in the Core and SDS of the game. In fact, suppose that there exists an equilibrium where* 4 *offers* $x_i^4 = \epsilon > 0$ *(or rejects offers below* $1 - \epsilon$*) and the other strategies are compatible, i.e., the acceptance threshold for the other type is* ϵ*. Then look at the deviation by the big player, who offers* $x_i^{4'} = (\epsilon - \delta)$ *such that* $\epsilon > (\epsilon - \delta) > 0$*. Given that the responder could not turn around to make a proposal to the big player, he would have no alternative offer to make respecting (7); hence the only option for the small player is to accept. Because this is true for every* $\epsilon > 0$*, the proposed outcome* $(1, 0)$ *is the only equilibrium one.*

Theorem 3 *Consider a game* $\gamma(n, w)$ *where the vector* w *is an equivalent homogeneous representation of the vector of weights, i.e., such that* $\sum_{i \in S} w_i = q\ \forall S \in \Omega^m$*. Given Assumptions 1 and 2, the Symmetric SSPE strategies of the game prescribe*

$$\begin{array}{lll} a_i^* = & \{S^{i*}(j), x^{i*}(S^{i*}(j)), \beta^{i*}(S^{i*}(j))\} \quad \textit{whenever} & i = P_i \textit{ and} \\ & & j \textit{ is the previous proposer} \\ & \textit{yes} & i \in S^j \textit{ and } x_i^j \geq \frac{w_i}{q} \\ & \textit{no} & i \in S^j \textit{ and } x_i^j < \frac{w_i}{q} \end{array} \tag{9}$$

where $S^{i*}(j) \in M^i \cap \{\Omega^m \backslash M^j\}$*,* $\beta^{i*}(S^{i*}(j))$ *is any order of response of the players in* $S^{i*}(j)$*, and* $x_h^{i*}(S^{i*}(j)) = \frac{w_h}{q}\ \forall h \in S^{i*}(j)$*.*

The unique stable demand vector α^* shown in theorem 1 corresponds to the x^* of theorem 3. Which MWC prevails depends on who is the first proposer in the game, while in the cooperative approach this dependence was obviously unspecified. The game $\gamma(n, w)$ implements the SDS, giving the first non-cooperative content to the proposals and objections of the cooperative approach. In the Apex Game, for example, it is clear that **I** (the MWC with the big player and one small player) will be the prevailing MWC if 5 is the first proposer, while the MWC **II**, together with the proportional payoff division, can be the outcome of the coalitional bargaining game only if there is some chance for a small player to be the first proposer. If *ex ante* every player has the same probability of being chosen as first proposer, and if every player is indifferent between coalitions that give her the same payoff (absence of ideology), it can be easily seen that the MWC including the big player has probability $\frac{3}{5}$, and the homogeneous MWC has probability $\frac{2}{5}$.

6 Algorithm for Symmetric Stationary SPE

Lemma 5 *If $m^i \leq 1$ for some i and if the first proposer is the player with the largest number of votes, no SSPE of $\gamma(n, w)$ can contain a payoff greater than 0 for such a player i.*

Remark 5 *Lemma 5 extends to any player j whenever $\{\Omega^m \setminus M^k\} \cap M^j = \emptyset$ (where M^k is the set of MWCs containing the first proposer, P_k).*

This section shows that the equilibrium of a game $\gamma(n, w)$ is sometimes not unique, but the set of equilibria is always well defined and computable. The algorithm to find the equilibria is as follows.

Step 1: Assume that the first proposer P_k is the one with the most votes. Denote by $W^j(x)$ and $M^j(x)$ the set of WCs and MWCs (resp.) that remain feasible for j given a vector x of acceptance thresholds. Even if the proposer only proposes payoffs for the players participating in her coalitional proposal, we can imagine, for the sake of constructing the algorithm, that she actually has to choose explicitly a payoff for everyone. Given this interpretation caveat, *take* any MWC $S^k \in M^k$, and choose the payoff vector that solves the following program:

$$\min_{x \in X} \sum_{i \in S^k, \neq P_k} x_i \tag{10}$$

subject to

$$\sum_{h \in S^j} x_h \geq 1 \;\; \forall S^j \in \{\Omega^m \setminus M^k\} \cap M^j, \;\; \forall j \in S^k;^{22} \tag{11}$$

and subject to

$$m^j(x) \geq 2 \;\; \forall j \in N \setminus S^k \textit{ such that } \{\Omega^m \setminus M^k\} \cap M^j \neq \emptyset \textit{ and } m^j \geq 2. \tag{12}$$

The last condition that has to hold is:

$$x_j = 0 \;\; \forall j : \; m^j \leq 1 \;\; \text{or } \{\Omega^m \setminus M^k\} \cap M^j = \emptyset.^{23} \tag{13}$$

Step 2: Given the result of step 1 for every possible $S^k \in M^k$, the proposer chooses a coalition S^{k*} such that

$$x_k^{k*} = x_k^k(S^{k*}) \geq x_k^k(S^k) \;\; \forall S^k \in M^k.$$

Theorem 4 *A strategy profile constitutes a Symmetric SSPE for $\gamma(n, w)$ (given Assumptions 1 and 2) if and only if the* acceptance thresholds *of all players satisfy the program (10) subject to (11), (12) and (13) for some S^k chosen by some P_k.*

[22] This constraint says that the vector x must be such that the winning outcome is exhausted or exhaustable by any MWC available in principle to anyone who rejects (and that therefore cannot include P_k in any new proposal).

[23] See lemma 5 and remark 5.

Example 8 *Consider a game where*

$$n = 7,\ w_1 = w_2 = w_3 = w_4 = 1,\ w_5 = w_6 = 2,\ w_7 = 3;$$

$p = 11$ *and* $q = 6$*. This game is not homogeneous because there exists a MWC with* 7 *votes.*

Let us find the equilibria of the game in example 8, starting from the application of step 1 to the MWC including one player of each type (i.e., with one of the first four players, player 7, and one of the remaining two). For simplicity, denote by x_{t1}, x_{t2}, x_{t3} the (symmetric) acceptance thresholds for the players of the three different types, where the four players with 1 vote are type $t1$, and so on. The program, when the big player is the first proposer, works as follows:

$$\min_{x \in X} x_{t2} + x_{t1}$$

subject to

1. $2x_{t1} + 2x_{t2} \geq 1;$
2. $x_{t2} + 4x_{t1} \geq 1;$
3. $x_{t1} + x_{t2} \leq 2x_{t2};$
4. $x_{t1} + x_{t2} \leq 3x_{t1}.$

The first two constraints correspond to (11); the other two are there to make sure that the selected MWC minimizes payments for P_7, guaranteeing that step 2 is not violated.

From this system of inequalities we obtain

$$\frac{1}{6} \leq x^*_{t1}(S^*) \leq \frac{1}{4};\ x^*_{t2}(S^*) = \frac{1}{2} - x^*_{t1}(S^*);\ x^*_{t3} = \frac{1}{2}.$$

To see that any vector x^*, solving this system, is indeed an equilibrium, note that, because x^* satisfies constraints 1 and 2, there is no alternative MWC in $\Omega^m \setminus M^k$ that any of the players of type 1 or 2 can find that would improve upon the initial proposal; constraints 3 and 4 make sure that there is no deviation for P_7 either.

Suppose now that the MWC with two positions offered to the players of type 2 is chosen; then the payoff x_{t2} offered to each player of type 2 should satisfy:

$$\min x_{t2}$$

subject to

1. $2x_{t1} + 2x_{t2} \geq 1;$
2. $x_{t2} + 4x_{t1} \geq 1;$
3. $2x_{t2} \leq x_{t1} + x_{t2};$
4. $2x_{t2} \leq 3x_{t1};$
5. $m^j(x) \geq 2 \quad \forall j \text{ of type } 1.$

The first four constraints have the same interpretation of the four constraints for the previous case, while the last one corresponds to (12). The only solution of this program is

$$x_{t2} = \frac{1}{4};\ x_{t3} = \frac{1}{2},\ x_{t1} = \frac{1}{4}.$$

As shown, P_7 is *indifferent* between this equilibrium and any of the equilibria where the first S^* is selected.

Following exactly the same steps, one could also verify that the only equilibrium compatible with P_7 offering the coalition with three small players, is $x_{t1} = \frac{1}{6}$, which again makes P_7 indifferent. P_7 obtains 1/2 in every equilibrium, when he is the first proposer. Player 7 is, however, not necessarily in the equilibrium winning coalition if he is not the first proposer.

In example 8 the *proportional payoff distribution* $(3/6, 2/6, 1/6)$ is an equilibrium, but there are also other equilibria where players of type 1 receive slightly more (upto 1/4).

By applying the algorithm for every possible first proposer chosen at random, the proportional payoff $(3/6, 2/6, 1/6)$ still constitutes an equilibrium vector of thresholds. The vector $(3/6, 2/6, 1/6)$ is also the unique demand vector of the pairs in the SDS. By comparing the algorithm of theorem 2 with the one in theorem 4 one can see that if a vector solves the former, it is definitely a solution for the latter. Hence, when the game does not admit a homogeneous representation, the set of Symmetric SSPE outcomes *contains* the SDS allocations.

7 Implementation by Demand Commitment Bargaining

Consider a weighted majority game that admits homogeneous representations, and, for notational convenience, take the homogeneous representation where each w_i is the *fraction* of the total number of votes that is held by party i (with $\sum_{i=1}^{n} w_i = 1$). Let ρ denote the order of play, where $\rho(i) = l$ means that player i is the l-th to move. Consider a perfect information extensive form game $\Gamma(n, w, \rho)$, where players move only once, sequentially, according to the order ρ. For any ρ, let $p(\rho, w)$ be the number such that

$$\sum_{i:\rho(i)<p(\rho,w)} w_i < q, \qquad \sum_{i:\rho(i)\leq p(\rho,w)} w_i \geq q.$$

$\forall i : \rho(i) < p(\rho, w)$ the only action available is a *demand* $x \in [0, 1]$. When the game gets to the player in position $p(\rho, w)$, then, if $\exists S \subseteq \{i : \rho(i) < p(\rho, w)\}$ such that $\sum_{i\in S} x_i \leq 1$, she can choose whether to form the winning coalition with S (demanding $x \leq 1 - \sum_{i\in S} x_i$) or just make the demand and let the next player move. If instead $\sum_{i\in S} x_i > 1$ $\forall S \subseteq \{i : \rho(i) < p(\rho, w)\}$, obviously she has only the option of making a demand.

For any stage $l > p(\rho, w)$ reached by the game, player $\rho^{-1}(l)$ has the same set of possibilities as those just described for $\rho^{-1}(p)$. The game ends as soon as one player makes the complementary demand to form a winning coalition, or, if nobody has done it when all players have moved, the game ends anyway and they all get 0. For any given order of play, the finite game $\Gamma(n, w, \rho)$ has a unique Subgame Perfect Equilibrium outcome $S^*(\rho), x^*(\rho)$, and the following theorem establishes the implementability of the SDS through demand bargaining.

Theorem 5 *In any game $\Gamma(n, w, \rho)$ where $n \geq 3$ and $\sum_{i \in S} w_i = q$ for every $S \in \Omega^m(w)$, the* unique *equilibrium payoff distribution is*

$$x_i^* = \frac{w_i}{q} \quad \forall i: \ \rho(i) \leq p(\rho, w) \text{ and } M^i(w) \neq \emptyset.$$

This theorem confirms that a proportional payoff distribution is a robust prediction for weighted majority games. For every solution in the SDS there exists an order of play ρ such that the unique SPE of $\Gamma(n, w, \rho)$ determines the same distributional outcome.

Moreover, the non-cooperative game defined in Selten (1981) implements (in SPE) the set of semi-stable demand vectors, while the game form used here implements the set of stable demand vectors.[24] Bennet & Van Damme (1991) used a similar model for Apex games, but they obtained a unique prediction only using the refinement of "Credible" SPE, while with our game form no refinement is necessary.

8 Concluding Remarks

In this paper we have explored the cooperative as well as the non-cooperative implications of the methodological standpoint that payoff distribution and coalition formation should be studied simultaneously. Since Value concepts give only an *ex ante* evaluation of the prospectives of different players, they cannot be used to predict the *ex post* payoff distribution in an equilibrium coalition structure. Solution concepts that keep the spirit of Core-like competition, respecting individual rationality as well as group rationality, seem

[24] For the class of games considered here a stable demand vector of Selten is stable also in the sense defined in this paper.

more appropriate for this task. However, no existing solution concept of this kind could provide a unique prediction for every proper simple game. The Stable Demand Set gives a prediction of payoff distribution that is sharper than any other concept in the Core-like tradition. Moreover, selecting undominated coalition structures, the Stable Demand Set allows for inefficient outcomes when the game is superadditive but not balanced.

The connection highlighted in this paper between the bargaining process underlying the Stable Demand Set and the rules of the games implementing it may be very useful and lead to predictions that are more grounded and have more positive justifications than those of approaches where this link is not created. Also, the stable demands of the games studied are monotonic in the bargaining power of players, which is intuitively an important property for any positive theory of coalition formation, cooperative or non-cooperative. Our analysis deals with bargaining power explicitly, and this allows us to map endowments into payoff allocations directly. In a weighted majority game the bargaining power of a player depends on the number of winning coalitions she can belong to, and determines the types of players, both when the players can negotiate effectively (cooperative case) and when they cannot (non-cooperative case). In this way the bargaining power of each player is uniquely defined.

We have provided a unique characterization of the solutions in the Stable Demand Set for majority games, relating them to the Symmetric SSPE of an alternate proposals bargaining game and to the SPE outcomes of a sequential demand game. Especially the latter connection between the Stable Demand Set and sequential demand games can be fruitfully explored for other classes of games. In future research we intend to do so, and we will use the results of this paper in some applications to political economy issues, such as the problem of party formation. In different electoral systems, voters' preferences determine the distribution of seats in the parliament in different ways, and in order to study the incentives to party formation in each system it is important to predict the distribution of payoffs for any possible outcome of the elections. Other potential applications are activity analysis, the representation of economies with indivisibilities in production, and the resolution of collective bargaining problems.

Appendix

PROOF OF LEMMA 1:

Suppose that $\mu^i = \mu^j$ and $w_i > w_j$. Consider all the WCs that contain either i or j: $\{W^i \cup W^j\} \setminus \{W^i \cap W^j\}$. If we take every single coalition in $W^j \setminus \{W^i \cap W^j\}$ and substitute j with i, we always obtain a WC with i, while, when substituting i with j in every coalition in $W^i \setminus \{W^i \cap W^j\}$ this is not the case: $w_i > w_j$ plus homogeneity implies that there exists at least one MWC containing i but not j with exactly q votes, and this in turn implies that after the substitution the coalition would not be winning anymore. Hence $\mu^i > \mu^j$. Contradiction. $\square$

PROOF OF LEMMA 2:

One direction, i.e., if $w_i = w_j$ then $m^i = m^j$ is obvious. In order to show the other direction, we use an argument by contradiction. Suppose that $m^i = m^j$ but $w_i \neq w_j$; without loss of generality, take $w_i > w_j$.

Notice first that $w_i + w_j \leq q$, otherwise the coalition $\{i, j\}$ would be a MWC, violating homogeneity. Let us consider the set $M^j \setminus \{M^i \cap M^j\}$ of all the MWCs containing only j and not i.

Claim 1 *The set $M^j \setminus \{M^i \cap M^j\}$ must be non-empty.*

Proof: In fact, being empty would mean that every MWC containing j would have to contain i as well; knowing that M^i and M^j are non-empty, this would mean that there would exist a MWC S where i and j are together in S; but then the coalition $\{N \setminus S\} \cup \{j\}$

is a WC (guaranteed by Assumptions 1 and 2); from this, one can get to a MWC containing j by subsequent elimination of players. **End proof of claim 1**

For each MWC in $M^j \setminus \{M^i \cap M^j\}$ then, take out j and replace her with i. Since $w_i > w_j$, the new coalition could not be a MWC because the game is assumed to be homogeneous. However, the new WC containing i can be reduced to a MWC containing i by definition of MWCs. At least one MWC always exists, and hence $m^i \geq m^j$ (even if we have considered only the MWCs containing i derived from M^j by substitution).

To see that actually $m^i > m^j$, take any $S \in \{M^j \setminus \{M^i \cap M^j\}\}$. As argued above, there exists $T \in M^i$ obtained by substitution, and, given homogeneity, i must replace j *and* a player (or set of players) owning $w_i - w_j$ votes. But then, given Assumption 1, there exists another MWC Z, containing the player(s) owning $w_i - w_j$ votes, containing i, and containing a weak subset of the set $N \setminus S \setminus \{i\}$. For example, if $w_i - w_j = 1$, then $N \setminus S \setminus \{i\}$ has $q - 1 - w_i$ votes, and if we sum those to the votes owned by i and to the $w_i - w_j$ votes mentioned above, we get exactly q votes.

□

Proof of Lemma 3:

Suppose that for some i there exists *a* MWC $S \in M^i$. Our assumptions guarantee $w_i < q \ \forall i$, and that there is no veto player. Knowing that S has $q = \lfloor \frac{p}{2} + 1 \rfloor$ votes, the coalition $T \equiv \{N \setminus S\} \cup \{i\}$ must be a winning coalition, because $p - q + w_i \geq q$. If $w_i = 1$ then T is a MWC and hence $m^i \geq 2$. If $w_i > 1$ then $T \notin \Omega^m$, otherwise homogeneity would be violated; however, there must exist $Z \subset T$ such that $Z \in \Omega^m$, by definition of MWC, and such a coalition Z must contain i, otherwise S would not have been winning in the first place; so, again, $m^i \geq 2$. □

Proof of Proposition 1:

Since the set of acceptable objections is restricted (with respect to Core theory), while the set of proposals is not, obviously the SDS contains the Core. To see that the SDS is also generically contained in the Mas-Colell Bargaining Set, consider a generic game where every player is of a different type, and where therefore anonymity does not bite. Consider a pair (α, σ) generating an allocation in the SDS. This means, by definition, that there are no acceptable objections to it; i.e., if $\exists T, y : y_i > \alpha_i^\sigma \ \forall i \in T$ then $\exists Z : \alpha_i(Z) > y_i \ \forall i \in Z \cap T$. But this implies that the allocation (α, σ) is also in the Mas-Colell Bargaining Set, because with transferable utility it would always be possible to find a redistribution of the payoffs in Z to make every member of such a counter-objecting coalition strictly better off. $\square$

Proof of Lemma 4:

First of all it is easy to see that pairs (α, σ), with a σ that does not contain *any* winning coalition, are ruled out: in fact, (1) if α is feasible for some WC, then such a coalition can block, and constitute an acceptable objection; (2) if α is not feasible for any WC, then any feasible distribution within any WC constitutes an acceptable objection.

The second thing to show is that no (α, σ), with σ containing a WC S in $\Omega \setminus \Omega^m$, can belong to the SDS. To see this, notice first that if α is feasible for such a WC, then there exists for sure a MWC $T \subset S$, where everybody can be assigned more than in α, yielding therefore an "easy" acceptable objection. If α were not feasible for S, but feasible for some other WC, then such a WC would constitute an acceptable objection, and if α is not feasible for any coalition, then any coalition with a feasible distribution of payoffs is an acceptable objection.

The last thing to show is that even when a pair (α, σ) contains a σ with a MWC, the set of pairs in the SDS is restricted further, to those where α is *coalition balanced* for σ, i.e., where $\sum_{i \in S \in \Omega^m} \alpha_i = V(S)$. Well, if $\sum_{i \in S} \alpha_i < V(S)$, S itself would be the blocking coalition, with every player receiving a positive share of the surplus. $\square$

Proof of Theorem 1:

Notice first that all the MWCs can distribute the total payoff of winning (normalized to 1) using α^*. Consider then a pair where there is a MWC $S \in \sigma$ and the demand vector is α^*. We want to show first that (α^*, σ) is in the SDS of the game.

Any blocking coalition T must contain some agents in common with S (i.e., $T \cap S \neq \emptyset$), because the agents in $N \setminus S$ own $p - q < q$ votes. T can make an objection to (α^*, σ) only if there exists a payoff vector y feasible for T such that $y_i > \frac{w_i}{q} \;\; \forall i \in T \cap S$. Then there must be agent(s) $j \in T \cap \{N \setminus S\}$ receiving $y_j < \frac{w_j}{q}$. To see this, one needs only to notice that T must be a winning coalition, and hence

$$\sum_{j \in T} w_j \geq q;$$

dividing by q we have:

$$\sum_{j \in T} \frac{w_j}{q} = \sum_{j \in T} \alpha_j^* \geq 1;$$

however, knowing that feasibility implies

$$\sum_{j \in T} y_j \leq 1,$$

the claim that if someone in T gets more than α^* somebody else in T must get less than that, trivially follows.

We can now show that there always exists a counter-objection using the demande vector α^*. Consider a MWC Z, containing at least some of the agents j who receive less than α_j^* in T, and a set of agents taken from $N \setminus T$. Let us show that this blocking coalition Z would always exist and could always use the vector α^*. (1) Suppose first that y is a coalition balanced payoff vector (i.e., $\sum_{i \in T} y_i = V(T)$). Consider a blocking coalition T where there are some agents (weak subset of $N \setminus S$) receiving less than α^*. Knowing that any blocking coalition T must be a winning coalition, we can say that T holds, in general, $q + c$ votes, with $c \geq 0$. If there is only one player j with $y_j < \frac{w_j}{q}$, y_j must actually be less than $\frac{w_j}{q} - \frac{c}{q}$, which is positive only if $w_j > c$. Thus, given that $N \setminus T$ has $p - q - c = q - 1 - c$ votes, the set $\{j\} \cup \{N \setminus T\}$ has a number of votes greater than or equal to q (in fact $q - 1 - c + w_j \geq q$). A MWC Z, distributing exactly α^*, can then be found: Lemma 3 implies that j must be in at least one other MWC with other players from $N \setminus T$, and homogeneity guarantees that all MWCs have q votes. When there is more than one player receiving less than α^* in T, the argument is very similar: if there are J agents receiving less than they do in α^*, then it must be the case that $\sum_{j=1}^{J} w_j > c$, which again implies that all those agents plus the agents in $N \setminus T$ have enough votes to form winning blocking coalitions. In particular, a MWC Z, distributing α^*, can again be found.[25] (2) Suppose now (for completeness) that y is not a coalition balanced vector.[26] If an agent i receiving a positive payoff y_i in $N \setminus T$ receives $y_i < \frac{w_i}{q}$, then agent i could always be included in any of the counter-objections discussed in the previous case (1). If instead, $y_i \geq \frac{w_i}{q}$ for some $i \in N \setminus T$, then the following happens. Call G the set of I agents receiving more than α^* outside T. We know that the J agents receiving less than α^* in T must have at least

$$\sum_{j=1}^{J} w_j > c + \sum_{i=1}^{I} w_i$$

votes; thus, we know that the union of the J agents in T (who were assigned less than α^*) plus the agents in $\{N \setminus T\} \setminus G$ have enough votes to form winning coalitions ($q - 1 - c - \sum_{i=1}^{I} w_i + \sum_{j=1}^{J} w_j \geq q$). Can one of these coalitions be a coalition Z with exactly q votes, distributing α^*? The answer is again positive, because $\{N \setminus T\} \setminus G$ contains at least one MWC, with at least one of the J agents, who can all gain by reproposing α^* (which is feasible because of homogeneity).

[25] A MWC $Z \subset \{\cup_{j=1}^{J}\{j\} \cup \{N \setminus T\}\}$ needs some players among those J, and homogeneity guarantees the possibility of proportional payoffs.

[26] This would simply mean that there are some agents in $N \setminus T$ receiving positive payoffs.

We have shown that no objection to (α^*, σ) can be acceptable, because every objection would have to include a blocking coalition T which, as we have just seen, would lead to a further objection using α^* once again. In order to complete the proof, we now have to show that any pair (α, σ) with $\alpha \neq \alpha^*$, has acceptable objections.

First of all we know that we can limit ourselves to considering (α, σ) where $S \in \sigma$ has q votes, and $\alpha \neq \alpha^*$ is a coalition balanced payoff vector for σ (see Lemma 4). Since $\alpha \neq \alpha^*$, there are agents receiving $\alpha_j < \alpha_j^*$ in S. We can then show that we can always choose a MWC T such that the subsequent objections could never use α. To show this, recall that α is a n-dimensional vector, to be applied contingent on belonging to a winning coalition and conditional on feasibility for it; consider then the following exhaustive list of possibilities, denoting by A the set of agents $j : \alpha_j < \alpha_j^*$ and by B the set of agents $h : \alpha_h \geq \alpha_h^*$.

(1) Consider the case in which A is large enough to contain at least one MWC $T \subseteq A$; in this case we could choose the objection (α^*, T), so that any subsequent objection could not use α because at least one of the players in T would have to be in Z as well and would have to be assigned more than α.

(2) Consider the case in which A is not large enough to contain any $T \in \Omega^m$; in this case choose T containing A and a large enough number of players from $\{N \setminus S\} \cap B$, and take as objection (α^*, T). Any WC Z, part of any objection to (α^*, T), would then have to assign to at least one player from T more than α^*, and hence something less than that to some player from $N \setminus T$; but since $N \setminus T \subset B$, the payoff could not be derived from α, and hence (α^*, T) is an acceptable objection. □

Proof of Theorem 2:

If there is a veto player, the veto player obviously gets everything and the SDS coincides with the Core.[27] If there is no veto player, the argument is as follows.

[27] In fact, by definition of veto player, each pair containing a winning coalition has the veto player in it,

Step 0: Notice first that $\forall j : m^j \leq 1$ the component $\hat{\alpha}_j$ of the solution to the program must be 0.

Step 1: There exists at least one σ, containing a MWC S, such that $\sum_{i \in S} \hat{\alpha}_i = 1$, which implies that $(\hat{\alpha}, \sigma)$ is a feasible proposal. In fact, if that wasn't true, we could reduce $\sum \alpha_i$ without violating (6). In particular, $\hat{\alpha}$ must clearly be feasible at least for the MWC(s) with the smallest number of votes.

Step 2: Consider the set Σ_l of coalition structures containing a MWC with the *least* number of votes; consider then the set $\Sigma_r \subseteq \Sigma_l$ of structures containing a MWC with the largest number of players among the MWCs with the least number of votes; as a final restriction, consider the set $\Sigma_t \subseteq \Sigma_r \subseteq \Sigma_l$ of structures where the MWC has the largest number of types among those with the largest number of players.[28] We can show that $\forall \sigma \in \Sigma_t$, $(\hat{\alpha}, \sigma)$ is in the SDS. To see this, notice first that for (y, T) to constitute an objection to $(\hat{\alpha}, \sigma)$, it must be the case that

$$y_i > \hat{\alpha}_i \quad \forall i \in S \cap T$$

and hence

$$y_j < \hat{\alpha}_j \text{ for some } j \in T \cap \{N \setminus S\}.$$

This implies, by step 0, that $m^j \geq 2$; moreover, since j was not belonging to $S \in \sigma \in \Sigma_t$, and since none of the players in $S \cap T$ can be of the same type as j (otherwise a violation of anonymity would arise), the number of MWCs, where he belongs, that are feasible given $\hat{\alpha}$ $(m^j(\hat{\alpha}))$ is also ≥ 2. But then there exists a counter-objection $(\hat{\alpha}, Z)$ where $Z \in \Omega^m$ contains a weak subset of the set $\{j : y_j < \hat{\alpha}_j\} \cup \{N \setminus T\}$.[29] $\square$

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and no objections exist to such pairs. It is also clear that reducing the payoff to the veto player, giving some to other players in some MWCs, would violate (5).

[28] It should be clear that the MWCs belonging to structures in Σ_t contain the maximum possible number of "small" players that can be possibly contained in a MWC.

[29] For example, consider a game with 7 players: one with 3 votes, two with 2 votes, and four with 1 vote;

Proof of Theorem 3:

To see that the proposed strategy profile constitutes a SSPE, suppose not; i.e., suppose that someone has an incentive to deviate from a^*. In other words, suppose that P_k is the first proposer and consider the possibility that an agent i rejects P_k's proposal. For a rejection to make sense, he must then propose a coalition T^i together with a payoff vector $x^i(T^i)$ that assigns to himself more than $\frac{w_i}{q}$. But then the payoff vector $x^i(T^i)$ has to assign $x^i_h < \frac{w_h}{q}$ to some h (for T^i to be feasible). Then, since $M^h \setminus \{M^i \cap M^h\}$ is non-empty (see claim 1 in Lemma 2), this cannot be a profitable deviation for i, because player h in T^i would reject and propose a coalition $S^h(i)$ in the set $M^h \setminus \{M^i \cap M^h\}$ with payoff x^*.[30]

Let us now show that the Symmetric SSPE is actually unique, in the sense that it prescribes only a^*, and hence only x^* as the vector of "acceptance thresholds". It is clear that any strategy profile not containing a MWC (as a coalitional proposal) leads to easy deviations. We have to check that it is always true that any $a' \neq a^*$ would induce profitable deviations. To see this, consider a profile where the proposal by P_k is $\{x^{k'}, S^{k'}, \beta^{k'}\}$, and the acceptance thresholds are such that $x_i^{k'} < \frac{w_i}{q}$ for some player(s) i in $S^{k'}$, and $x_i^{k'} \geq \frac{w_i}{q}$ for the other members of $S^{k'}$. Take a player in the set $\{i : x_i^{k'} < \frac{w_i}{q}\}$, with the maximum "absolute" distance from the proportional payoff, (call this player j), and notice that he has a profitable deviation: he can reject[31] and propose a coalition S^j containing the maximum possible number of players from the set $\{i : x_i^{k'} < \frac{w_i}{q}\}$, plus some of the "least greedy"

$p = 11$ and $q = 6$; this game is not homogeneous (since one MWC has 7 votes). If one takes σ containing the MWC S with the four "small" players and one 2-vote player, together with $\hat{\alpha} = (3/6, 2/6, 1/6)$, it is easy to check that no acceptable objections exist to such a pair $(\hat{\alpha}, \sigma)$. This example is considered again when applying the algorithm of theorem 4.

[30] In particular, given homogeneity, it is always possible to find $S^h(i) \in \Omega^m$ where $S^h(i) \subset \{N \setminus T^i\} \cup \{h : x^i_h < \frac{w_h}{q}\}$.

[31] If the deviation is from the first proposer, obviously there is no rejection before the deviating proposal.

players from $N \setminus S^{k'}$; he can offer $x_i^j = x_i^{k'} + \epsilon \;\; \forall i \in S^{k'} : x_i^{k'} < \frac{w_i}{q}$; in this way the latter players would definitely accept the new proposal, and for ϵ small enough, the others would accept too, because if they reject they cannot include P_j and they have to replace him with a player who wants a larger share, given that P_j already selected all the "least greedy" players in the first place.[32] □

Proof of Lemma 5:

If $m^i = 0$ the claim is obvious. When $m^i = 1$ for some i, such a player cannot be the one with the largest number of votes: in fact, $\max_{i \in N} w_i < q$ by assumption; hence, if the proposer P_k is the one with the largest number of votes and $P_k \cup T$ constitutes a MWC, then $P_k \cup \{N \setminus T\}$ too has more than q votes. This establishes that i must be of some "smaller" type. But then, if the first proposer is the player with the largest number of votes, the latter never chooses to offer a positive payoff to members of the set $\{i : m^i = 1\}$, because they have no "outside option". □

Proof of Theorem 4:

Sufficiency: Let us first show that whenever a vector x is compatible with the algorithm, it can be interpreted as the vector of thresholds characterizing an equilibrium strategy profile. To see this, it is enough to check that whenever x satisfies the program, there does not exist any profitable deviation for any player. First of all, it is clear that whenever (11) is satisfied, no player in the proposed coalition can deviate by proposing a new coalition where he gets more than the proposed payoff and nobody else in the new coalition gets less than in x. The only deviations that are left to check are those where some player - say j - in the new coalition is offered *strictly* less than in x; let us see what

[32] Notice that if there was another player (or group of players) in $N \setminus S^{k'}$, who could be used to replace j, thereby paying less, P_k should have done it in the first place, and hence there cannot exist any continuation equilibrium after the rejection of S^j, where the one who rejects it obtains a strictly greater payoff.

happens in this case:

(1) If $j \in N \setminus S^k$, (12) guarantees that j would reject in turn, proposing one of the other coalitions in $M^j(x)$, and hence the first deviation cannot be profitable.

(2) If $j \in S^k$, (12) is silent, and so one could worry that in this case j could be offered something less than in x without being able to reject; however, this problem would arise only if x_j was strictly positive and $m^j(x) = 1$, but this is impossible because under the algorithm the rational proposer would not be maximizing, because by reducing the payoff to j by ϵ (giving this ϵ to herself) all constraints would be still satisfied and she would be better off. So, only when x assigns 0 to j is it possible to have $m^j(x) = 1$ (for those who have $m^j \geq 2$), but in this case no such deviation is possible in the first place.

(3) If $m^j < 2$, then (13) guarantees that the payoff should be 0, and hence the undercutting would again be impossible.

Finally, (10) makes sure that there is no deviation by the proposer either, provided that step 2 of the algorithm is performed for the selection of the right MWC (and as it is part of the algorithm, this selection occurs).

Necessity: Now let us show that every SSPE of the game $\gamma(N, W)$ must imply a vector of thresholds that satisfies the algorithm. First of all, it is clear that every equilibrium has to satisfy (11): if not, i.e., if for some $j \in S^k$ there exists some other coalition $S \in M^j(x)$: $\sum_{h \in S} x_h < 1$, then j can reject the first proposal and propose S, making everybody better off.

In order to show that (12) is necessary too, let us suppose that there exists an equilibrium where the threshold vector x is such that $m^j(x) = 1$ for some $j \in N \setminus S^k$ among those players with $m^j \geq 2$; then, given the definition of MWC, the only feasible MWC - say T - in $M^j(x)$, must have at least one player - say h - in common with S^k; but then h can reject, propose T, reducing by ϵ the payoff for j and increasing by ϵ that for himself;

that proposal would be accepted for ϵ small enough (given that j has no alternatives and the other payoffs are untouched). $\square$

Proof of Theorem 5:

Let us show, first of all, the reason why only players who belong to some MWC ($M^i(w) \neq \emptyset$) can receive positive payoffs in equilibrium. Consider, as an example, a four-player game where three parties have $\frac{2}{7}$ of the votes each, and one has $\frac{1}{7}$; this vector w is homogeneous, but the player with $\frac{1}{7}$ does not belong to any MWC. Therefore, even if the latter player could move first, there would be no demand greater than 0 that he could make with hope to be included in the prevailing coalition. This reasoning obviously extends to any other situation of this kind. It follows that the strategy of player $i : M^i(w) = \emptyset$ is irrelevant, and can be ignored henceforth.

The unique equilibrium strategy profile a^* is:

1.
$$a^*_{\rho^{-1}(l)} = x_{\rho^{-1}(l)} = \frac{w_{\rho^{-1}(l)}}{q}, \ 1 \leq l \leq p(\rho, w) - 1; \tag{14}$$

2. $\forall l : p(\rho, w) \leq l < n$ the strategies are:

   (a)
$$a^*_{\rho^{-1}(l)} = \left\{ (1 - \sum_{i \in S^*(x,l)} x_i), \{\rho^{-1}(l) \cup S^*(x,l)\} \right\}, \ \textit{if}\ 1 - \sum_{i \in S^*(x,l)} x_i \geq \frac{w_{\rho^{-1}(l)}}{q} \tag{15}$$

   where
$$S^*(x,l) = \arg \min_{S \subseteq \{i:\rho(i)<l\}: S \cup \rho^{-1}(l) \in \Omega} \sum_{i \in S} x_i;$$

   (b) if (off the equilibrium path)
$$1 - \sum_{i \in S^*(x,l)} x_i < \frac{w_{\rho^{-1}(l)}}{q},$$

then

$$a^*_{\rho^{-1}(l)} = \left\{x^*_{\rho^{-1}(l)}(x),\ \phi\right\} \tag{16}$$

where $x^*_{\rho^{-1}(l)}(x) \geq \frac{w_{\rho^{-1}(l)}}{q}$ is the maximum demand $\rho^{-1}(l)$ can make without risking to be excluded from the prevailing MWC, and obviously depends on the vector of demands:

$$x^*_{\rho^{-1}(l)}(x) = \max x \text{ S.T. } \rho^{-1}(l) \in S^*(x,p),\ l < p \leq n;$$

3. for the last player (if the game does not end before)

$$a^*_{\rho^{-1}(n)} = \left\{(1 - \sum_{i \in S^*(x,n)} x_i),\ \{\rho^{-1}(n) \cup S^*(x,n)\}\right\}. \tag{17}$$

Checking node by node starting from the last one, one can verify that there are no profitable deviations. In fact, given the strategies of the players moving after $p(\rho, w)$, which are clearly optimal at all histories, no player moving before $p(\rho, w)$ will find it convenient to demand more than the proportional share, because by doing so she would automatically be out of $S^*(l \geq p(\rho, w))$. $\square$

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