Discounting an Uncertain Future

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Abstract

We discuss the selection of the socially optimal discount rate for public investment projects that entail costs and benefits in the very long run. More specifically, we examine in an expected utility framework how the uncertainty on the growth rate of the GNP per head affects this rate. Under various conditions on preferences, as positive prudence, decreasing relative risk aversion or decreasing absolute risk aversion, we prove that (1) the fact that growth is uncertain reduces the optimal discount rate, and that (2) this discount rate should be smaller the longer the time horizon is. We characterize the asymptotic value of the discount rate. We also examine the case of Kreps-Porteus social welfare functions.

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1 Introduction

Much of life is made of investments. Costly actions are taken today in the prospect of future benefits. In the presence of efficient financial markets, the investment decision process is based on the classical concept of the Net Present Value (NPV). If the investment is risk free, one discounts future benefits by using the interest rate of the risk free asset with the corresponding duration. One then compares this discounted benefit to the short term cost. The argument sustaining this decision rule is based on the notion of opportunity costs. Instead of doing the planned investment, one could invest in the financial markets. As a consequence, the return of the planned investment should yield at least the risk free rate. For standard investment projects, this rule is equivalent to having a positive NPV. If financial markets are frictionless and if agents are paternalistic towards future generations, the use of the observed risk free rate to discount public investment projects leads to a socially efficient level of investment.

The analysis is less easy to perform when benefits and costs of the set of current potential actions are expected to last in the long run. The carbon dioxide that one emits today will not be recycled for a couple of centuries, yielding long term costs like global warming. Some nuclear wastes like Plutonium have half-life in the tens of thousands years. Obviously, financial markets are not very helpful to provide a guideline for investing in technologies that prevent this kind of long-lasting risks to occur. There simply does not exist any financial instrument with such large durations. For the sake of comparison, U.S. Treasury Bonds have time horizons that do not exceed 30 years. We are thus forced to use a model to value the distant future. The aim of this valuation model is to provide a socially optimal discount rate for such long durations.

The selection of the discount rate for long term investments is both crucial and controversial. It is crucial because of the exponential nature of discounting. It is controversial for the same reason, since even the smallest positive discount rate leads ultimately to a disenfranchising of future generations.[1] So, the question often raised is why the discount rate is not zero. Why does the mere displacement in time change one's values? The name of Böhm-

[1] See for example Heal (1997) for a discussion of this phenomenon, and for possible alternatives to the NPV criterion.

Bawerk is intimately related to these questions. The first reason offered by Böhm-Bawerk is purely psychological. Namely, agents may have a pure preference for the present. We do not think that the pure time preference is a good argument when several generations of agents are involved. In short, we do not see how the impatience to consume by one generation does justify transferring more consumption from generations that follow it.

The second reason to discount the future is related to a wealth effect. We expect that future generations will be better off than we are. After all, in the western world at least, we experienced an uninterrupted growth during the two last centuries. Each generation enjoyed a larger level of consumption than the previous one. Given decreasing marginal utility, an investment which gives one unit of the consumption good in the future in exchange for one unit of the consumption good in the present should not be acceptable. Thus, there must exist a relationship between the socially optimal discount factor and the growth rate of the economy. In the case of a time separable logarithmic utility function with no pure time preference, the socially optimal discount rate equals the growth rate per capita of the economy. The growth rate of GNP per capita in the western world has been roughly 2% per year since the beginning of the industrial revolution. Only investment projects that have an internal rate of return (IRR) not smaller than the growth rate should be implemented. For projects with a positive IRR that is less than the growth rate, implementing them would indeed increase the overall consumption. But they should be rejected because the increase in utility tomorrow does not compensate the reduction in utility today, given the difference in marginal utilities originating from the wealth effect.

A problem arises with the wealth effect if the growth rate is not known with certainty. Estimating the growth rate for the coming year is already a difficult task. No doubt, any estimation of growth for the next century/millennium is subject to potentially enormous errors. The history of the western world before the industrial revolution is full of important economic slumps, as the one due to the invasion of the roman empire, or the one due to the Black Death during the middle ages. The recent debate on the notion of a sustainable growth is an illustration of the degree of uncertainty we face to think about the future of Society. Some will argue that the effects of the improvements in information technology have yet to be realized, and the world faces a period of more rapid growth. On the contrary, those who emphasize the effects of natural resource scarcity will see lower growth rates in the future. Some even suggest a negative growth of the GNP per head in the future, due to the deterioration of the environment, population growth and decreasing returns to scale. They claim that the wealth effect goes the other direction, so that everything should be made to improve the future. This uncertainty at least casts some doubt on the relevance of the wealth effect to justify the use of a large discount rate.

In this paper, we provide an analysis of the effect of the uncertainty on growth on the socially optimal discount factor. Instrumental to this analysis is the concept of prudence that has been formalized by Kimball (1990). An agent is prudent if his willingness to save increases in the face of an increase in his future income risk. Technically, an agent is prudent if the third derivative of his utility function is positive. As shown in this paper, prudence justifies taking a discount rate that is less than the one that would have been obtained by assuming a certain growth. The magnitude of the effect depends upon the degree of prudence and the degree of uncertainty on growth. This analysis is provided in section 2. Some numerical simulations are performed in section 3.

In section 4, we examine the relationship between the time horizon and the socially optimal discount rate. Namely, do longer time horizons justify selecting a smaller discount rate? An intuitive argument would rely on the increased risk of longer horizons due to the accumulation of period to period growth risks. The longer the horizon is, the larger is the uncertainty on future wealth, the smaller should the discount rate be. We show in this paper that this intuition is correct only if relative risk aversion is decreasing. To prove this result, we use some properties of log-supermodular functions. This section is related to the literature on the term structure of interest rate. A reinterpretation of our work is that, if the growth rate of consumption is stationary over time, the yield curve is increasing or decreasing depending upon whether relative risk aversion is increasing or decreasing.

Section 4 is also related to a recent paper by Martin Weitzman (1998)[2] who also proves that the discount rate should be decreasing with time horizon. Weitzman's conclusion is obtained in a much different framework, with risk neutral agents together with a simple early revelation of future uncertain productivity of capital. His conclusion relies on the fact that the NPV is a convex function of the discount rate, whereas our result relies on more com-

[2]See also Gollier (1998) for a discussion and extensions of Weitzman (1998).

plex assumptions on preferences. The two approaches are complementary.

Because the standard framework of intertemporal expected utility does not distinguish between risk aversion and the resistance to intertemporal substitution, we explore the case of a Kreps-Porteus social welfare function in section 5.

2 The socially optimal discount rate

In this section, we consider a model with one period and two dates, $t = 0, 1$. Seen from date $t = 0$, date $t = 1$ is far distant, so there is no credible way to compensate some costs borne at $t = 1$ by promising to increase the availability of goods at that date. There is a social planner who maximizes a weighted sum W of the expected utility of the generations living at dates 0 and 1:

$$W = u(c) + \beta E v(\tilde{c}_1). \tag{1}$$

Functions u and v are the twice-continuous increasing and concave von Neumann-Morgenstern utility functions of the representative agent living respectively at dates 0 and 1. They consume respectively c and $\tilde{c}_1 = c(1+\tilde{g})$, where $\tilde{g}$ is the per capita growth rate of consumption. The support of $\tilde{g}$ is in $]-1, +\infty[$. The distribution of the growth rate is taken as exogenous in our model.[3] Weight β is also exogenous, as it describes the planner's ethical attitude towards future generations.

Suppose that the planner is considering the possibility to make an investment today that would benefit to the future generation. Suppose that the return of the investment be r, i.e. each dollar invested at $t = 0$ generates $1 + r$ dollars at $t = 1$ with certainty.[4] This project can be to reduce the

[3] This is of course a questionable assumption. Growth is the consequence of capital accumulation and innovation. The first is obviously endogenous, whereas recent developments in growth theory tend to suggest that innovation is also partly endogenous. However, we may question the validity of growth theory to predict with enough precision the relationship that exists between current macro variables and far-distant generations' consumption.

[4] In many instances, the future benefits are to reduce risks borne by future generations. When the benefits are uncertain, but independent of the growth rate of the economy, the same model can be used with $1 + r$ denoting the certainty equivalent benefit at date

pollution of the air, or the preservation of animal species.[5] Should Society invest in this project? At the margin, the answer is yes if

$$-u'(c) + (1+r)\beta E v'(\tilde{c}_1) \geq 0,$$

or, equivalently, if

$$-1 + \frac{1+r}{1+\delta} \geq 0, \tag{2}$$

with

$$\delta = \frac{u'(c)}{\beta E v'(c(1+\tilde{g}))} - 1. \tag{3}$$

Notice that δ would be the equilibrium risk-free rate in this exchange economy. The left-hand side of condition (2) can be interpreted as the net present value of the project, with discount rate δ. Thus, the project should be implemented, at least at the margin, if and only if its NPV with discount rate δ be positive. The socially optimal δ depends upon social values (β), together with the attitude towards risk and the elasticity of substitution, as is apparent in equation (3).

We want to evaluate the effect of the uncertain growth on the socially optimal discount rate. To do this, we compare δ to δ^c, the socially optimal discount rate in an economy which faces a sure growth rate equaling $E\tilde{g}$:

$$\delta^c = \frac{u'(c)}{\beta v'(c(1+E\tilde{g}))} - 1. \tag{4}$$

Obviously, we obtain that

$$\delta \leq \delta^c \iff E v'(c(1+\tilde{g})) \geq v'(c(1+E\tilde{g})). \tag{5}$$

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1. Notice that the presence of the background uncertainty on the future wealth level affects this certainty equivalent. Under risk vulnerability (see Gollier and Pratt (1996)), a condition on preferences that is satisfied by most familiar utility functions, the presence of an uncertain future wealth increases the certainty equivalent benefit of a risk-reducing investment. This provides another argument to select a smaller discount rate.

[5] There is a scientific uncertainty about the benefit of these strategies that will hopefully be reduced over time. Also, many of such actions have irreversible effects. Gollier, Jullien and Treich (1997) examine the effects of the scientific uncertainty and irreversibilities on the optimal timing of the prevention effort.

The latter condition holds for any c and for any distribution of $\tilde{g}$ if and only if v' is convex. The convexity of marginal utility is a well-known condition since Leland (1968) and Drèze and Modigliani (1972). It is a necessary and sufficient condition for an increase in future risk to increase (precautionary) savings. Kimball (1990) coined the term "prudent" to characterize individuals who behave in this way.

Proposition 1 *The uncertainty affecting the growth rate of the economy should induce Society to select a smaller (resp. greater) discount rate if agents living in the future are prudent (resp. imprudent).*

One way to quantify the effect of the uncertain growth on the socially efficient growth rate is to define the "precautionary equivalent"[6] growth rate, the certain growth rate that yields the same socially efficient discount rate. The precautionary equivalent growth rate ρ is defined as

$$Ev'(c(1+\tilde{g})) = v'(c(1+\rho)). \tag{6}$$

As suggested by Kimball (1990), the certainty equivalent growth rate ρ can be approximated by

$$\rho \cong E\tilde{g} - \frac{1}{2}\sigma_{\tilde{g}}^2 \psi(c), \tag{7}$$

where $\psi(c)$ is the coefficient of relative prudence, i.e.

$$\psi(c) = -\frac{cv'''(c)}{v''(c)}.$$

Condition (7) indicates that the effect of the uncertainty on growth on the efficient discount rate is the same as a sure reduction of the growth rate by the product of half its variance by relative prudence.

Another consequence of this analysis is that the wealth effect may well go in the opposite direction if the uncertainty on the growth rate is large enough with respect to its expectation. As indicated by approximation (7),

[6]This should not be confounded with the certainty equivalent growth rate, the certain growth rate that generates the same expected utility of the representative agent in the future.

an economy with a positive expected growth can be better off doing as if the growth rate be zero or negative, but certain, when deciding which discount rate to use. The more prudent people are[7], the smaller is the precautionary equivalent growth rate, and the smaller is the optimal discount rate.

The objective of this paper is to provide a guideline for the selection of the discount rate when probabilistic scenarios on the future growth rate are provided. In order to isolate the wealth effect from other effects like the ethical and demographic ones, we will hereafter make two simplifications: *first, we will assume that the attitude toward risk of the future generation is the same as the current generation, i.e.* $v \equiv u$. *Second, we will take the ethical position to treat the two generations equally, i.e.* $\beta = 1$. Under these conditions, we get

$$\delta = \frac{v'(c)}{v'(c(1+\rho))} - 1. \tag{8}$$

When v is logarithmic, we obtain $\delta = \rho$. But, in general, the optimal discount rate will differ from the precautionary equivalent growth rate. If ρ is small, we can use a first-order Taylor approximation for the denominator of (8) that yields

$$\delta \cong \phi(c)\rho, \tag{9}$$

where ϕ is the degree of relative concavity of v, i.e.

$$\phi(c) = \frac{-cv''(c)}{v'(c)}. \tag{10}$$

The role of ϕ is to measure the resistance to intertemporal substitution of consumption. Approximation (9) suggests that the optimal discount factor is approximately the product of two numbers, the resistance to intertemporal substitution and the precautionary equivalent growth rate. This was already observed in the case of certainty, as in Nordhaus (1994), who examines a continuous-time model with isoelastic utility functions. It implies in particular that, for small $\rho > 0$ at least, the optimal discount rate is larger or smaller

[7] The concept of "more prudence" has a precise meaning. An agent with utility v_1 is more prudent than an agent with utility v_2 if v_1' is obtained by a convex transformation of v_2'.

than the precautionary equivalent growth rate depending upon whether relative risk aversion is larger or smaller than unity. In the following, we show that this is true in general.[8]

Proposition 2 *The socially optimal discount rate satisfies the following properties:*

1. $\delta \geq \rho$ when $(\phi - 1)\rho \geq 0$;

2. $\delta \leq \rho$ when $(\phi - 1)\rho \leq 0$.

To sum up, two different characteristics of the utility function affect the level of the optimal discount rate. The first question is to determine the effect of uncertainty. Which certain growth rate should we consider as equivalent to the uncertain growth rate we face? We showed that the degree of relative prudence determines the impact of the riskiness of growth on the precautionary equivalent growth rate. The more prudent we are, the smaller should the equivalent certain growth rate be. Prudence is measured by the degree of convexity of v'. The second question is to determine by how much we should substitute consumption today by consumption tomorrow in the face of this precautionary equivalent growth rate. This depends upon the degree of resistance to intertemporal substitution. This is measured by the degree of concavity of v. The more resistant to intertemporal substitution we are, the larger should the discount rate be, for a given certainty equivalent growth rate.

[8]The proof of Proposition 2 is let to the reader. This result is in fact a corollary of the following Proposition: Consider two economies $j = 1, 2$ with the same precautionary equivalent growth rate, but with respectively utility functions v_1 and v_2, v_2 being more concave than v_1 in the sense of Arrow-Pratt. Let δ_j denote the socially optimal discount rate in economy j. We have:

- $\delta_2 \geq \delta_1$ if $\rho > 0$;
- $\delta_2 \leq \delta_1$ if $\rho < 0$.

Since $\delta = \rho$ for the logarithmic function which exhibits a relative risk aversion ϕ equaling 1, this result directly yields Proposition 2.

3 Numerical simulations

The first step consists in replacing a random growth rate by its precautionary equivalent. The second step is to translate the precautionary equivalent growth rate into an optimal discount rate through the standard Böhm-Bawerk argument of the wealth effect. We provide in Table 1 some simple numerical simulations for the precautionary effect and the substitution effect. We consider the case of power utility functions

$$v'(z) = (z-k)^{-\gamma},\ z \geq k, \tag{11}$$

for some $\gamma \geq 0$ and $k \geq 0$, which is some minimum subsistence level. For those functions, prudence is positive, and

$$\phi(c) = \frac{\gamma c}{c-k},\ \text{and}\ \psi(c) = \frac{(\gamma+1)c}{c-k}.$$

Relative risk aversion and relative prudence are decreasing from plus infinity at $c = k$ to respectively γ and $\gamma+1$ as c tends to infinity. Normalizing current consumption to unity, we consider a minimum subsistence level at 50% of the current level of subsistence, i.e. $k = 0.5$. There is a great uncertainty on which relevant values to consider for γ. To explain the observed demand for insurance, Drèze (1981) suggests to take ϕ between 2 and 4. But in order to explain the large equity premium observed on financial markets, Mehra and Prescott (1985) need to assume ϕ between 20 and 40.[9] Since $\phi(1) = 2\gamma$, and we tried γ taking values 0.5, 1, 2 and 6. To help the reader to appreciate the degree of realism of these values, let us consider a power-utility individual with $k = 0.5$ and a wealth normalized to unity,with the risk of gaining or losing 10% of it with equal probability. For $\gamma = 0.5$, the individual would not be ready to pay more than 0.50% of her wealth to escape the risk. On the contrary, the individual with $\gamma = 6$ would be ready to pay as much as 5.20% of her wealth to escape the risk.

We now turn to the description of the potential scenarios on the growth rate per capita. Maddison (1991) estimates the GNP per capita in western countries to be 1034 (1985 US dollars) in 1820, and 14413 in 1989. This makes a growth rate of 1.6% per year. In the following, we consider a period of twenty years. We provide some historical data for Australia, France and

[9]See Kocherlakota (1996) for a recent survey on the equity premium puzzle.

	1870-90	1890-10	1910-30	1930-50	1950-70	1970-90
Australia	25.4%	16.8%	-14.2%	50.1%	62.3%	38.2%
France	23.6%	23.1%	49.3%	16.3%	121.4%	54.0%
United States	40.0%	45.7%	24.2%	51.8%	48.3%	42.0%

Table 1: Historical growth rates per capita (Source: Maddison (1991)).

the United States over the period 1870-1990 in Table 1. For France, the average growth per capita over a period of twenty years is 48%, i.e. 1.97% per year, and the standard deviation is 36%.

Three scenarios are considered. All these scenarios assume that the expected GNP per capita increases by 48% in expectation. In the first scenario, the growth rate is 48% with certainty, so only the effect of the resistance to intertemporal substitution emerges. In the two other set of scenarios, we introduce some uncertainty on the growth rate. In the first probabilistic scenario, we assume that $\tilde{g}$ takes values 0% or 75% with respectively probability .36 and .64. In the other probabilistic scenario, the uncertainty is increased in the sense of third-order dominance. *These two probabilistic scenarios are compatible with the two first moments of the historical data for France over the period 1870-1990.*

As an illustration, when $\gamma = 2$ and with the second scenario, the precautionary equivalent growth rate is 1.06%, almost half of the expected growth rate. But if we consider the third scenario, the certainty equivalent growth rate goes down to 0.77%. Under the same scenario, but with $\gamma = 6$, the optimal discount rate becomes even negative, although the GNP per capita is expected to increase by almost 2% per year over the period!

We see that the precautionary equivalent growth rate is decreasing in the degree of risk and in γ, since relative prudence is increasing in γ. The optimal discount rate is decreasing in the degree of uncertainty. The effect of a change in γ on δ is ambiguous, as an increase in γ also raises the resistance to intertemporal substitution. As is well-known, an important deficiency of expected utility is the inability to disentangle the willingness of the planner to smooth consumption across states and his willingness to smooth consumption across time. In section 5, we explore the case of a Kreps-Porteus social welfare function.

	Scenario on growth rates		
	48% for sure (1.97% per year)	0% with probability .36 75% with probability .64	−10% with probability .28 70.56% with probability .72
$\gamma = 0.5$	$\rho = 1.97\%$ $\boldsymbol{\delta = 1.70}\%$	$\rho = 1.53\%$ $\boldsymbol{\delta = 1.35}\%$	$\rho = 1.42\%$ $\boldsymbol{\delta = 1.27}\%$
$\gamma = 1$	$\rho = 1.97\%$ $\boldsymbol{\delta = 3.42}\%$	$\rho = 1.36\%$ $\boldsymbol{\delta = 2.45}\%$	$\rho = 1.20\%$ $\boldsymbol{\delta = 2.19}\%$
$\gamma = 2$	$\rho = 1.97\%$ $\boldsymbol{\delta = 6.96}\%$	$\rho = 1.06\%$ $\boldsymbol{\delta = 3.93}\%$	$\rho = 0.77\%$ $\boldsymbol{\delta = 2.93}\%$
$\gamma = 6$	$\rho = 1.97\%$ $\boldsymbol{\delta = 22.26}\%$	$\rho = 0.44\%$ $\boldsymbol{\delta = 5.20}\%$	$\rho = -0.03\%$ $\boldsymbol{\delta = -0.35}\%$

Table 2: The precautionary equivalent growth rate per year (ρ) and the optimal discount rate per year (δ), with $k = 0.5$.

4 Compound growth rates

When considering environmental risks, we are not only interested in discounting benefits and costs occurring in 20 years from now, but also for much longer time horizons. Obviously, the uncertainty prevailing on the GNP per head prevailing t periods from now is an increasing function of t.[10] In this section, we address the question of how this increased uncertainty affects the optimal discount rate per period. In fact, we address the question of whether the yield curve should be decreasing or increasing.

Empirical evidence suggests that the discount rate used by individuals to value the future is a decreasing function of the time horizon. Lowenstein and Thaler (1989) for example obtained discount rates ranging around 15% for the short run (less than 5 years), and then dropping to a level as low as 2% for the long run (100 years). Can this be due to the increase of uncertainty about one's future wealth?

In accordance with this empirical evidence, our intuition is that the interaction between intertemporal growth risks is an aggravating factor. A longer time horizon should induce a smaller discount rate per period. There

[10] As indicated by Samuelson (1963), compounding risks borne at different points in time, even if they are independent, is not a good way to diversify these risks.

is a growing literature on how one should behave in the presence of multiple independent risks. Pratt and Zeckhauser (1987) examine whether one should reject the combination of two independent lotteries that are individually undesirable. Gollier and Pratt (1996) show how the presence of a zero-mean risk to wealth affects the attitude towards another independent risk. To solve these questions, one needs to check conditions on the third and fourth derivatives of the utility function. Here, our problem is similar, since we examine how two independent growth risks interact with each other.

Let us consider a two-period model with three dates $t = 0, 1, 2$. The growth rate from $t = 0$ to $t = 1$ is $\tilde{g}_1$, whereas the growth rate from $t = 1$ to $t = 2$ is $\tilde{g}_2$. We assume that random variables $\tilde{g}_1$ and $\tilde{g}_2$ are independently and identically distributed. They are distributed as $\tilde{g}$. Therefore, the socially optimal rate δ_1 to discount costs and benefits occurring at date $t = 1$ is as in section 2:

$$1 + \delta_1 = \frac{v'(c)}{Ev'(c(1+\tilde{g}))}. \tag{12}$$

The growth rate over the two periods is $(1+\tilde{g}_1)(1+\tilde{g}_2) - 1$. The socially optimal discount rate per period δ_2 for projects yielding cash flows two periods later should be such that

$$(1 + \delta_2)^2 = \frac{v'(c)}{Ev'(c(1+\tilde{g}_1)(1+\tilde{g}_2))}. \tag{13}$$

δ_2 is the long term discount rate, whereas δ_1 is the short term one. Usually, δ_2 is not equal to δ_1, except for the isoelastic utility function. Indeed, we obtain in this case

$$\begin{aligned} (1+\delta_1)^2 &= [E(1+\tilde{g})^{-\phi}]^{-2} \\ &= \left[E(1+\tilde{g}_1)^{-\phi} E(1+\tilde{g}_2)^{-\phi}\right]^{-1} \\ &= \left[E((1+\tilde{g}_1)(1+\tilde{g}_2))^{-\phi}\right]^{-1} = (1+\delta_2)^2. \end{aligned} \tag{14}$$

In this case, the increased risk on future GNP per head is not an argument to select a smaller discount rate for a longer horizon. The intuition is straightforward. Since $v'(c)/Ev'(c(1+\tilde{g}_2))$ is independent of c for isoelastic functions, the discount rate that will be used next period is known in advance and is equal to δ_1. By a standard arbitrage argument, it implies that δ_2 must equal δ_1 in that case.

More generally, using conditions (13) and (15), δ_2 is less than δ_1 if and only if

$$v'(c)Ev'(c(1+\tilde{g}_1)(1+\tilde{g}_2)) \geq Ev'(c(1+\tilde{g}_1))Ev'(c(1+\tilde{g}_2)). \qquad (15)$$

Let function h from R_+^2 to R be defined as $h(x_1,x_2)=v'(cx_1x_2)$. Suppose that this function be log supermodular.[11] This means that

$$h(\min(x_1,x_1'),\min(x_2,x_2'))\ h(\max(x_1,x_1'),\max(x_2,x_2')) \geq h(x_1,x_2)\ h(x_1',x_2') \qquad (16)$$

for all (x_1,x_2) and (x_1',x_2') in R_+^2. Taking $x_1=x_2'=1$, the log supermodularity of h implies that

$$h(1,1)\ h(x_1',x_2) \geq h(1,x_2)\ h(x_1',1) \qquad (17)$$

for all x_1' and x_2 that are both larger than 1. This inequality is equivalent to

$$v'(c)v'(c(1+g_1)(1+g_2)) \geq v'(c(1+g_1))v'(c(1+g_2)) \qquad (18)$$

for all $g_1=x_1'-1$ and $g_2=x_2-1$ that are both positive. Suppose now that the growth rate $\widetilde{g}_t$ per period is positive almost surely. Then, the log supermodularity of h is sufficient to guarantee that condition (18) holds almost everywhere. Taking the expectation of this inequality directly yields inequality (15), which implies in turn that δ_2 is less than δ_1.

If the utility function is three time differentiable, the log supermodularity of h also means that the cross derivative of $\log h$ is positive. It is easily seen that this is equivalent to require that relative risk aversion is decreasing (DRRA). Notice that all inequalities from (15) to (18) are reversed if relative risk aversion is increasing.

Proposition 3 *Suppose that the growth rate of consumption is nonnegative almost surely. The long term discount rate is smaller (resp. larger) than the short term one if relative risk aversion is decreasing (resp. increasing).*

Decreasing relative risk aversion is compatible with the well-documented observation that the relative share of wealth invested in risky assets is an increasing function of wealth. Kessler and Wolf (1991) for example show that

[11] For an analysis of the usefulness of this concept in the economics of uncertainty, see Athey (1998).

the portfolios of U.S. households with low wealth contains a disproportionately large share of risk free assets. Measuring by wealth, over 80% of the lowest quintile's portfolio was in liquid assets, whereas the highest quintile held less than 15% in such assets. Guiso, Jappelli and Terlizzese (1996), using cross-section of Italian households, observed portfolio compositions which are also compatible with decreasing relative risk aversion.

Thus one should select a smaller rate to discount far distant benefits than the rate to discount benefits realized in the near future. Growth risks are mutually aggravating. To illustrate, let us consider again the scenario in which the growth rate of the GNP per capita is either 0% or 75% with respectively probability 0.36 and 0.63 in each period of 20 years. We also consider the family of DRRA power utility functions (11). Let us consider more specifically the case $\gamma = 1$. Current consumption is normalized to unity.

We computed the socially optimal discount rate for horizons up to 45 periods, using different values of the parameter k. The results are drawn in Figure 1. As a benchmark, consider the case $k = 0.5$ for which relative risk aversion equals 2 at current consumption and goes to 1 as consumption goes to infinity. The discount rate per year that should be used to value benefits arising at the end of the first period is 2.45%. For benefits arising in 10 periods (200 years) from today, it is reduced to less than 2%.

An important requirement of the above Proposition is that the growth rate of consumption per capita is nonnegative almost surely. Because inequality (18) also holds when g_1 and g_2 are both negative under DRRA, DRRA is also sufficient for the yield curve to be decreasing when $\widetilde{g}_t$ is nonpositive almost surely. A difficulty arises when $\widetilde{g}_t$ alternate in sign, since inequality (18) is reversed when $g_1 < 0 < g_2$. When there is a risk of recession in an economy with a positive expected growth, it is not true anymore that DRRA is sufficient for the yield curve to be decreasing. Let us illustrate this point by the following example, using One-Switch utility functions introduced by Bell (1988). Take $v'(z) = a + z^{-b}$ with $a > 0$ and $b > 0$. It yields $-zv''(z)/v'(z) = b\left[az^{-b} + 1\right]^{-1}$, which is decreasing in z. In addition, take $a = b = 1$ and $\widetilde{g}_t \sim (-50\%, 1/3; +100\%, 2/3)$. In such a situation, straightforward computations generate $\delta_1 = \delta_2 = 0$: the yield curve is flat in spite of DRRA! This result does not contradict the above Proposition, since $\widetilde{g}_t$ is not positive with probability 1. Thus, extending the analysis to economies with

Figure 1: The discount rate as a function of time horizon.

a risk of a recession requires restricting the set of DRRA utility functions to guarantee that the socially optimal discount rate be decreasing with time horizon.

The case of One-Switch utility functions is interesting also because there is a simple proof for the property that the yield curve is not increasing: when $v'(z) = a + z^{-b}$, condition (15) is rewritten as

$$(a+c)(a+cE(1+\widetilde{g}_1)^{-\gamma}E(1+\widetilde{g}_2)^{-\gamma}) \geq (a+cE(1+\widetilde{g}_1)^{-\gamma})(a+cE(1+\widetilde{g}_1)^{-\gamma}), \tag{19}$$

This can be rewritten as

$$1 + \left[E(1+\widetilde{g})^{-\gamma}\right]^2 \geq 2E(1+\widetilde{g})^{-\gamma}, \tag{20}$$

or

$$\left[1 - E(1+\widetilde{g})^{-\gamma}\right]^2 \geq 0. \tag{21}$$

This is always true, which implies that the yield curve is nonincreasing for the set of One-Switch utility functions. This means that restricting the set of

DRRA functions to One-Switch functions is sufficient to guarantee that the yield curve be nonincreasing, even with a risk of recession.[12] Notice that the numerical example above has been built on the basis that $E(1+\widetilde{g})^{-\gamma}=1$, which implies that the three above conditions are actually equalities. This is the limit case where the yield curve is flat. It would be decreasing for any random variable with $E(1+\widetilde{g})^{-\gamma}\neq 1$.

Up to now, we have not been able to fully characterize the set of utility functions generating a nonincreasing yield curve, whatever the distribution of $\widetilde{g}_t$. We know that DRRA is necessary. In the next Proposition, we provide another necessary condition.

Proposition 4 *Define function H as $H(x,y)=v'(c)v'(cxy)-v'(cx)v'(cy)$. Suppose that v exhibits DRRA. A necessary condition for the yield curve to be nonincreasing independent of the distribution of $\widetilde{g}_t$ is written as*

$$[H(x,y)]^2\leq H(x,x)H(y,y) \tag{22}$$

for all (x,y) such that $x<1<y$. If we limit the analysis to binary distributions for $\widetilde{g}_t$ makes condition (22) necessary and sufficient.

Proof: See the Appendix.

The reader can check that, symmetrically, increasing relative risk aversion together with condition $[H(x,y)]^2\geq H(x,x)H(y,y)$ for all $x<1<y$ are necessary for the yield curve to be nondecreasing. It can also easily be checked that condition (22) is satisfied as an equality for One-Switch utility functions. Finally, observe that a sufficient condition for (22) is that H itself be log supermodular. This condition should not be confounded with DRRA, which means that $h(x,y)=v'(cxy)$ be log supermodular.

4.1 The asymptotic value of the discount rate

It is interesting to determine the asymptotic value of the per-period discount rate when time horizon recedes to infinity. The rate δ_t to discount the net

[12] Another example of this is for exponential utility functions, a particular case of functions with *increasing* relative risk aversion. It is easily shown that the yield curve is increasing for that subset of functions, even with a risk of recession.

benefit at date t is implicitly defined by the following equation:

$$(1+\delta_t)^t = \frac{v'(c)}{Ev'(c\prod_{i=1}^{t}(1+\tilde{g}_i))} \tag{23}$$

where $\widetilde{g}_i$ is the growth rate per head between date $i-1$ and i. Suppose first that, at each period i, there is a positive probability that a recession ($g_i < 0$) occurs. Suppose moreover that there exists a positive k such that $\lim_{z\longrightarrow k} v'(z) = +\infty$. In that case, the denominator in the right-hand side of equation (23) tends to infinity for large t. This is possible only if δ_t is negative. We conclude that the risk of repeated recessions over long periods eventually induces the selection of a nonpositive rate to discount far-distant futures. The low probability of such events is overweighted by the large marginal utility of wealth in these bad states.

We now turn to the more interesting case where there is no risk of recession in long periods ($\Pr[\widetilde{g} \leq 0] = 0$). Remember that $\phi(z) = -zv''(z)/v'(z)$ denote the relative risk aversion function, with $\phi_\infty = \lim_{z\to\infty}\phi(z)$. The following Proposition shows that convergence of the socially efficient discount rate as the horizon becomes distant occurs if ϕ_∞ exists.

Proposition 5 *Suppose that $\widetilde{g}, \widetilde{g}_1, \widetilde{g}_2, ...$ are i.i.d. variables whose support is in R^+. Suppose also that $\phi_\infty = \lim_{z\to\infty} -zv''(z)/v'(z)$ exists. It implies that the socially efficient discount rate δ_t defined by equation (23) tends to δ_∞ for very far-distant futures, with*

$$\delta_\infty = \left[E(1+\widetilde{g})^{-\phi_\infty}\right]^{-1} - 1. \tag{24}$$

Proposition 6 *Suppose that $\widetilde{g}_1, \widetilde{g}_2, ...$ are i.i.d. variables whose support is in R_+. Suppose also that $\phi_\infty = \lim_{z\to\infty} -zv''(z)/v'(z)$ exists. It implies that the socially efficient discount rate δ_t defined by equation (23) tends to δ_∞ for very far-distant futures, with*

$$\delta_\infty = \left[E(1+\widetilde{g})^{-\phi_\infty}\right]^{-1} - 1. \tag{25}$$

Proof: See the Appendix.

The Proposition states that the discount rate for far-distant futures tends to the short-term interest rate in an economy with agents having a constant relative risk aversion equaling ϕ_∞. For example, it implies that all curves in Figure 1 tend to a discount rate of 1.62%, independent of the value of k. This result is related to turnpike Theorems in finance. Huberman and Ross (1983) for example showed that the optimal portfolio strategy converges to the one of the CRRA agent with $\phi(z) = \phi_\infty$ as the horizon becomes distant.

5 Kreps-Porteus preferences

In this section we explore a generalized version of the social welfare function (1), but we limit the analysis to the one-period version of the model. As shown in section 2, an important problem related to this formulation is its inability to distinguish between the aversion to risk and the resistance to intertemporal substitution. Kreps and Porteus (1978) and Selden (1979) proposed two equivalent preference functionals that allow for disentangling these two concepts. The simplest way of presenting this family of preference functionals is to first define the certainty equivalent consumption m at date $t = 1$ as is standard under expected utility:

$$v(m) = Ev(\tilde{c}_1). \tag{26}$$

Then the social planner evaluates the intertemporal welfare as

$$u(c) + U(m). \tag{27}$$

In words, the planner first computes the certainty equivalent consumption of the future generation by using its attitude toward risk characterized by the concavity of v. Then the planner aggregates consumptions of the two generations in a nonlinear way. The concavity of u and U characterizes the planner's resistance to intertemporal substitution of consumption. The intertemporal expected utility model is obtained by taking $U \equiv v$. Another particular case is when u and U are linear. In that case, the social welfare function is a weighted sum of the certainty equivalent consumption, i.e. certainty equivalent consumptions are perfect substitutes.

Which investment projects should be realized under this framework? As in section 2, take a marginal project which costs 1 at date $t = 0$ and which brings a sure benefit $1 + r$ at date $t = 1$. This sure benefit increases the certainty equivalent consumption at date $t = 1$ by

$$\Delta m = \frac{Ev'(\tilde{c}_1)}{v'(m)}(1+r).$$

The social welfare function (27) is increased by the investment if

$$-u'(c) + U'(m)\Delta m \geq 0.$$

This is equivalent to condition (2) with

$$\delta = \frac{u'(c)}{U'(m)} \frac{v'(m)}{Ev'(c(1+\tilde{g}))} - 1. \tag{28}$$

Thus, the socially optimal discount rate δ takes now a more complex form. The myopic planner who would not take into account of the uncertainty of growth would rather use δ^c, with

$$\delta^c := \frac{u'(c)}{U'(c(1+E\tilde{g}))} - 1, \tag{29}$$

since the certainty equivalent of the sure consumption $c(1+E\tilde{g})$ is just $c(1+E\tilde{g})$.

The socially optimal discount rate δ is smaller than δ^c if and only if

$$\frac{u'(c)}{U'(m)} \frac{v'(m)}{Ev'(c(1+\tilde{g}))} \leq \frac{u'(c)}{U'(c(1+E\tilde{g}))}. \tag{30}$$

Under Kreps-Porteus preferences, the uncertainty on growth has two effects on the socially optimal discount rate:

- By risk aversion (concavity of v), the uncertain growth has a negative impact on the certainty equivalent consumption of the future generation: $m \leq c(1+E\tilde{g})$. It implies that $U'(m) \geq U'(c(1+E\tilde{g}))$. This has the unambiguous effect to reduce δ below δ^c: more sacrifices should be made today. This is the standard Böhm-Bawerk's wealth effect. This effect is increasing in the risk aversion of the future generation, and in the resistance to intertemporal substitution as measured by $-mU''(m)/U'(m)$. In particular, this effect is zero if U is linear.

- The uncertainty on growth also affects the sensitivity of the certainty equivalent to a sure increase in consumption, expressed as $Ev'(\tilde{c}_1)/v'(m)$. Under certainty, this sensitivity equals 1. As observed for example by Kimball and Weil (1992) and Gollier and Kimball (1994), a necessary and sufficient condition for this sensitivity to be larger than 1 under uncertainty is decreasing absolute risk aversion. Indeed, decreasing absolute risk aversion is the condition for a sure increase in consumption to decrease the aversion to risk, thereby increasing certainty equivalents. Notice that δ is inversely related to this sensitivity. We conclude that this sensitivity effect goes the same direction as the wealth effect presented above if and only if absolute risk aversion is decreasing. Observe that decreasing absolute risk aversion is stronger than prudence.

Proposition 7 *Under Kreps-Porteus preferences, the uncertainty affecting the growth rate of the economy should induce Society to select a smaller discount rate if agents living in the future are decreasingly absolute risk-averse. Moreover, this condition is necessary if U is linear.*

Decreasing absolute risk aversion is necessary for decreasing relative risk aversion, and is sufficient for prudence. It is a standard assumption in the economics of uncertainty.

If we compare this Proposition with Proposition 1, we see that we end up here with a stronger condition to guarantee that growth uncertainty affects the discount rate negatively. This is not a surprise, since Kreps-Porteus preferences are more general than expected utility. This is the weakest sufficient condition one can get without restricting U. Indeed, when U is linear, only the sensitivity effect holds.

6 Conclusion

When public investment projects entail costs and benefits in the very long run, a question arises about the selection of the relevant discount rate to use for the cost-benefit analysis. Indeed, financial markets do not provide any guideline in this case. The main argument for using a positive discount rate is the fact that the GNP per head is expected to grow over time. Therefore, projects whose costs today are as large as benefits in the future are clearly not desirable, since we do not see why current generations should sacrifice part of their consumption today for the benefit of future generations who will already be better off.

But growth is an uncertain phenomenon. The recognition of this fact should induce Society to take this argument with caution/prudence. By how much has been the question we tried to answer in this paper. We showed that the answer mainly depends upon the degree of relative prudence and upon the degree of resistance to intertemporal substitution. For commonly accepted levels of these indexes, the effect of uncertainty on the socially optimal discount rate may be very large. In particular, the potential of even a small slump in the economic growth could justify the selection of a zero discount rate, despite of a positive growth in expectation. Another important message is that the discount rate to be used for long-lasting investments should be a decreasing function of their duration. This is due to the negative effect of accumulating the per period growth risk in the long run.

The french Commissariat au Plan recommends to use a constant 8% per year as the discount rate for all public investments, and most developed countries use a rate between 5% and 8%. From our simulations, we feel that this range of rates is too large with respect that what would be socially efficient, given our current expectation on growth, and the uncertainty that prevails on it. We recommend using the risk free rate that is observable on financial markets for short time horizons. A discount rate not larger than 5% should be used in the medium run (between 50 and 100 years), whereas a decreasing rate down to around 1.5% would be relevant for flows of benefits and costs occurring in the very long run (more than 200 years).

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Appendix: Proof of Proposition 4

We normalize c to unity. Let $1+\widetilde{g}$ be distributed as $(x,p;y,1-p)$. Condition (15) is rewritten as

$$v'(1)\left[p^2v'(x^2)+2p(1-p)v'(xy)+(1-p)^2v'(y^2)\right] \geq \left[pv'(x)+(1-p)v'(y)\right]^2, \tag{31}$$

or, equivalently,

$$K(x,y,p) \equiv p^2H(x,x)+2p(1-p)H(x,y)+(1-p)^2H(y,y) \geq 0. \tag{32}$$

Remember that DRRA means that $(x-1)(y-1)H(x,y) \geq 0$. In particular, it means that $H(x,x)$ and $H(y,y)$ are nonnegative. When x and y are both larger than unity, we also have that $H(x,y)$ is nonnegative, yielding $K(x,y,p)$ ≥ 0. This is another proof of Proposition 3 for binary distributions of $\widetilde{g} \geq 0$. Again, the difficulty is when $x<1<y$, implying $H(x,y)<0$. This is the case that we examine now.

Observe that K is quadratic with $\partial^2K/\partial p^2 = H(x,x)-2H(x,y)+H(y,y) \geq 0$. It is minimum at

$$p^* = \frac{H(y,y)-H(x,y)}{H(x,x)-2H(x,y)+H(y,y)} \tag{33}$$

which is between 0 and 1. The minimum value of $K(x,y,p)$ for $p \in [0,1]$ is

$$K(x,y,p^*) = \frac{H(x,x)H(y,y)-[H(x,y)]^2}{H(x,x)-2H(x,y)+H(y,y)}. \tag{34}$$

It will be nonnegative for all $x<1<y$ if and only if condition (22) is satisfied.■

Appendix: Proof of Proposition 5

By definition of function ϕ, we have

$$v'(z) = K \exp\left[-\int_0^z \frac{\phi(s)}{s} ds\right]. \tag{35}$$

It implies that

$$\frac{v'(ce^x)}{v'(c)} = \exp\left[-\int_c^{ce^x} \frac{\phi(s)}{s} ds\right] = \exp\left[-\int_0^x \phi(ce^u) du.\right] \tag{36}$$

Take any scalar $\varepsilon > 0$. Define

$$q_-(x) = \int_0^x \left[\phi(ce^u) - \phi_\infty - \varepsilon\right] du \tag{37}$$

and

$$q_+(x) = \int_0^x \left[\phi(ce^u) - \phi_\infty + \varepsilon\right] du. \tag{38}$$

Function $q_-(x)$ tends to minus infinity when x tends to infinity. It implies that q_- has a maximum in R^+ that is denoted M. It implies in turn that the right-hand side of condition (36) is bounded below by $\exp\left[-(\phi_\infty + \varepsilon)x - M\right]$. Similarly, q_+ has a minimum m in R^+, implying that the right-hand side of condition (36) is bounded above by $\exp\left[-(\phi_\infty - \varepsilon)x - m\right]$. To sum up, we know that for any $\varepsilon > 0$, there exists two scalars $A = e^{-M}$ and $B = e^{-m}$ such that, for any $x \in R^+$,

JCR, don't we need function ϕ to be bounded?

$$A \exp\left[-(\phi_\infty + \varepsilon)x\right] \leq \frac{v'(ce^x)}{v'(c)} \leq B \exp\left[-(\phi_\infty - \varepsilon)x\right]. \tag{39}$$

Observe now that condition (23) can be rewritten as

$$(1 + \delta_t)^{-t} = E \frac{v'(ce^{\widetilde{x}_t})}{v'(c)} \tag{40}$$

where $\widetilde{x}_t = \sum_{i=1}^t \widetilde{h}_i$ and $\widetilde{h}_i = \log(1 + \widetilde{g}_i)$, whose support is in R^+. Combining conditions (39) and (40) yields

$$A\left[E \exp\left[-(\phi_\infty + \varepsilon)\widetilde{h}\right]\right]^t \leq (1 + \delta_t)^{-t} \leq B\left[E \exp\left[-(\phi_\infty - \varepsilon)\widetilde{h}\right]\right]^t, \tag{41}$$

or, equivalently,

$$-\frac{1}{t}\log B - \log\left[E\exp\left[-(\phi_\infty - \varepsilon)\widetilde{h}\right]\right] \leq \log(1+\delta_t) \tag{42}$$

$$\leq -\frac{1}{t}\log A - \log\left[E\exp\left[-(\phi_\infty + \varepsilon)\widetilde{h}\right]\right]. \tag{43}$$

Taking the limit when t tends to infinity yields

$$-\log\left[E\exp\left[-(\phi_\infty - \varepsilon)\widetilde{h}\right]\right] \leq \lim_{t\to\infty}\log(1+\delta_t) \leq -\log\left[E\exp\left[-(\phi_\infty + \varepsilon)\widetilde{h}\right]\right] \tag{44}$$

Since this condition holds for any $\varepsilon > 0$, it implies that

$$\lim_{t\to\infty}\log(1+\delta_t) = -\log\left[E\exp\left[-\phi_\infty\widetilde{h}\right]\right], \tag{45}$$

or, equivalently,

$$\lim_{t\to\infty}(1+\delta_t) = E\left[(1+\widetilde{g})^{-\phi_\infty}\right]^{-1}. \tag{46}$$

This concludes the proof.■