A SIMPLE MODEL OF

OPTIMAL SUSTAINABLE GROWTH

by

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ABSTRACT

This paper presents a simple model of optimal sustainable growth in which the environmental stock enters consumers' preferences and production possibilities depend upon the use of produced physical capital and the use of a flow of productive services provided through the exploitation of the environmental stock. Endogenous growth is obtained by making the productivity growth of the environmental resources dependent on past capital accumulation. Both the effect of the environmental preservation on the consumers' utility function and the effect of past accumulation on the productivity of the environmental services are considered as externalities which are internalized along an optimal growth path. Optimal growth is sustainable when the use of the environmental asset for production is equal to the regeneration capacity of the environment. We demonstrate that the optimal balanced-growth path, whenever it exists, is always a saddle point and that the optimal trajectories converging to the asymptotic growth rate are always locally unique.

RIASSUNTO

Questo articolo presenta un semplice modello di crescita ottima sostenibile nell'ambito del quale lo stock di risorse ambientali influenza le preferenze dei consumatori mentre le possibilità produttive dipendono sia dall'impiego del capitale fisico prodotto sia dall'uso del flusso di servizi produttivi ottenuti attraverso lo sfruttamento dello stock ambientale. Il processo di crescita endogena viene ottenuto attraverso l'assunzione che la crescita produttiva delle risorse ambientali dipenda dall'accumulazione passata del capitale. Entrambi l'effetto della preservazione dell'ambiente sulle preferenze dei consumatori e l'effetto dell'accumulazione passata sulla produttività dei servizi ambientali vengono considerati come esternalità che sono internalizzate lungo il sentiero di crescita ottima. La crescita ottima è sostenibile quando l'uso della risorsa ambientale nel processo produttivo è uguale alla capacità di rigenerazione dell'ambiente. Si dimostra che il sentiero di crescita ottima bilanciata, qualora esista, è sempre un punto di sella e che le traiettorie ottimali che convergono verso il tasso di crescita asintotico sono sempre localmente uniche.

1. **Introduction**.

In a number of models studied by the contemporary literature, economic growth is obtained by endogenizing the productivity of labor services. This goal is typically achieved making the contribution to production due to the use of those factors depend upon the past accumulation of capital, be this physical, as in Romer (1986) and Barro (1990), or human, as in Lucas (1988). The purpose of this paper is to mimic such a technique to get sustainable growth in a simple model characterized by the presence of an environmental asset.

We shall consider environment as a stock which is valuable when preserved, but which provides production services when exploited. Emission flows constitute a clear example of these services. Sustainability, in this context, means that environmental production services must be equal to the regenerative capacity of the environmental asset so as to keep the stock of environment constant and therefore preserved at the level considered optimal from the intertemporal society's welfare point of view.

The rate of growth of the productivity of the environmental services is assumed to depend on the past accumulation of the physical capital stock. This assumption corresponds to the view that capital accumulation embodies new technologies requiring a decreasing emission coefficient per unit of output.

While the effect of the past capital accumulation on the productivity of the environmental services is a positive externality for each unit of production, and therefore treated as given at individual level, the stock of preserved environment provides utility to consumers and can be assimilated to a public good which delivers a non-marketed benefit. A rational environmental policy aiming to achieve a socially optimal sustainable growth path should internalize both external effects.

In the next section we shall introduce the model, whereas in section 3 we shall assess the issue of the existence and uniqueness of the sustainable balanced-growth path. In particular we shall demonstrate that, when the assimilative capacity of environment depends on the environmental stock, the conditions that must be imposed on discounting to get sustainable growth are not independent of the stock of environment. It is therefore possible to establish a trade-off between the sustainable balanced growth rate and the sustainable environmental stock. In section 4 we finally discuss the local stability properties of the balanced-growth path.

2. **A model of optimal sustainable growth**.

We consider a continuous-time, infinite horizon economy which is endowed with two assets: a physical capital stock, K, and an environmental asset, E. Following Becker (1982), the stock of environmental resources is defined as the difference between a maximum tolerable pollution stock ! > 0 and the current pollution stock P:

$$E = ! - P \qquad (2.1)$$

that is

$$! = - !. \qquad (2.2)$$

A constant proportion m > 0 of the pollution stock is assumed to be assimilated in each instant t by the natural factors governing the environment. The pollution flow (emissions) Z expresses, in this context, the rate at which the environmental asset is used as a source of productive services. It follows that the pollution stock changes according to the following rule

$$! = Z - mP. \qquad (2.3)$$

Using (2.3) and (2.1), equation (2.2) can be rewritten as

$$! = A(E) - Z, \qquad (2.4)$$

where

$$A(E) = m(! - E). \qquad (2.5)$$

Equation (2.5), in other words, represents the assimilative capacity of the environment as a decreasing linear function of the environmental stock.

The economy, which is also endowed with a large number of identical competitive firms, produces a unique consumption good using a standard aggregate Cobb-Douglas technology defined upon the fraction of the physical capital stock devoted to production, K_1, and labor in efficiency units, hL, i. e.

$$Y = BK!(hL)^{1-\alpha} \tag{2.6}$$

where B > 0 is a scale parameter and h represents the efficiency of the total labor force L which is normalized and set to be equal to 1. h acts as an externality for each individual firm and is assumed to depend on the past accumulation of the total capital stock used in production according to the following expression:

$$h = K_1. \tag{2.7}$$

Ex post, therefore, the aggregate production function is linear in K_1:

$$Y = BK_1. \tag{2.8}$$

The production process, however, entails polluting emissions Z according to the emission function given by

$$Z = eY \tag{2.9}$$

where the emission-output coefficient e is supposed to be reduced through the exploitation of the remaining fraction K_2 of the total physical capital stock according to the function

$$e = !_2. \tag{2.10}$$

In view of (2.10), equation (2.9) becomes

$$Z = !_2. \tag{2.11}$$

Full factor employment requires, as usual,

$$K_1 + K_2 = K. \tag{2.12}$$

Hence, the whole economy's capital stock is completely utilized either to increase production, or to reduce the emission coefficient and, therefore, pollution.

Defining $u \equiv K_2/K$ and using (2.12), we can rewrite (2.11) as

$$Z = B(! - 1). \tag{2.13}$$

Assuming that capital lasts forever, the capital accumulation constraint is given by

$$! = B(1 - u)K - C \tag{2.14}$$

where C represents aggregate consumption.

A constant population of identical consumers endowed with an infinite lifetime profile is assumed to derive satisfaction from both consumption and the environment asset according to the following intertemporal utility function:

$$W(C, E) = ! e^{-\delta t} !?!?! (CE)^{1-\eta} dt \qquad (2.15)$$

where $\delta > 0$ is the standard subjective rate of time preference.

A benevolent central planner maximizes (2.15) subject to the constraints (2.4) and (2.14).

The Hamiltonian function associated with this program is

$$H = !!!!!!! + v[BK(1 - u) - C] + \lambda[A(E) - B(1/u - 1)]. \qquad (2.16)$$

The first order necessary conditions for an optimum are

$$C^{-\eta}E^{1-\eta} = v \qquad (2.17)$$

$$vK = \lambda u^{-2} \qquad (2.18)$$

$$! = \delta - (1 - u)B \qquad (2.19)$$

$$! = \delta + m - ! C^{1-\eta}E^{-\eta} \qquad (2.20)$$

whereas the transversality conditions at infinity are given by

$$\lim_{t\to\infty} e^{-\delta t}vK = 0 \qquad (2.21)$$

$$\lim_{t\to\infty} e^{-\delta t}\lambda E = 0. \qquad (2.22)$$

The variables v and λ designate the shadow prices of the capital and the environmental stocks, respectively. Conditions (2.17) and (2.18) are the temporary equilibrium requirements establishing the equality of the marginal utility of consumption and the marginal product of the environmental services to their respective prices. Condition (2.19) is the standard arbitrage condition which requires the rate of time preference to be equal to the marginal product of capital plus the capital gains. Finally, condition (2.20) is a modified version of the Hotelling rule requiring the rate of return from preserving the environmental stock to be equal to the rate of time preference.

Equations (2.17)-(2.20) constitute a four dimensional dynamical system which fully describes the equilibrium paths of the economy. Following a number of authors (Alogoskoufis-van der Ploeg (1991), Buiter (1992), and Mulligan-Sala-i-Martin (1991)), that system can be studied appealing to the introduction of some auxiliary variables. If we introduce the notation $\tau \equiv \lambda/(vK)$, and $x \equiv C/K$, the original system can be reduced by one dimension. From equation (2.18) one gets

$$u = \tau^{1/2} \tag{2.23}$$

and, substituting the latter expression in (2.13),

$$Z = B(\tau^{-1/2} - 1). \tag{2.24}$$

Inserting (2.23) in (2.19) yields

$$! = \delta - B(1 - \tau^{1/2}). \tag{2.25}$$

Using (2.17), equation (2.21) can be expressed as

$$! = \delta + m - ! !. \qquad (2.26)$$

Differentiating (2.17) with respect to time and making use of (2.25) one obtains

$$! = !?!?! \; ! + ![B(1 - \tau^{1/2}) - \delta]. \qquad (2.27)$$

Plugging again (2.23) in the accumulation equation (2.14) gives

$$! = B(1 - \tau^{1/2}) - x \qquad (2.28)$$

whereas substitution of (2.24) in (2.4) yields

$$! = A(E) - B(\tau^{-1/2} - 1). \qquad (2.29)$$

Subtracting (2.28) from (2.27) and using (2.29) leads to

$$! = !?!?! \; ![A(E) - B(\tau^{-1/2} - 1)] + !?!?!B(1 - \tau^{1/2}) - ! + x. \quad (2.30)$$

Taking into account that

$$! = ! - ! - !$$

and using (2.25), (2.26) and (2.28), one finally obtains

$$! = m - ! \; ! + x. \qquad (2.31)$$

A sustainable balanced-growth path requires all the variables defining the dynamical system (2.29)-(2.31) to be constant in the long-run. This in turn implies that consumption, physical capital, and output all grow at the same positive growth rate, whereas the stock of environmental resources E remains constant.

3. Existence and uniqueness of the sustainable balanced-growth path.

An (interior) equilibrium sustainable balanced-growth path is a positive triplet (x^*, E^*, τ^*) such that ! = ! = ! = 0, for all t, which satisfies, in addition, the transversality conditions (2.22) and (2.23). In view of the definition, it is clear that (x^*, E^*, τ^*), whenever it exists, must be a fixed point of the stationary system given by

$$E = ! - !(\tau^{-1/2} - 1) \tag{3.1}$$

$$x = ![\delta - (1 - \eta)B(1 - \tau^{1/2})] \equiv x(\tau) \tag{3.2}$$

$$E = !!!?!?!!. \tag{3.3}$$

Since each variable is defined only for non-negative values along the stationary loci (3.1)-(3.3), it is necessary to impose some restrictions on the structural parameters to ensure that (x, E, τ) is actually non-negative. Establishing existence and uniqueness through the choice of appropriate boundary conditions becomes relatively easy if one assumes that the state-variable x, i. e. C/K, is always non-negative for all τ, where the domain of $\tau \equiv u^{1/2}$ is the open interval (0, 1), as $0 < u < 1$ along each stationary locus. From equation (3.2) one sees that, if $\eta \geq 1$, then $x(\tau) > 0$, for all τ in (0, 1). When $\eta < 1$, however, $x(\tau)$ is strictly increasing when τ is increased from 0 to 1. As a result,

$x(\tau) \geq 0$, for all τ in (0, 1) if and only if $\delta \geq (1 - \eta)B$. The following assumption, therefore, ensures that $x(\tau) \geq 0$, for all τ in (0, 1).

Assumption. $\delta \geq (1 - \eta)B$, i. e. $x(\tau)$ is non-negative for all τ in the open interval (0, 1).

Consider now equation (3.1). In that case, $E \geq 0$ if and only if

$$! \geq !(\tau^{-1/2} - 1) \tag{3.4}$$

that is, if and only if

$$1 > \tau \geq [B/(!m + B)]^2 \equiv \tau_1 > 0. \tag{3.5}$$

It follows that the domain of E as a function of τ is restricted to the open interval $(\tau_1, 1)$. In view of the above assumption and after imposing the constraint in (3.5) to guarantee the non-negativity of (x, E, τ) along each stationary locus, we can now address the existence issue without incurring in ambiguities. Substituting equations (3.1) and (3.2) in (3.3) yields

$$\tau! - !(\tau^{1/2} - \tau) = !?!!!!!! \tag{3.6}$$

which is an equation in τ only. Finding the zeros of (3.6) is then equivalent to find the number of stationary solutions of the system (3.1)-(3.3). Let the left and the right-hand side of (3.6) be defined as $\Omega(\tau)$ and $\Gamma(\tau)$ respectively. To determine the zeros of (3.6) we consider, first, the function $\Omega(\tau)$. Direct inspection shows that, as

τ increases from τ_1 to 1, $\Omega(\tau)$ increases monotonically from 0 to !. Moreover, it is strictly convex. To study $\Gamma(\tau)$ we must distinguish two cases.

Case 1: $\eta \geq 1$. In such a case $\Gamma(\tau)$ decreases monotonically from $x(\tau_1)/[m + x(\tau_1)] > 0$ to $\delta/(\eta m + \delta) < 1$. A solution τ^* in $(\tau_1, 1)$ of equation (3.6), whenever it exists, is then unique. To ensure existence, and consequently uniqueness, it is sufficient to choose the appropriate boundary condition. This reduces, in the present context, to setting ! > $\delta/(m\eta + \delta)$, as shown in Fig. 1.

Case 2: $\eta < 1$, with $\delta \geq B(1 - \eta)$. In such a case, $\Gamma(\tau)$ increases monotonically from $x(\tau_1)/[m + x(\tau_1)] > 0$ to $\delta/(\eta m + \delta) < 1$. In addition, it is strictly concave. Once again, there exists a unique τ^* in $(\tau_1, 1)$ that solves (3.6) if and only if ! > $\delta/(m\eta + \delta)$, as shown in Fig. 2.

Insert Fig. 1 and Fig. 2 here.

The above argument can be now summarized in the following proposition.

Proposition 1. Assume that $\delta > B(1 - \eta)$. Then an interior solution (x^*, E^*, τ^*) for the system (3.1)-(3.3) exists and is also unique if and only if ! > $\delta/(\eta m + \delta)$.

In light of the above proposition, it is possible to state that the unique solution (x^*, E^*, τ^*) of the stationary system (3.1)-(3.3) truly represents an interior equilibrium sustainable balanced-growth path if it also satisfies the transversality conditions (2.22) and (2.23), the requirement on the boundedness of the objective function (2.6), and the positive growth condition. Easy computations show that the integral in (2.6) is bounded if

$$\delta > (1 - \eta)B(1 - \tau^{*1/2}). \qquad (3.7)$$

The condition in (3.7) also implies the fulfillment of (2.22) and (2.23). As far as positive steady-state growth is concerned, one obtains, from equation (2.20),

$$g = ! = ![B(1 - \tau^{*1/2}) - \delta]. \qquad (3.8)$$

As a result, the long-run equilibrium growth rate is strictly positive if and only if

$$B(1 - \tau^{*1/2}) > \delta. \qquad (3.9)$$

Combining (3.7) and (3.9) gives

$$B(1 - \tau^{*1/2}) > \delta > (1 - \eta)B(1 - \tau^{*1/2}), \qquad (3.10)$$

where the second inequality is always fulfilled because of the assumption $\delta > B(1 - \eta)$.

4. Local dynamics.

To investigate the local stability properties of the sustainable balanced-growth path, we shall, as usual, analyze the sign of the eigenvalues associated with the Jacobian of the three dimensional system (2.29)-(2.30) evaluated at the steady-state (x^*, E^*, τ^*). Since the dynamical system possesses one predetermined variable (E) and two non-predetermined variables, (τ and x), we conclude, in view of the Blanchard-Khan Theorem, that (x^*, E^*, τ^*) is locally unique if the Jacobian has two characteristic roots with positive real parts and one root with negative real part.

The Jacobian evaluated at the steady-state is given by

! ! ! !! - !! ! !!!!! !!!!!! !!!!! !!!!! ! !!!!!

where $F^* \equiv A(E^*) + B$. The trace and the determinant of J are, respectively,

$$Tr(J) = 2x^* > 0 \quad (4.2)$$

and

$$Det(J) = - m^2x^* - mx^{*2} - ! F^*\tau^*(m + x^*) + m^{2\,!}\,\tau^*F^* \; !?!?! \quad (4.3)$$

Direct inspection of (4.3) shows that $Det(J) < 0$ if $\eta \geq 1$. When $\eta < 1$, on the other hand, to conclude that $Det(J) < 0$ is equivalent to prove that

$$- m^2x^* + m^{2\,!}\,\tau^*F^* \; !?!?! < 0 \quad (4.4)$$

that is

$$x^* > !?!?! \; ! \; \tau^{*1/2}. \quad (4.5)$$

Plugging in (4.5) the expression of x^* yields

$$\delta - (1 - \eta)B + ! \; (1 - \eta)B\tau^{*1/2} > 0. \quad (4.6)$$

Since δ - (1 - η)B > 0 by assumption, one concludes that Det(J) < 0, and, therefore, that the sustainable balanced-growth path is always locally unique, i. e. there are two eigenvalues with positive real parts and one eigenvalue with negative real part. All this leads to the following result.

Proposition 2. Suppose that Proposition 1 holds. Then the sustainable balanced-growth path is locally saddle-point stable, i. e. given the initial condition $E_0 > 0$, there exists a unique choice of τ_0 and x_0 in a neighborhood of (x^*, E^*, τ^*) that places the economy on the unique converging path.

5. Conclusions.

We have presented a simple model to achieve sustainable economic growth. This means that the model we have analyzed is able to explain both balanced persistent growth in national product, consumption and capital, and the possibility of a constant stock of the environmental asset at the steady-state. An intertemporal optimality concept is required to choose the level at which the environmental stock has to be maintained. A constant environmental stock, on the other hand, implies a constant pollution stock and, hence, a constant emission flow within the assimilative capacity of the environment itself. The only way to generate both economic growth and constant emissions in the long-run equilibrium consists of a continuously decreasing emission coefficient per unit of output; this in turn requires the firms to devote an appropriate share of capital to improve the environmental use of technologies. An optimal price of the environmental use is needed to obtain this appropriate share of capital. It is shown, indeed, that, along the optimal sustainable growth path, the optimal environmental price must grow at the balanced rate of growth. As a result, one gets a constant ratio of the value of the environment to national output.

We have finally showed that, under the very simple assumption made, the optimal sustainable growth equilibrium is locally saddle-point stable.

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