FONDAZIONE ENI ENRICO MATTEI

Fuzzy Measures, Choquet Integral and Preferences' Elicitation

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Outline

Fundamentals

- Motivation/Intuition
- Fuzzy Measures
- Choquet Integral

Behavioral Analysis

Approaches to Identify Fuzzy Measures

- Direct
- Indirect (Data Learning Set)
- Least Squares

FEEM Applications

- FEEM SI

  or

- ICCG Climate Think Tank Ranking

Motivations

Case 1

Scenario	Object A	Object B	Overall Satisfaction
1	1	0	
2	0	1	

Independency

Traditional Weights & Simple Mean

Motivations

Case 2

Scenario	Object A	Object B	Overall Satisfaction
1	1	0	
2	0	1	

Substitutability

Non Additive Measures & Choquet Integral

Motivations

Case 3

Scenario	Object A	Object B	Overall Satisfaction
1	1	0	
2	0	1	

Complementarity

Non Additive Measures & Choquet Integral

1.4 Mathematical Intuition

Assigning a "weight" (measure) not only to any single object/criterion/characteristic under consideration in the evaluation process, but to any coalitions formed by them too.

Example

Two criteria *A, B*:

$w(A); w(B); w(A,B)$

Three criteria *A, B, C*:

$w(A); w(B); w(C); w(A,B),\ w(A,C),\ w(B,C),\ w(A,B,C)$

Non Additive Measures

Suppose again two criteria A and B:

- If A and B are complements:

$$\mu(A) = \mu(B) = 0 \rightarrow \mu(A) + \mu(B) = 0 < \mu(A,B) \equiv 1$$

Subadditive

- If A and B are substitutes:

$$\mu(A) = \mu(B) = 1 \rightarrow \mu(A) + \mu(B) = 2 > \mu(A,B) \equiv 1$$

Superadditive

Traditional weights are instead additive measure:

$$w(A,B) = w(A) + w(B)$$

Fuzzy Measures

A fuzzy measures or NAM [*Choquet (1953), Sugeno (1974) –> Mori, Murofushi, Grabisch (1990)*], defined over the set of criteria $N = \{1,2,\dots,n\}$, is a set function $\mu: 2^N \to [0,1]$ satisfying the following boundary and monotonicity conditions:

$$\begin{cases} \mu(\emptyset) \equiv 0 \\ \mu(N) \equiv 1 \\ \mu(S) \leq \mu(T) \leq 1 \ \ \forall\, S,T \subseteq N \end{cases}$$

Example

$$N = \{A,B,C\}$$
$$\mu(A) \leq \mu(A,B) \leq \mu(A,B,C) \equiv 1$$
$$\mu(A) \leq \mu(A,C) \leq \mu(A,B,C) \equiv 1$$
$$\dots$$

Möbius Representation

Given a NAM μ, its Möbius representation is the following set function:

$$m(S) = \sum_{T \subseteq S} (-1)^{s-t} \mu(T), \forall S, T \subseteq N$$

where $s = card(S)$, $t = card(T)$.

Moreover the following boundary and the monotonicity conditions are required:

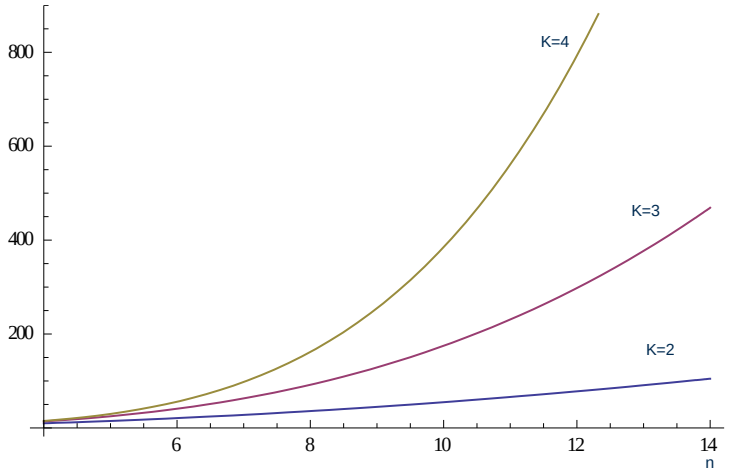
$$\begin{cases} m(\emptyset) \equiv 0 \\ \sum_{T \subseteq N} m(T) \equiv 1 \\ \sum_{\substack{T \subseteq S \\ T \ni i}} m(T) \geq 0 \quad \forall S \subseteq N, \quad \forall i \in S \end{cases}$$

A capacity μ on N is said to be k-additive if its Möbius representation satisfies $m(T) = 0 \ \forall T \subseteq N$ such that $t > k$, and there exists at least one subset T with $card(T) = k$ such that $m(T) \neq 0$.

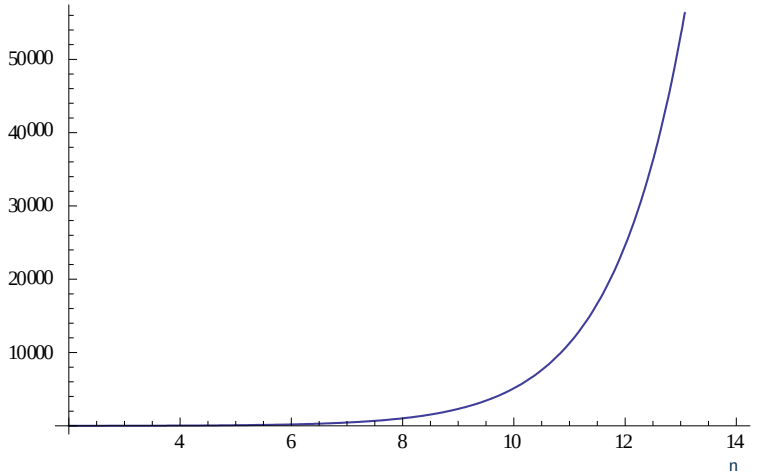
Limitations

k	n	Möbius	Constraints
2	2	3	4
	3	6	12
	4	10	32
	…	…	…
	10	55	5120
	12	78	24576
3	2	-	4
	3	7	12
	4	14	32
	…	…	…
	10	175	5120
	12	298	24576
4	2	-	4
	3	-	12
	4	15	32
	…	…	…
	10	385	5120
	12	793	24576
12	12	4095	24576

$$N°Möbius = \sum_{j=1}^{k} \binom{n}{j}$$



$$N°Cons. = n \cdot 2^{(n-1)}$$



Choquet Integral

Let μ be a NAM defined on N and $X = \{x_1, x_2, \dots, x_n\}$ the scores of the criteria belonging to N; the (discrete) Choquet integral with respect to μ is given by:

$$C_\mu(x_1, \dots, x_n) = \sum_{i=1}^{n} (x_{(i)} - x_{(i-1)}) \mu(A_{(i)})$$

where (i) means that the indices have been permutated in such a way that $x_{(1)} \leq \cdots \leq x_{(n)}$, while $A_{(i)} = \{x_{(i)}, \dots, x_{(n)}\}$ and $x_{(0)} = 0$.

Using the Möbius representation the Choquet integral can be written as:

$$C_m(x_1, \dots, x_n) = \sum_{T \subseteq N} m(T) \bigwedge_{i \in T} x_i$$

Behavioural Analysis

Behavioural Analysis

- **Importance Index** for criteria i [*Shapley (1953)*]

$$\varphi(m;i) = \sum_{T \ni i} \frac{1}{t} m(T) \quad \forall \; T \subseteq N$$

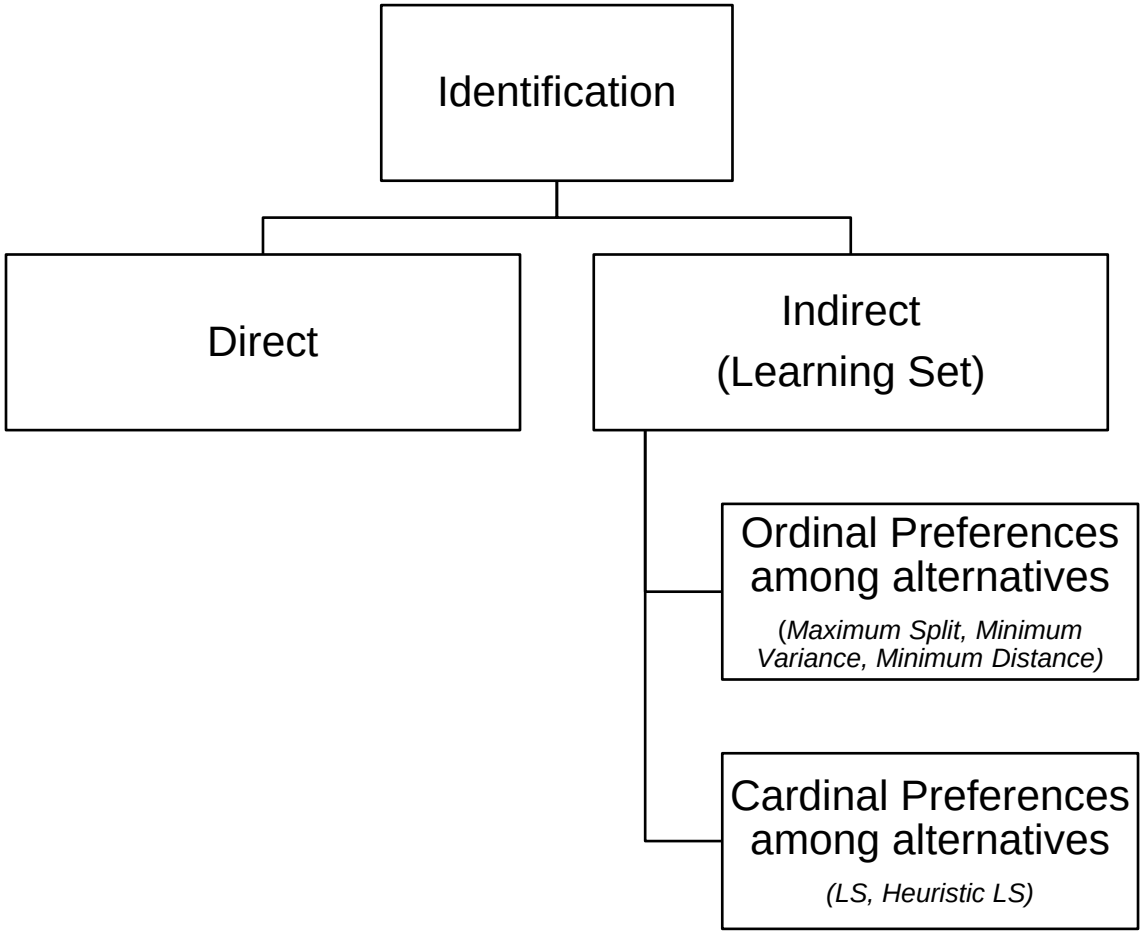
- **Interaction Index** among a combination S of criteria [*Morofushi, Soneda (1993), Grabisch, Roubens (1997)*] :

$$I(m;S) = \sum_{T \supseteq S} \frac{1}{t-s+1} m(T) \quad \forall \; S \subseteq N$$

- **Orness / Andness**
- **Veto / Favor**
- **…**

Identification of Fuzzy Measures

Identification of Fuzzy Measures



Direct Identification

Student	Math.	Stat.	Liter.	Choquet
a	10	5	8	8.6
b	8	5	10	7.9
c	10	8	5	7.4
d	8	10	5	7.4
e	6	7	10	7.2

$$\begin{cases} \mu(M) = \mu(S) = 0.45 \\ \mu(L) = 0.1 \\ \mu(M,S) = 0.5 \\ \mu(M,L) = \mu(S,L) = 0.9 \\ \mu(M,S,L) \equiv 1 \end{cases}$$

$$0.9 = \mu(M) + \mu(S) > \mu(M,S) = 0.5 \rightarrow Sub.$$
$$0.55 = \mu(M) + \mu(L) < \mu(M,L) = 0.9 \rightarrow Com.$$
$$0.55 = \mu(S) + \mu(L) < \mu(S,L) = 0.9 \rightarrow Com.$$

Indirect Identification: Ordinal Based

Student	*Math.*	*Stat.*	*Liter.*	*DM Pref.*
a	10	5	8	1°
b	8	5	10	2°
c	10	8	5	3°
d	8	10	5	4°
e	6	7	10	5°

Alternatives **Criteria Scores Matrix** **DM Ordinal Preferences**

Indirect Identification: Cardinal Based

Student	*Math.*	*Stat.*	*Liter.*	*DM Score*
a	10	5	8	8.6
b	8	5	10	7.9
c	10	8	5	7.4
d	8	10	5	7.4
e	6	7	10	7.2

Alternatives **Criteria Scores Matrix** **DM Cardinal Preferences**

Cardinal Based: Least Squares Approach

$$\textbf{LS}: MIN_{m(T)} \sum_{a=1}^{v} \left(C_m(a) - y(a)\right)^2$$

$$s.t.$$

$$\begin{cases} \sum_{T \subseteq S} m(T) \geq 0 \;\; \forall\, S \subseteq N, \forall\, i \in S : Monotonicity \\ \sum_{\substack{T \subseteq N \\ T \ni i}} m(T) = 1 : Boundary \end{cases}$$

Cardinal Based: Least Squares Approach

In matrix notation the LS approach can be formulated as:

$$\mathbf{y} = \mathbf{ch} + \boldsymbol{\varepsilon}$$

$$\mathbf{ch} \equiv \mathbf{A} \cdot \mathbf{m}$$

$$\mathbf{A} = [\mathbf{X} \quad \boldsymbol{\Lambda}] = \begin{bmatrix} x_{11} & \cdots & x_{1n} & \Lambda_1 \\ x_{21} & \cdots & x_{2n} & \Lambda_2 \\ \vdots & \vdots & \vdots & \vdots \\ x_{v1} & \cdots & x_{vn} & \Lambda_v \end{bmatrix}$$

$$\Lambda_i = \Lambda\, x_i(T)\ \forall T \subseteq N$$

$$Min_{\mathbf{m}} \boldsymbol{\varepsilon}' \boldsymbol{\varepsilon} = [\mathbf{y} - \mathbf{A} \cdot \mathbf{m}]' \cdot [\mathbf{y} - \mathbf{A} \cdot \mathbf{m}]$$

$$s.t.$$

$$\begin{cases} \boldsymbol{\Psi} \cdot \mathbf{m} \geq \mathbf{0} & Monotonicity\ Conditions \\ \mathbf{1}' \cdot \mathbf{m} = 1 & Boundary\ Condition \end{cases}$$

Limitations Again

k	n	Möbius	Constraints
2	2	3	4
	3	6	12
	4	10	32
	…	…	…
	10	55	5120
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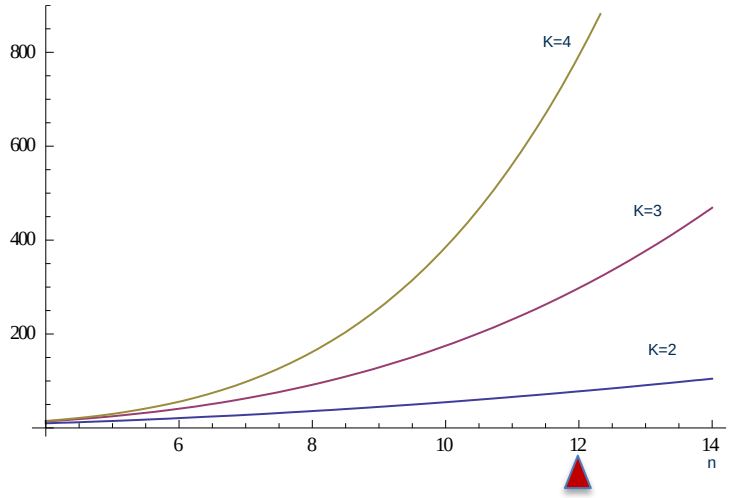
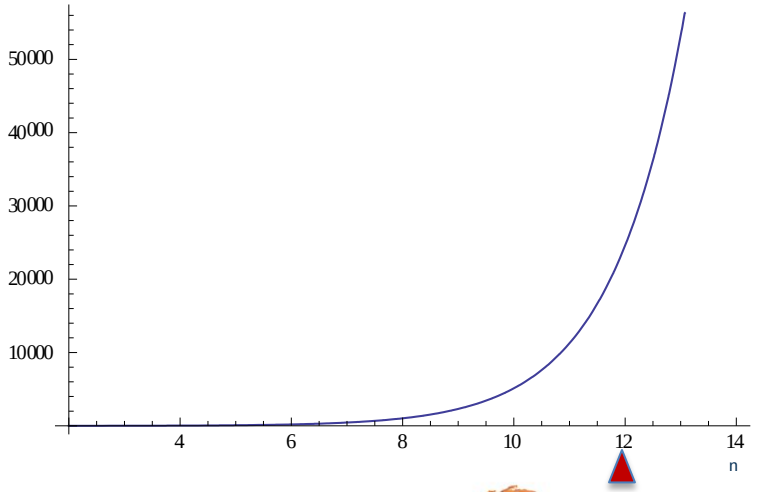
$$N°Möbius = \sum_{j=1}^{k} \binom{n}{j}$$



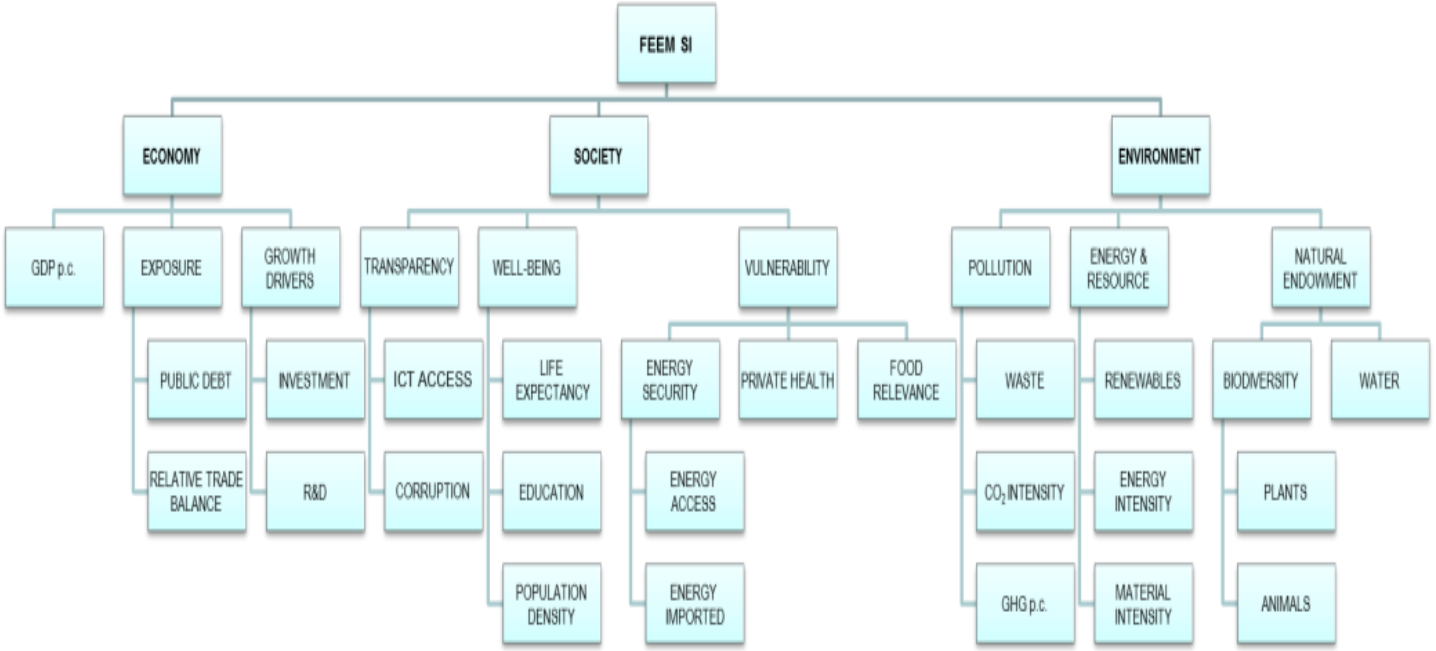
$$N°Cons. = n \cdot 2^{(n-1)}$$



FEEM Applications

FEEM Sustainability Index

4.2 2013 FEEM Sustainability Index

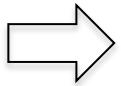


Questionnaire Design

Qualitative Scale		Numerical Scale
Criteria Performance	*Expert Evaluation*	
Very bad	Very Dissatisfied	0
Bad	Dissatisfied	0.25
Fair	Nor Diss./ Sat.	0.5
Good	Satisfied	0.75
Excellent	Very Satisfied	1

Alternative	Criteria *A*	*B*	*C*	Expert *Overall Evaluation*
1	Excellent	Good	Bad	-
2	Excellent	Bad	Good	-
3	Good	Excellent	Bad	-
4	Bad	Excellent	Good	-
5	Bad	Good	Excellent	-

Qualitative Scale



Alternative	Criteria *A*	*B*	*C*	Expert *Overall Evaluation*
1	1	0.75	0.25	-
2	1	0.25	0.75	-
3	0.75	1	0.25	-
4	0.25	1	0.75	-
5	0.25	0.75	1	-

Numerical Scale

FEEM SI – DM Preferences Aggregation

For each node of the decision tree the final Fuzzy Measures are the result of DM weighted preference

$$\mathbf{y}_j = \mathbf{ch}_j + \boldsymbol{\varepsilon}_j$$

$$g_j = \boldsymbol{\varepsilon}_j{}' \boldsymbol{\varepsilon}_j$$

$$h_j = e^{-\rho g_j}$$

$$w_j = \frac{h_j}{\sum_{j=1}^{d} h_j}$$

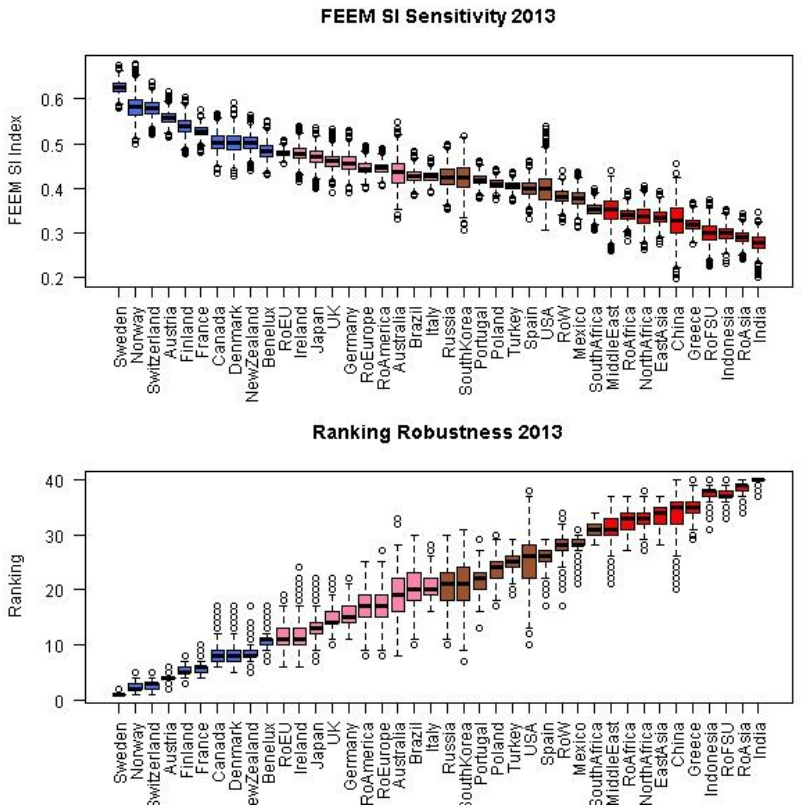
$$m^*\{T\} = \sum_{j=1}^{d} w_j m_j\{T\} \quad \forall\, T \subseteq N \text{ and } t \leq k$$

Pillar	*Node*	*Criteria*	*Shapley (%)*
	FEEM SI	Environment	35.70
		Society	38.60
		Economy	25.70
Environmental Pillar	*Environment*	Natural Endowment	35.59
		Energy & Resources	30.70
		Pollution	33.71
	Natural Endowment	Water	48.59
		Biodiversity	51.41
	Biodiversity	Animals	51.07
		Plants	48.93
	Energy & Resources	Material Intensity	32.01
		Energy Intensity	31.93
		Renewables	36.06
	Pollution	GHG p.c.	37.33
		CO2 Intensity	33.65
		Waste	29.02
Social Pillar	*Society*	Vulnerability	29.47
		Well-Being	41.19
		Transparency	29.34
	Vulnerability	Food Relevance	33.91
		Private Health	32.28
	Energy Security	Energy Security	33.81
		Energy Imported	29.69
		Energy Access	70.31
	Well-Being	Population Density	21.04
		Education	49.09
		Life Expectancy	29.87
	Transparency	Corruption	70.47
		ICT Access	29.53
Economy Pillar	*Economy*	Growth Drivers	38.34
		Exposure	31.42
		GDP p.c.	30.24
	Growth Drivers	R&D	56.92
		Investment	43.08
	Exposure	Relative Trade Balance	57.69
		Public Debt	42.31

Final Shapley Values

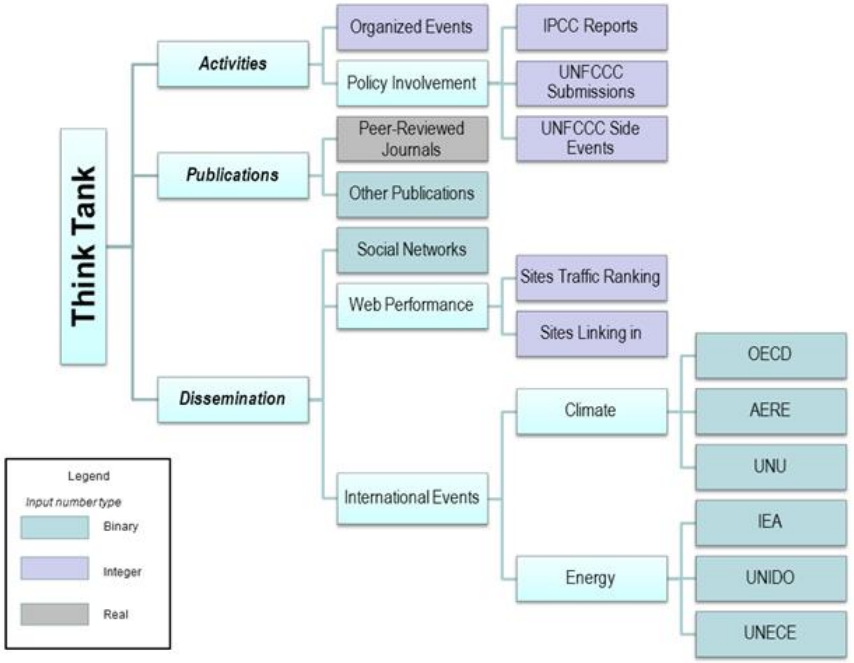
FEEM SI Results

	Year 2013									
	FEEM SI Sensitivity					Ranking Robustness				
Region	*Mean*	*Min*	*Max*	*St.Dev*	*Sensitivity*	*Mean*	*Best Rank*	*Worst Rank*	*St.Dev*	*Robustness*
Sweden	0.626	0.578	0.676	0.014	Medium	1.011	1	2	0.102	High
Norway	0.582	0.497	0.679	0.027	Medium	2.566	1	5	0.717	Medium
Switzerland	0.579	0.520	0.638	0.017	Medium	2.580	1	5	0.583	Medium
Austria	0.559	0.513	0.615	0.014	Medium	3.951	2	6	0.456	Medium
Finland	0.539	0.475	0.605	0.019	Medium	5.161	3	8	0.613	Medium
France	0.527	0.479	0.575	0.012	Medium	5.958	4	10	0.795	Medium
Canada	0.502	0.435	0.568	0.021	Medium	8.282	6	17	1.446	Medium
Denmark	0.502	0.428	0.592	0.023	Medium	8.334	5	17	1.702	Medium
NewZealand	0.501	0.439	0.564	0.017	Medium	8.418	5	17	1.345	Medium
Benelux	0.484	0.430	0.551	0.019	Medium	10.625	7	17	1.338	Medium
RoEU	0.479	0.454	0.508	0.008	Low	11.436	6	19	2.471	Medium
Ireland	0.478	0.414	0.538	0.019	Medium	11.872	6	24	2.212	Medium
Japan	0.469	0.401	0.535	0.019	Medium	12.988	7	22	2.056	Medium
UK	0.461	0.392	0.532	0.017	Medium	14.654	10	22	1.591	Medium
Germany	0.457	0.390	0.529	0.019	Medium	15.472	11	22	1.734	Medium
RoEurope	0.445	0.400	0.494	0.014	Medium	16.851	8	27	2.995	Medium
RoAmerica	0.445	0.405	0.490	0.013	Medium	16.802	8	25	2.509	Medium
Australia	0.434	0.332	0.548	0.031	High	19.115	8	33	3.966	Low
Brazil	0.427	0.382	0.482	0.015	Medium	20.395	10	30	3.190	Low
Italy	0.426	0.391	0.467	0.011	Low	20.405	16	28	1.834	Medium
Russia	0.425	0.352	0.499	0.024	Medium	20.786	10	30	3.388	Low
SouthKorea	0.425	0.305	0.517	0.032	High	20.879	7	31	4.230	Low
Portugal	0.419	0.379	0.462	0.013	Medium	21.821	13	29	1.987	Medium
Poland	0.410	0.376	0.442	0.010	Low	23.785	17	30	1.920	Medium
Turkey	0.405	0.371	0.436	0.010	Low	24.837	19	29	1.861	Medium
Spain	0.400	0.330	0.460	0.018	Medium	25.569	17	29	1.546	Medium
USA	0.400	0.307	0.539	0.035	High	25.223	10	38	4.476	Low
RoW	0.381	0.325	0.441	0.016	Medium	27.876	17	34	1.998	Medium
Mexico	0.377	0.313	0.437	0.018	Medium	28.335	21	31	1.199	Medium
SouthAfrica	0.354	0.308	0.399	0.014	Medium	30.908	28	34	0.990	Medium
MiddleEast	0.352	0.261	0.439	0.030	Medium	31.193	21	37	2.173	Medium
RoAfrica	0.341	0.282	0.394	0.014	Medium	32.679	27	37	1.852	Medium
NorthAfrica	0.337	0.263	0.405	0.022	Medium	33.198	27	38	1.642	Medium
EastAsia	0.335	0.276	0.392	0.017	Medium	33.384	28	37	1.701	Medium
China	0.328	0.198	0.456	0.038	High	34.285	20	40	3.003	Low
Greece	0.319	0.275	0.369	0.015	Medium	35.332	29	40	1.833	Medium
RoFSU	0.300	0.225	0.376	0.023	Medium	37.452	33	40	1.271	Medium
Indonesia	0.299	0.232	0.352	0.017	Medium	37.378	31	40	1.091	Medium
RoAsia	0.290	0.241	0.342	0.015	Medium	38.480	34	40	1.031	Medium
India	0.278	0.200	0.345	0.019	Medium	39.736	37	40	0.539	High



ICCG Climate Think Tank Ranking

ICCG Climate Think Tank Ranking

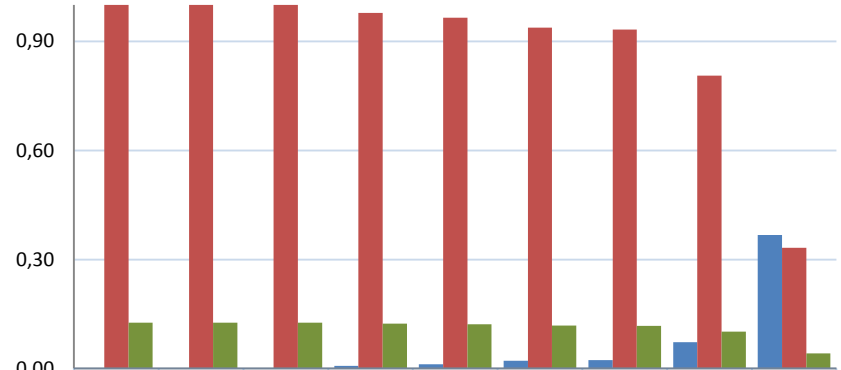


ICCG Ranking – DM Weighting Scheme

$$g_j = \boldsymbol{\varepsilon}_j{}' \boldsymbol{\varepsilon}_j$$

$$h_j = e^{-\rho g_j}$$

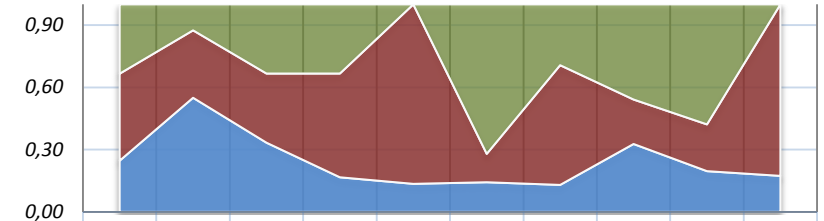
$$w_j = \frac{h_j}{\sum_{j=1}^{d} h_j}$$



	DM_1	DM_2	DM_3	DM_4	DM_5	DM_6	DM_7	DM_8	DM_9
g(j)	0,00	0,00	0,00	0,01	0,01	0,02	0,02	0,07	0,37
'h(j)	1,00	1,00	1,00	0,98	0,96	0,94	0,93	0,81	0,33
'w(j)	0,13	0,13	0,13	0,12	0,12	0,12	0,12	0,10	0,04

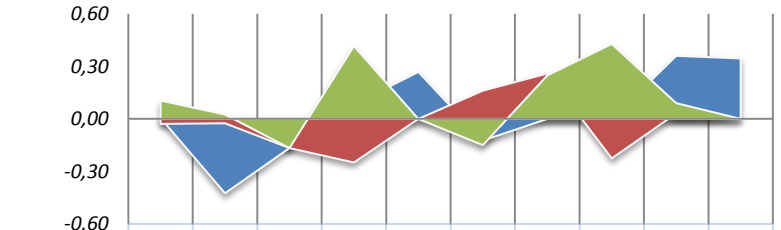
Weighting Scheme (Main Node)

4.11 ICCG Ranking – DM Preferences



	Agg. value	DM_1	DM_2	DM_3	DM_4	DM_5	DM_6	DM_7	DM_8	DM_9
Diss.	0,33	0,13	0,33	0,33	0,00	0,72	0,29	0,46	0,58	0,00
Publ.	0,42	0,33	0,33	0,50	0,87	0,14	0,58	0,21	0,23	0,83
Activ.	0,25	0,55	0,33	0,17	0,13	0,14	0,13	0,33	0,20	0,17

Shapley Values (Main node)



	Agg. value	DM_1	DM_2	DM_3	DM_4	DM_5	DM_6	DM_7	DM_8	DM_9
Act&Publ.	0,01	-0,42	-0,17	0,08	0,27	-0,12	0,00	0,00	0,36	0,35
Act&Diss.	-0,03	-0,03	-0,17	-0,25	0,00	0,16	0,26	-0,23	0,03	0,00
Publ&Diss.	0,11	0,02	-0,17	0,42	0,00	-0,15	0,25	0,43	0,09	0,00

Interaction Indices (Main node)

ICCG Ranking – DM Preferences Aggregation

For each node of the decision tree the final Fuzzy Measures are the result of DM weighted preference

$$g_j = \boldsymbol{\varepsilon}_j{}' \boldsymbol{\varepsilon}_j$$

$$h_j = e^{-\rho g_j}$$

$$w_j = \frac{h_j}{\sum_{j=1}^{d} h_j}$$

$$m^*\{T\} = \sum_{j=1}^{d} w_j m_j\{T\} \quad \forall\, T \subseteq N \text{ and } t \leq k$$

Pillar	*Node*	*Criteria*	*Shapley*
Main	Think Tank	Activities	0.25
		Publications	0.42
		Dissemination	0.33
Activities	Activities	Organized Events	0.28
		Policy Involvement	0.72
	Policy Involvement	IPCC Reports	0.47
		UNFCCC Submission	0.22
		UNFCCC Side Events	0.31
Public.	Publications	Peer-Review Journals	0.68
		Other Publications	0.32
Dissemination	Dissemination	Social Network	0.15
		Web Performance	0.48
		International events	0.37
	Web Performance	Sites Traffic Ranking	0.54
		Site Linking in	0.46
	International Events	Climate Events	0.49
		Energy Events	0.51
	Climate Events	OECD	0.30
		AERE	0.35
		UNU	0.35
	Energy Events	IEA	0.35
		UNIDO	0.32
		UNECE	0.34

Final Shapley Values

Results

Rank	Think-Tank	Score
1	Woods Hole Research Center (WHRC)	0.418
2	Mercator Research Institute on Global Commons and Climate Change (MCC)	0.400
3	Resources for the Future (RFF)	0.365
4	Basque Centre for Climate Change (BC3)	0.355
5	Potsdam Institute for Climate Impact Research (PIK)	0.336
6	Plymouth Marine Laboratory (PML)	0.332
7	Center for Global Development*	0.313
8	Worldwatch Institute	0.313
9	International Institute for Applied Systems Analysis (IIASA)	0.307
10	Union of Concerned Scientists (UCS)	0.296
11	Centre for International Forestry Research (CIFOR)	0.295
12	Institute for Ecological Economy Research (IÖW)	0.293
13	Fondazione Eni Enrico Mattei (FEEM)	0.288
14	Centre for Policy Research (CPR) *	0.282
15	Centre for European Economic Research (ZEW) *	0.280
16	Rainforest Alliance (RA)	0.279
17	Wetlands International	0.271
18	International Centre for Climate Change and Development (ICCCAD)	0.268
19	Motu Economic and Public Policy Research*	0.267
20	Environmental Defense Fund (EDF)	0.266
21	James Hutton Institute	0.266
22	Brighter Green	0.261
23	National Institute for Water & Atmospheric Research (NIWA)	0.261
24	Centre for European Policy Studies (CEPS) *	0.258
25	Climate Interactive	0.251
26	Centro Euro-Mediterraneo sui Cambiamenti Climatici (CMCC)	0.250
27	Helmholtz Centre for Environmental Research (UFZ)	0.248
28	Fundacion Bariloche*	0.244
29	Institute of International and European Affairs (IIEA) *	0.240
30	Red Cross / Red Crescent Climate Centre (RCCC)	0.230

Standardized Ranking

Rank	Think-Tank	Score
1	The Nature Conservancy (TNC)	0.545
2	Helmholtz Centre for Environmental Research (UFZ)	0.483
3	International Institute for Applied Systems Analysis (IIASA)	0.461
4	Environmental Defense Fund (EDF)	0.453
5	Potsdam Institute for Climate Impact Research (PIK)	0.401
6	Natural Resources Defense Council (NRDC)	0.398
7	World Resources Institute (WRI)	0.369
8	James Hutton Institute	0.351
9	Conservation International	0.348
10	Union of Concerned Scientists (UCS)	0.332
11	Overseas Development Institute (ODI)	0.327
12	The Energy and Resources Institute (TERI)	0.317
13	Energy Research Centre of the Netherlands	0.305
14	International Institute for Sustainable Development (IISD)	0.305
15	Centro Euro-Mediterraneo sui Cambiamenti Climatici (CMCC)	0.303
16	Stockholm Environment Institute (SEI)	0.300
17	Centre for International Forestry Research (CIFOR)	0.299
18	Global Green Growth Institute (GGGI)	0.297
19	International Center for Tropical Agriculture (CIAT) *	0.294
20	Rainforest Alliance (RA)	0.294
21	RAND Corporation *	0.292
22	Institute for Global Environmental Strategies (IGES)	0.291
23	Centre for European Economic Research (ZEW) *	0.289
24	The Climate Group (TCG)	0.287
25	National Institute for Water & Atmospheric Research (NIWA)	0.285
26	Fondazione Eni Enrico Mattei (FEEM)	0.284
27	International Institute for Environment and Development (IIED)	0.283
28	Chatham House *	0.269
29	Woods Hole Research Center (WHRC)	0.266
30	International Centre for Trade and Sustainable Development (ICTSD)	0.263

Absolute Ranking

Potential Applications (?)

- **On your work (?)**
- **Economic Modelling**
  “Fuzzy” Utility Function
  “Fuzzy” Production Function
- **Microeconometrics**
  Choice Modelling/Stated Preferences
- **Engineering Studies**

Thank you for your time